RANK 2 LOCAL SYSTEMS, BARSOTTI-TATE GROUPS, AND SHIMURA CURVES

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Abstract. We develop a descent criterion for K-linear abelian categories. Using recent advances in the Langlands correspondence due to Abe, we build a correspondence between certain rank 2 local systems and certain Barsotti-Tate groups on complete curves over a finite field. We conjecture that such Barsotti-Tate groups "come from" a family of fake elliptic curves. As an application of these ideas, we provide a criterion for being a Shimura curve over $\mathbb{F}_q$. Along the way we formulate a conjecture on the field-of-coefficients of certain compatible systems.

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1. Introduction

A consequence of Drinfeld's first work on the Langlands correspondence is the following:

Theorem. *(Drinfeld) Let $C/\mathbb{F}_q$ be a smooth affine curve with compactification $\overline{C}$. Let $\mathscr{L}$ be a rank 2 irreducible $\overline{\mathbb{Q}}_l$-local system on C such that*

- *$\mathscr{L}$ has infinite monodromy around some $\infty \in \overline{C}\backslash C$*
- *$\mathscr{L}$ has determinant $\overline{\mathbb{Q}}_l(-1)$*
- *The field of Frobenius traces of $\mathscr{L}$ is $\mathbb{Q}$*

Then $\mathscr{L}$ "comes from a family of elliptic curves", i.e. there exists a map

$$\begin{array}{ccc} & & \mathcal{E} \\ & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{f} & \mathcal{M}_{1,1} \end{array}$$

such that $\mathscr{L} \cong f^*(R^1\pi_*\mathbb{Q}_l)$. Here $\mathcal{M}_{1,1}$ is the moduli of elliptic curves with universal elliptic curve $\mathcal{E}$.

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See the recent article of Snowden-Tsimerman [37, Proposition 19, Remark 20] for how to recover this result from Drinfeld's work.

Definition 1.1. Let D be an indefinite non-split quaternion algebra over $\mathbb{Q}$ of discriminant d and let $\mathcal{O}_D$ be a fixed maximal order. Let k be a field whose characteristic is prime to d. A *fake elliptic curve* is a pair (A, i) of an abelian surface A over k together with an injective ring homomorphism $i : \mathcal{O}_D \to \mathrm{End}_k(A)$.

D has a canonical involution, which we denote ι. Pick $t \in \mathcal{O}_D$ with $t^2 = -d$. Then there is another associated involution $*$ on D:

$$x^* := t^{-1} x^\iota t$$

There is a unique principal polarization λ on A such that the Rosati involution restricts to $*$ on $\mathcal{O}_D$. We sometimes refer to the triple (A, λ, i) as a fake elliptic curve, suppressing the implicit dependence on $t \in \mathcal{O}_D$.

Just as one can construct a modular curve parameterizing elliptic curves, there is a Shimura curve X^D parameterizing fake elliptic curves with multiplication by $\mathcal{O}_D$. Over the complex numbers, these are compact hyperbolic curves. Explicitly, if one chooses an isomorphism $D \otimes \mathbb{R} \cong M_{2\times 2}(\mathbb{R})$, look at the image of $\Gamma = \mathcal{O}^1_D$ of elements of $\mathcal{O}^*_D$ of norm 1 (for the standard norm on $\mathcal{O}_D$) inside of $SL(2, \mathbb{R})$. This is a discrete subgroup and in fact acts properly discontinuously and cocompactly on $\mathbb{H}$. The quotient $X^D = [\mathbb{H}/\Gamma]$ is the complex Shimura curve associated to $\mathcal{O}_D$. In fact, X^D has a canonical integral model and may therefore be reduced modulo p for almost all p. See [6] or [4, Section 2] for a thorough introduction to moduli spaces of fake elliptic curves.

Conjecture 1.2. *Let $C/\mathbb{F}_q$ be a smooth projective curve. Let $\mathscr{L}$ be a rank 2 irreducible $\overline{\mathbb{Q}}_l$-local system on C such that*

- *$\mathscr{L}$ has infinite image*
- *$\mathscr{L}$ has determinant $\overline{\mathbb{Q}}_l(-1)$*
- *The field of Frobenius traces of $\mathscr{L}$ is $\mathbb{Q}$*

Then, $\mathscr{L}$ "comes from a family of fake elliptic curves", i.e. there exists

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "\pi"] \\ C \arrow[r, "f"] & X^D \end{tikzcd}$$

where X^D is a moduli of fake elliptic curves with universal family $\mathcal{A}$ such that $(\mathscr{L}^{\oplus 2}) \cong f^*(R^1\pi_*\mathbb{Q}_l)$.

In this article we prove the following, which perhaps provides some evidence for 1.2.

Theorem. *(Theorem 10.10) Let C be a smooth, geometrically connected, complete curve over $\mathbb{F}_q$ with q a square. There is a natural bijection between the following two sets*

$$\left\{\begin{array}{l} \overline{\mathbb{Q}}_l\text{-local systems } \mathscr{L} \text{ on } C \text{ such that} \\ \bullet\, \mathscr{L} \text{ is irreducible of rank 2} \\ \bullet\, \mathscr{L} \text{ has trivial determinant} \\ \bullet\, \text{The Frobenius traces are in } \mathbb{Q} \\ \bullet\, \mathscr{L} \text{ has infinite image,} \\ \text{up to isomorphism} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} p\text{-divisible groups } \mathscr{G} \text{ on } C \text{ such that} \\ \bullet\, \mathscr{G} \text{ has height 2 and dimension 1} \\ \bullet\, \mathscr{G} \text{ is generically versally deformed} \\ \bullet\, \mathbb{D}(\mathscr{G}) \text{ has all Frobenius traces in } \mathbb{Q} \\ \bullet\, \mathscr{G} \text{ has ordinary and supersingular points,} \\ \text{up to isomorphism} \end{array}\right\}$$

such that if $\mathscr{L}$ corresponds to $\mathscr{G}$, then $\mathscr{L} \otimes \mathbb{Q}_l(-1/2)$ is compatible with the F-isocrystal $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q}$.

Remark. The condition that q is a square is to ensure that the character $\overline{\mathbb{Q}}_l(1/2)$ on $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ has Frobenius acting as an integer. Therefore $\mathscr{L}(-1/2)$ also has Frobenius traces in $\mathbb{Q}$ and determinant $\overline{\mathbb{Q}}_l(-1)$, exactly as in Conjecture 1.2.

Remark. If $A \to C$ were a hypothetical fake elliptic curve verifying Conjecture 1.2, then $A[p^\infty] \cong \mathscr{G} \oplus \mathscr{G}$ for $\mathscr{G}$ as in Theorem 10.10.

Remark. In particular, given $\mathscr{L}$ as in Theorem 10.10, there is a natural effective divisor on C (possibly empty), whose underlying set of points are those $c \in |C|$ over which $\mathscr{G}$ is not versally deformed. If Conjecture 1.2 is true, this divisor is the ramification divisor of a generically separable map

$$C \to X^D$$

realizing $\mathscr{L}$ as "coming from a family of fake elliptic curves."

Theorem 10.10 arose from the following question: is there a "purely group-theoretic" characterization of proper Shimura curves over $\mathbb{F}_q$? As an application of our techniques, we have the following criterion, largely motivated by an article of Mochizuki [33].

Theorem. *(Theorem 12.10) Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be an étale correspondence of smooth, geometrically connected, complete curves without a core over $\mathbb{F}_q$ with q a square. Let $\mathscr{L}$ be a $\overline{\mathbb{Q}}_l$-local system on X as in Theorem 10.10 such that $f^*\mathscr{L} \cong g^*\mathscr{L}$ as local systems on Z. Suppose the $\mathscr{G} \to X$ constructed via Theorem 10.10 is everywhere versally deformed. Then X and Z are the reductions modulo p of Shimura curves.*

We remark that the conditions of the Theorem 12.10 are met in the case of the reduction modulo p of an l-adic Hecke correspondence of Shimura curves parametrizing fake elliptic curves, as long as $l \nmid pd$. We conjecture that if one drops the hypothesis of being "everywhere versally deformed" in Theorem 12.10, X and Z are obtained from Shimura curves by an "Igusa level structure" construction (see Examples 10.4, 12.7.)

Finally, as a basic observation from our analysis, we prove the following:

Theorem. *(Theorem 11.6) Let X be a smooth, geometrically connected curve over $\mathbb{F}_q$. Let $\mathcal{E} \in$ **F-Isoc**$^\dagger(X)$ be an overconvergent F-isocrystal on X with coefficients in $\mathbb{Q}_p$ that is rank 2, absolutely irreducible, and has trivial determinant. Suppose further that the field of traces of $\mathcal{E}$ is $\mathbb{Q}$. Then $\mathcal{E}$ has finite monodromy.*

We now briefly discuss the sections. The goal of Sections 3-10 is to prove Theorem 10.10.

- Section 3 sets up a general machinery extension-of-scalars and Galois descent for K-linear abelian categories. For other references, see Deligne [16] or Sosna [38].
- Section 4 constructs a Brauer-class obstruction for descending certain objects in an abelian K-linear category. This is used to prove a criterion for descent which will be useful later.
- Section 5 reviews background on F-isocrystals over perfect fields k of characteristic p and explicitly describes the base-changed category $F\text{-Isoc}(k)_L$ for L a p-adic local field.
- Section 6 specializes the discussion of Section 5 to finite fields. Many of the results are likely well-known, but we could not find always find a reference.
- Section 7 clarifies our conventions about "coefficient objects"; in particular, we discuss (overconvergent) F-crystals and isocrystals over varieties. For a more comprehensive survey, see Kedlaya's recent articles [25, 27].
- Section 8 is a brief discussion of Deligne's "Companions Conjecture" from Weil II [14]. The p-adic part of this conjecture has recently been resolved for X a curve by Abe [1, 2] by proving a p-adic Langlands correspondence. We will use the so-called "crystalline companion", an overconvergent F-isocrystal, extensively for the rest of the article.
- Section 9 is where the above steps come together. Given $\mathscr{L}$ as in Theorem 10.10, the p-adic companion is an F-isocrystal $\mathscr{E}$ on X with coefficients in $\overline{\mathbb{Q}}_p$. Using the descent criterion of Section 4 and the explicit description furnished in Section 6, we prove that if $p^2|q$, then $\mathscr{E}(-\frac{1}{2})$ descends to $\mathbb{Q}_p$. This is the content of Corollary 9.10. Using work of Katz [24], de Jong [11], and the highly nontrivial slope-bounds of V. Lafforgue [31], we then prove that $\mathscr{E}(-\frac{1}{2})$ is the Dieudonné isocrystal of a *(non-canonical)* height 2, dimension 1 Barsotti-Tate group $\mathscr{G}$ on X.

- Section 10 briefly discusses the deformation theory of BT groups. We then use work of Xia [43] to prove that there is a unique such $\mathscr{G}$ that is *generically versally deformed,* thereby proving Theorem 10.10.

BT groups are generally non-algebraic, but here we can show there are only finitely many that occur in Theorem 10.10 via the Langlands correspondence. In Section 11, we speculate on when height 2, dimension 1 BT groups $\mathscr{G}$ "come from a family of abelian varieties". We pose concrete instantiations of this question. Here, the information about the p-adic companion is vital, and we are optimistic that there will be many future applications of Abe's recent resolution of the p-adic part of Deligne's conjecture.

Finally, in Section 12, we discuss applications of our results to "characterizing" certain Shimura curves over $\mathbb{F}_q$ using results of Mochizuki [33] and Xia [43].

Acknowledgements. This work is based on Chapter 3 of my PhD thesis at Columbia University. I am very grateful to Johan de Jong, my former thesis advisor, for guiding this project and for countless inspiring discussions. Ching-Li Chai read my thesis very carefully and provided many valuable corrections and remarks; I think him. I also thank Marco d'Addezio, Hélène Esnault and Ambrus Pál for discussions on the subject of this article. During my time at Freie Universität Berlin I have been funded by an NSF postdoctoral fellowship, Grant No. DMS-1605825.

2. Conventions, Notation, and Terminology

For convenience, we explicitly state conventions and notations. These are in full force unless otherwise stated.

(1) A *curve* C/k is a separated geometrically integral scheme of dimension 1 over k. Unless otherwise explicitly stated, we assume $C \to \mathrm{Spec}(k)$ is smooth.
(2) A *variety* X/k is a geometrically integral k-scheme of finite type, also assumed to be smooth unless otherwise stated.
(3) A morphism of curves $X \to Y$ over k is a morphism of k-schemes that is non-constant, finite, and generically separable.
(4) A smooth curve C over a field k is said to be *hyperbolic* if $\mathrm{Aut}_{\overline{k}}(C_{\overline{k}})$ is finite.
(5) p is a prime number and $q = p^d$.
(6) $\mathbb{F}$ is a fixed algebraic closure of $\mathbb{F}_p$.
(7) If k is perfect, $W(k)$ denotes the Witt vectors and σ the canonical lift of Frobenius. $K(k) \cong W(k) \otimes \mathbb{Q}$.
(8) All p-adic valuations are normalized such that $v_p(p) = 1$.
(9) If $\mathcal{C}$ is a K-linear category and L/K is an algebraic extension, we denote by $\mathcal{C}_L$ the base-changed category.
(10) Let X/k be a smooth variety over a perfect field. Then the category $\mathbf{F\text{-}Isoc}^{\dagger}(X)$ is the category of *overconvergent F-isocrystals on X.*
(11) If $\mathscr{G} \to X$ is a p-divisible (a.k.a Barsotti-Tate) group, we denote by $\mathbb{D}(\mathscr{G})$ the *contravariant* Dieudonné crystal.

3. Extension of Scalars and Galois descent for abelian categories

We discuss extension-of-scalars of K-linear categories; in the case of K-linear abelian categories, we state Galois descent. We also carefully discuss absolute irreducibility and semi-simplicity. The material in this section is mostly well-known; we include it for the sake of completeness.

Definition 3.1. Let $\mathcal{C}$ be a K-linear additive category, where K is a field. Let L/K be a finite field extension. We define the base-changed category $\mathcal{C}_L$ as follows:

- Objects of $\mathcal{C}_L$ are pairs (M, f), where M is an object of $\mathcal{C}$ and $f : L \to End_{\mathcal{C}} M$ is a homomorphism of K-algebras. We call such an f *an L-structure on M.*
- Morphisms of $\mathcal{C}_L$ are morphisms of $\mathcal{C}$ that are compatible with the L-structure.

It is easy to check (and done in Sosna [38]; see also Deligne [13, Section 4] or the "official" version of that paper [16, Section 5]) that $\mathcal{C}_L$ is an L-linear additive category and that if $\mathcal{C}$ was abelian, so is $\mathcal{C}_L$. It is somewhat more involved to prove that if $\mathcal{C}$ is a K-linear rigid abelian $\otimes$ category, $\mathcal{C}_L$ is an L-linear rigid abelian $\otimes$ category, in particular with natural $\otimes$ and dual (see [13, Section 4.3] for elegant definitions of these.) Moreover, if $\mathcal{C}$ is a K-linear Tannakian category (not necessarily assumed to be neutral), then $\mathcal{C}_L$ is an L-linear Tannakian category. This last fact is a nontrivial theorem; see [13, Theorem 4.4] for an "elementary" proof.

Proposition 3.2. *If $\mathcal{C}$ and $\mathcal{D}$ are K-linear categories, then $(\mathcal{C} \oplus \mathcal{D})_L$ is equivalent to $\mathcal{C}_L \oplus \mathcal{D}_L$.*

Proof. Exercise. □

Remark 3.3. Deligne [13] defines $\mathcal{C}_L$ in a slightly different way. Given an object X of $\mathcal{C}$ and a finite dimensional K-vector space V, define $V \otimes X$ to represent the functor $Y \mapsto V \otimes_K Hom_K(Y, X)$. Given a finite extension L/K, an object with L-structure is defined to be an object X of $\mathcal{C}$ together with a morphism $L \otimes X \to X$ in $\mathcal{C}$. Then the base-changed category $\mathcal{C}_L$ is defined to be the category of objects equipped with an L-structure and with morphisms respecting the L-structure.

Note that there is an induction functor: $\mathrm{Ind}_K^L : \mathcal{C} \to \mathcal{C}_L$. Let $\{\alpha\}$ be a basis for L/K.

$$\mathrm{Ind}_K^L M = (\bigoplus_{\alpha} M, f)$$

where f is induced by the action of L on the $\{\alpha\}$. In Deligne's formulation, $\mathrm{Ind}_K^L M$ is just $L \otimes M$ and the L-structure is the multiplication map $L \otimes_K L \otimes M \to L \otimes M$. The induction functor has a natural right adjoint: Res_K^L which simply forgets about the L-structure. If M is an object of $\mathcal{C}$, we sometimes denote by M_L the object $\mathrm{Ind}_K^L M$ for shorthand. If C is a K-linear abelian category, induction and restriction are exact functors.

Proposition 3.4. *Let A and B be objects of a K-linear category $\mathcal{C}$ and let L be a finite extension of K. $Hom_{\mathcal{C}_L}(A_L, B_L) \cong L \otimes_K Hom_{\mathcal{C}}(A, B)$.*

Proof. By the adjunction, $\mathrm{Hom}_{\mathcal{C}_L}(A_L, B_L) \cong \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{Res}_{L/K}(B_L))$. On the other hand, $\mathrm{Res}_{L/K}(B_L)$ is naturally isomorphic to the object $L \otimes B$. By the defining property of $L \otimes B$, $\mathrm{Hom}_{\mathcal{C}}(A, \mathrm{Res}_{L/K}(B_L)) \cong L \otimes_K \mathrm{Hom}_{\mathcal{C}}(A, B)$ as desired. □

Remark 3.5. The induction functor allows us to describe more general extensions-of-scalars. Namely, let E be an *algebraic* field extension of K. We define $\mathcal{C}_E$ to be the 2-colimit of the categories $\mathcal{C}_L$ where L ranges over the subfields of E finite over K. In particular, we can define a category $\mathcal{C}_{\overline{K}}$ and an induction functor

$$Ind_K^{\overline{K}} : \mathcal{C} \to \mathcal{C}_{\overline{K}}$$

Definition 3.6. Let $\mathcal{C}$ be a K-linear abelian category. We say that an object M is *absolutely irreducible* if $Ind_K^{\overline{K}}(M)$ is an irreducible object of $\mathcal{C}_{\overline{K}}$.

If L/K is a Galois extension, note that we have a natural (strict) action of $G := \mathrm{Gal}(L/K)$ on the category $\mathcal{C}_L$ by twisting the L-structure. That is, if $g \in G$, ${}^g(C, f) := (C, f \circ g^{-1})$. The group G "does nothing" to maps: a map $\phi : (C, f) \to (C', f')$ in $\mathcal{C}_L$ is just a map $\phi_K : C \to C'$ in $\mathcal{C}$ that commutes with the L-actions, and $g \in G$ acts by fixing the underlying ϕ_K while twisting the underlying L-structures f, f': ${}^g\phi : (C, f \circ g^{-1}) \to (C', f' \circ g^{-1})$.

A computation shows that if $\lambda \in L$ is considered as a scalar endomorphism of M, then the endomorphism ${}^g\lambda : {}^gM \to {}^gM$ is the scalar $g(\lambda)$. If $\mathcal{C}$ is a K-linear rigid abelian $\otimes$ category, then this action is compatible in every way imaginable with the inherited rigid abelian $\otimes$ structure on $\mathcal{C}_L$. For instance, there are canonical isomorphisms ${}^g(M^*) \cong ({}^gM)^*$ and ${}^g(M \otimes N) \cong ({}^gM) \otimes ({}^gN)$. Similarly, if $F : \mathcal{C} \to \mathcal{D}$ is a K-linear functor between K-linear categories, we have a canonical extension functor $F_L : \mathcal{C}_L \to \mathcal{D}_L$ and ${}^gF_L(M) \cong F_L({}^gM)$.

Definition 3.7. Suppose $\mathcal{C}$ is a abelian K-linear category and L/K is a finite Galois extension with group G. Let $\mathcal{C}_L$ be the base-changed category. We define the category of descent data, $(\mathcal{C}_L)^G$, as follows. The objects of $(\mathcal{C}_L)^G$ are pairs $(M, \{c_g\})$ where M is an object of $\mathcal{C}_L$ and the $c_g : M \to {}^gM$ are a collection of isomorphisms for each $g \in G$ that satisfies the cocycle condition. The morphisms of $(\mathcal{C}_L)^G$ are maps $(f_g : {}^gM \to {}^gN)$ that intertwine the c_g.

Lemma 3.8. *If $\mathcal{C}$ is an abelian K-linear category and L/K is a finite Galois extension, then Galois descent holds. That is, $\mathcal{C}$ is equivalent to the category $(\mathcal{C}_L)^G$.*

Proof. This is [38, Lemma 2.7]. □

Remark 3.9. We say objects and morphisms in the essential image of Ind_K^L "descend".

We may use the technology of descent to compare the notions of semi-simplicity (intrinsic to any abelian category) with absolute semi-simplicity.

Proposition 3.10. *Let $\mathcal{C}$ be a K-linear abelian category with K a characteristic 0 field. Suppose M is an object of $\mathcal{C}$ that is absolutely semi-simple, i.e. $M_{\overline{K}}$ is a semi-simple object of $\mathcal{C}_{\overline{K}}$. Then M is semi-simple.*

Proof. Let $i : N \hookrightarrow M$ be a sub-object of M in $\mathcal{C}$. By assumption, there is an $f : M_{\overline{K}} \to N_{\overline{K}}$ splitting i. This f is defined over some finite Galois extension L/K with group G. Averaging f over G (using that K is characteristic 0), we get a section of i_L that is invariant under G. By Lemma 3.8, this section is defined over K, as desired. □

Proposition 3.11. *Let $\mathcal{C}$ be a K-linear abelian category with K a perfect field. Suppose M is a semi-simple object of $\mathcal{C}$ and that $End_{\mathcal{C}}(M)$ is a finite dimensional K-algebra. Then $M_{\overline{K}}$ is a semi-simple object of $\mathcal{C}_{\overline{K}}$.*

Proof. We may suppose M is irreducible. If $M_{\overline{K}}$ were irreducible, we would be done, so suppose it has is not irreducible. The hypothesis on the endomorphism algebra ensures that $M_{\overline{K}}$ has a finite Jordan-Holder filtration, so in particular $M_{\overline{K}}$ has proper simple sub-object N, defined over some finite Galois extension L/K (this is where we use K being perfect.) Then all of the Galois conjugates are all are also simple sub-objects of $M_{\overline{K}}$. The sum $S_{\overline{K}}$ of all simple sub-objects of $M_{\overline{K}}$ therefore descends to an object S of $\mathcal{C}$ by Lemma 3.8. As we assumed M was irreducible, S is M. On the other hand, $S_{\overline{K}}$ is the maximal sub-object of $M_{\overline{K}}$ that is semi-simple. Therefore $M_{\overline{K}}$ is semi-simple, as desired. For a proof with slightly different hypotheses on $\mathcal{C}$, see [13, Lemme 4.2]. □

Corollary 3.12. *Let $\mathcal{C}$ be a K-linear abelian category with K a characteristic 0 field. Let M be an object of $\mathcal{C}$ with $End_{\mathcal{C}}M$ a finite dimensional K-algebra. Then M is semi-simple if and only if it is absolutely semi-simple.*

We record for later a generalization of Schur's Lemma.

Lemma 3.13. *Let $\mathcal{C}$ be a K-linear abelian category such that the Hom groups of $\mathcal{C}$ are finite dimensional K-vector spaces. Let M be an absolutely irreducible object of $\mathcal{C}$. Then $End_{\mathcal{C}}(M) \cong K$.*

Proof. As M is absolutely irreducible, $\mathrm{End}_{\mathcal{C}_{\overline{K}}} M_{\overline{K}}$ is a division algebra over $\overline{K}$. The hypothesis ensures that this division algebra is finite dimensional over $\overline{K}$ and hence is exactly $\overline{K}$. Now, by Proposition 3.4, $\mathrm{End}_{\mathcal{C}}(M) \otimes_K \overline{K} \cong \overline{K}$ as algebras. By descent, $\mathrm{End}_{\mathcal{C}}(M) \cong K$ as desired. □

4. A 2-cocycle obstruction for descent

Suppose $\mathcal{C}$ is an abelian K-linear category and L/K a finite Galois extension with group G. Let $\mathcal{C}_L$ be the base-changed category. Suppose an object $M \in \mathrm{Ob}(\mathcal{C}_L)$ is isomorphic to all of its twists by $g \in G$ and that the natural map $L \tilde{\to} \mathrm{End}_{\mathcal{C}_L} M$ is an isomorphism. (This latter restriction will be relaxed later in the important Remark 4.7.) We will define a cohomology class $\xi_M \in H^2(G, L^*)$ such that $\xi_M = 0$ if and only if M descends to K. This construction is well-known in other contexts, see for instance [42].

Definition 4.1. For each $g \in G$ pick an isomorphism $c_g : M \to {}^gM$. The function $\xi_{M,c} : G \times G \to L^*$, depending on the choices $\{c_g\}$, is defined as follows:

$$\xi_{M,c}(g,h) = c_{gh}^{-1} \circ {}^gc_h \circ c_g \in Aut_{\mathcal{C}_L}(M) \cong L^*$$

Proposition 4.2. *The function $\xi_{M,c}$ is a 2-cocycle.*

Proof. We need to check

$${}^{g_1}\xi(g_2,g_3)\xi(g_1,g_2g_3) = \xi(g_1g_2,g_3)\xi(g_1,g_2)$$

We may think of the right hand side as a scalar function $M \to M$, which allows us to write it as

$$\begin{aligned}
\xi(g_1g_2,g_3)\xi(g_1,g_2) =& c_{g_1g_2g_3}^{-1} \circ {}^{g_1g_2}c_{g_3} \circ c_{g_1g_2} \circ c_{g_1g_2}^{-1} \circ {}^{g_1}c_{g_2} \circ c_{g_1} \\
=& c_{g_1g_2g_3}^{-1} \circ {}^{g_1}({}^{g_2}c_{g_3} \circ c_{g_2}) \circ c_{g_1} \\
=& c_{g_1g_2g_3}^{-1} \circ {}^{g_1}c_{g_2g_3} \circ {}^{g_1}(c_{g_2g_3}^{-1} \circ {}^{g_2}c_{g_3} \circ c_{g_2}) \circ c_{g_1} \\
=& c_{g_1g_2g_3}^{-1} \circ {}^{g_1}c_{g_2g_3} \circ {}^{g_1}\xi(g_2,g_3) \circ c_{g_1} \\
=& c_{g_1g_2g_3}^{-1} \circ {}^{g_1}c_{g_2g_3} \circ c_{g_1} \circ {}^{g_1}\xi(g_2,g_3) \\
=& \xi(g_1,g_2g_3)\, {}^{g_1}\xi(g_2,g_3)
\end{aligned}$$

In the penultimate line, we may commute the c_{g_1} and ${}^{g_1}\xi(g_2,g_3)$ because the latter is in L. ☐

Remark 4.3. If $\xi_{M,c} = 1$, then the collection $\{c_g\}$ form a descent datum for M which is effective by Galois descent for abelian categories, Lemma 3.8.

Proposition 4.4. *Let $\mathcal{C}$ be an abelian K-linear category and L/K a finite Galois extension with group G. Let $M \in Ob(\mathcal{C}_L)$ such that*

- *$M \cong {}^gM$ for all $g \in G$*
- *The natural map $L \hookrightarrow End_L(M)$ is an isomorphism.*

If $\xi_{M,c}$ is a coboundary, then M is in the essential image of Ind_K^L, i.e. M descends.

Proof. If $\xi_{M,c}$ is a coboundary, there exists a function $\alpha : G \to L^*$ such that

$$\xi_{M,c}(g,h) = \frac{{}^g\alpha(h)\alpha(g)}{\alpha(gh)}$$

Now, set $c'_g = \frac{c_g}{\alpha(g)} : M \to {}^gM$ and note that the c'_g are a descent datum for M because the associated $\xi_{M,c'} = 1$. ☐

Proposition 4.5. *Given $M \in \mathcal{C}_L$ as in Proposition 4.4 and two choices $\{c_g\}$ and $\{c'_g\}$ of isomorphisms, $\xi_{M,c}$ and $\xi_{M,c'}$ differ by a coboundary and thus give the same class in $H^2(G,L^*)$. We may therefore unambiguously write ξ_M for the cohomology class associated to M.*

Proof. Note that $(c'_g)^{-1} \circ c_g : M \to M$ is in L^*. This ratio will be a function $\alpha : G \to L^*$ exhibiting the ratio $\frac{\xi_{M,c}}{\xi_{M,c'}}$ as a coboundary. ☐

Corollary 4.6. *Let $\mathcal{C}$ be an abelian K-linear category, let L/K be a finite Galois extension with group G, and let $\mathcal{C}_L$ be the base-changed category. Let M be an object of $\mathcal{C}_L$ such that*

(1) *$M \cong {}^gM$ for all $g \in G$*
(2) *The natural map $L \to End_{\mathcal{C}_L}(M)$ is an isomorphism.*

Then the cocycle ξ_M (as in Definition 4.1) is 0 in $H^2(K,\mathbb{G}_m)$ if and only if M descends.

Proof. Combine Propositions 4.4 and 4.5. ☐

Remark 4.7. We did not have to assume that $L \to End_{\mathcal{C}_L}(M)$ was an isomorphism for a cocycle to exist. A necessary assumption is that there exists a collection $\{c_g\}$ of isomorphisms such that $\xi_{M,c}(g,h) \in L^*$ for all $g,h \in G$. The key is that H^2 exists as long as the coefficients are abelian. Note, however, that in this level of generality there is no guarantee that cohomology class is unique:

it depends very much on the choice of the isomorphisms $\{c_g\}$. Therefore, this technique *will not be adequate* to prove that objects do not descend; we can only prove than an object does descend by finding a collection $\{c_g\}$ whose associated ξ_c is a coboundary.

Remark 4.8. Note that if $\mathcal{C}_{\mathbb{R}}$ is the category of real representations of a compact group and $\mathcal{C}_{\mathbb{C}}$ is the complexification, namely the category of complex representations of a compact group, then this 2-cocycle has a more classical name: "Frobenius-Schur Indicator". It tests whether an irreducible complex representation of a compact group with a real character can be defined over $\mathbb{R}$. If not, the representation is called quaternionic.

Now, suppose $\mathcal{C}$ is a K-linear rigid abelian $\otimes$ category. If $M \in \mathrm{Ob}(\mathcal{C}_L)$ such that ${}^gM \cong M$ for all $g \in G$ and $L \to End_{\mathcal{C}_L}M$ is an isomorphism, then the same is true for M^*. Moreover, choosing $\{c_g\}$ for M gives the natural choice of $\{(c_g^*)^{-1}\}$ for M^* so $\xi_M^{-1} = \xi_{M^*}$.

In general, if M and N are as above with choices of isomorphisms $\{c_g : M \to {}^gM\}$ and $\{d_g : N \to {}^gN\}$ with associated cohomology classes ξ_M and ξ_N respectively, then we can cook up a cohomology class to possibly detect whether $M \otimes N$ descends, $\xi_M\xi_N$, using the isomorphism $c_g \otimes d_g : M \otimes N \to {}^gM \otimes {}^gN \cong {}^g(M \otimes N)$. This is interesting because in general $M \otimes N$ might have endomorphism algebra larger than L; in particular, we weren't guaranteed the existence of a cohomology class $\xi_{M\otimes N}$, as discussed in Remark 4.7.

Lemma 4.9. *Let $\mathcal{C}$ be a K-linear rigid abelian $\otimes$ category, L/K a finite Galois extension with group G, and $\mathcal{C}_L$ the base-changed category. Let $M \in Ob(\mathcal{C}_L)$ have endomorphism algebra L and suppose ${}^gM \cong M$ for all $g \in G$. Then $\underline{\mathrm{End}}(M) \cong M \otimes M^*$ descends to $\mathcal{C}_K$.*

Proof. As noted above, if ξ is the cocycle associated to M, then ξ^{-1} is the cocycle associated to M^*. Then $1 = \xi\xi^{-1}$ is a cocycle associated to $M \otimes M^*$, whence it descends. □

Remark 4.10. A related classical fact: let V be a finite dimensional complex representation V of a compact group G. Then the representation $\underline{\mathrm{End}}(V) \cong V \otimes V^*$ is defined over $\mathbb{R}$. One proof of this uses the fact that $\underline{\mathrm{End}}(V)$ has an invariant symmetric form: the trace.

We now give two criteria for descent. Though the second is strictly more general than the first, the hypotheses are more complicated and we found it helpful to separate the two.

Lemma 4.11. *Let $F : \mathcal{C} \to \mathcal{D}$ be a K-linear functor between abelian K-linear categories. Let L/K be a finite Galois extension with group G and let $F_L : \mathcal{C}_L \to \mathcal{D}_L$ be the base-changed functor. Let $M \in Ob(\mathcal{C}_L)$ be an object such that ${}^gM$ is isomorphic to M for $g \in G$ and $End_{\mathcal{C}_L}M \cong L$. Suppose $End_{\mathcal{D}_L}F_L(M) \cong L$. Then M descends to $\mathcal{C}$ if and only if $F_L(M)$ descends to $\mathcal{D}$.*

Proof. We claim that if we choose isomorphisms $\{d_g : F_L(M) \to {}^gF_L(M) \cong F_L({}^gM)\}$ we can choose isomorphisms $\{c_g : M \to {}^gM\}$ such that $d_g = F_L(c_g)$. This follows because $End_{\mathcal{C}_L}M \tilde{\to} End_{\mathcal{D}_L}F_L(M)$ implies that

$$Hom_{\mathcal{C}_L}(M,\ {}^gM) \to Hom_{\mathcal{D}_L}(F_L(M),\ {}^gF_L(M))$$

is an isomorphism. Therefore the cocycles $\xi_{F(M)}$ and ξ_M are the same. □

Lemma 4.12. *Let $F : \mathcal{C} \to \mathcal{D}$ be a K-linear functor between K-linear abelian categories and let L/K be a finite Galois extension with group G. Let $F_L : \mathcal{C}_L \to \mathcal{D}_L$ be the base-changed functor. Suppose $M \in Ob(\mathcal{C}_L)$ with $L \cong End_{\mathcal{C}_L}M$ and ${}^gM \cong M$ for all $g \in G$. Further suppose $F_L(M) \cong N_1 \oplus N_2$ satisfying the following two conditions*

- *$L \cong End_{\mathcal{D}_L}N_1$*
- *There is no nonzero morphism $N_1 \to {}^gN_2$ in $\mathcal{D}_L$ for any $g \in G$.*

Then M (and N_2) descend if and only if N_1 descends.

Proof. The composition

$$\begin{aligned} Hom_{\mathcal{C}_L}(M, {}^gM) \to & Hom_{\mathcal{D}_L}(F_L(M), {}^gF_L(M)) \\ & \cong Hom_{\mathcal{D}_L}(N_1 \oplus N_2, {}^gN_1 \oplus {}^gN_2) \to Hom_{\mathcal{D}_L}(N_1, {}^gN_1) \end{aligned}$$

is a homomorphism of L-vector spaces. In fact, the map is nonzero because an isomorphism $c_g : M \to {}^gM$ is sent to the isomorphism $F_L(c_g)$. By the second assumption, this projects to an isomorphism $N_1 \to {}^gN_1$. By the first assumption on N_1, the total composition is an isomorphism. Therefore a collection $\{n_g : N_1 \to {}^gN_1\}$ is canonically the same as a collection $\{m_g : M \to {}^gM\}$ and thus we have $\xi_M = \xi_{N_1}$. □

We now examine the relation between the rank of M and the order of its induced Brauer class when $\mathcal{C}$ is assumed to be a K-linear Tannakian category. Recall that Tannakian categories are not necessarily neutral, i.e. they do not always admit a fiber functor to Vect_K. For the remainder of this section, K is supposed to be a field of characteristic 0. Recall that Tannakian categories have a natural notion of rank (see, for instance, Deligne [15].) If P is an object of rank 1, there is a natural diagram

$$End(P) \cong P \otimes P^* \rightleftarrows \mathrm{K}$$

where the top arrow is evaluation (i.e. the trace) and the bottom arrow comes from the K-vector-space structure of $End(P)$. As $P \otimes P^*$ has rank 1, these two maps identify $End(P)$ isomorphically with K.

Corollary 4.13. *Let K be a field of characteristic 0, let $\mathcal{C}$ be a K-linear Tannakian category and let P be a rank-1 object. Then $K \cong End(P)$.*

Proposition 4.14. *Let K be a field of characteristic 0 and let $\mathcal{C}$ be a K-linear Tannakian category which is further supposed to be neutral, i.e. comes equipped with fiber functor $F : \mathcal{C} \to Vect_K$. Let L/K be a finite Galois extension and let $\mathcal{C}_L$ be the base-changed category. Let $P \in Ob(\mathcal{C}_L)$ be a rank-1 object. If ${}^gP \cong P$ for all $g \in G$ then P descends.*

Proof. Denote by F_L the base-changed fiber functor. By definition, P being rank 1 means that $F_L(P)$ is a rank 1 L-vector space, so $L \cong End\big(F_L(P)\big) \cong End(P)$. All vector spaces descend, so by Lemma 4.11, P descends as well. □

Lemma 4.15. *Let K be a field of characteristic 0, let $\mathcal{C}$ be a K-linear Tannakian category, and let L/K be a finite Galois extension. Suppose $M \in Ob(\mathcal{C}_L)$ has rank r, ${}^gM \cong M$ for all $g \in G$, and $L \cong End_{\mathcal{C}_L} M$. If $\bigwedge^r M$ descends (which is automatically satisfied e.g. if $\mathcal{C}$ is neutral by Proposition 4.14), then ξ_M is r-torsion in $H^2(G, L^*)$. In particular, there exists a degree r extension of K over which M is defined.*

Proof. Pick $\{c_g : M \to {}^gM\}$ giving the cohomology class ξ_M. The isomorphisms

$$\{c_g^{\otimes r} : M^{\otimes r} \to ({}^gM)^{\otimes r} \cong {}^g(M^{\otimes r})\}$$

preserve the space of anti-symmetric tensors and restrict to give isomorphisms

$$c_g^{\otimes r} : \bigwedge^r M \to {}^g \bigwedge^r M$$

The cohomology class associated to $c_g^{\otimes r}$ is ξ^r, and this cohomology class is unique because $L \cong End \bigwedge M$ as $\bigwedge M$ is a rank 1 object. As we assumed $\bigwedge M$ descends, we deduce that $\xi^r = 0 \in H^2(G, L^*)$. □

5. F-isocrystals on Perfect Fields

Throughout this section, let k be a perfect field. We write $W(k)$ for the Witt vectors of k and $K(k)$ for the field of fractions of $W(k)$. We also denote σ the lift of the absolute Frobenius automorphism of k. This section constitutes a review of the basic theory of F-Isocrystals. The only possibly novel thing is a systematic discussion of F-Isocrystals with coefficients in L, a finite extension of $\mathbb{Q}_p$. Our references for this section will be Ehud de Shalit's notes [12] and Grothendieck's course [21].

Definition 5.1. An *F-crystal on k* is a pair (L, F) with L a finite free $W(k)$-module and $F : L \to L$ a σ-linear injective map. A *morphism of F-crystals* $\Phi : (L, F) \to (L', F')$ is a map $L \to L'$ making the square commute. We denote the category of F-crystals by F-Crys(k).

This category is a $\mathbb{Z}_p$-linear additive category with an internal $\otimes$: $(L, F)\otimes(L', F') := (L\otimes L', F\otimes F')$. It notably does not have dual objects. The category of F-isocrystals will remedy this.

Definition 5.2. An *F-isocrystal* on k is a pair (V, F) with V a finite dimensional vector space over $K(k)$ and $F : V \to V$ a σ-linear bijective map. A *morphism of F-isocrystals* $\Phi : (V, F) \to (V', F')$ is a map $V \to V'$ that makes the square commute. We denote the category of F-isocrystals on k by **F-Isoc**(k).

F-Isoc(k) has internal homs, duals and tensor products given by the natural formulas. For instance, $(V, F) \otimes (V', F') := (V \otimes V', F \otimes F')$. Likewise, $(V, F)^* := (V^*, F^*)$: here $F^*(f)(v) = f(F^{-1}v)$. The rank of an F-isocrystal (V, F) is $\dim_{K(k)}V$. The category of F-isocrystals on k is a $\mathbb{Q}_p$-linear Tannakian category with fiber functor to $Vect_{K(k)}$. This Tannakian category is neutral if and only if $k \cong \mathbb{F}_p$.

Given an F-isocrystal $E = (V, F)$ on k and a perfect field extension $i : k \hookrightarrow k'$, we define the pull back $E_{k'} = i^*E$ to be $(V \otimes_{K(k)} K(k'), F')$ where F' acts by F on V and σ on $K(k')$. The Dieudonné-Manin classification theorem classifies F-isocrystals over algebraically closed fields by their "slopes".

Definition 5.3. Let $\lambda = \frac{s}{r}$ be a rational number with $r > 0$. We define the F-isocrystal E^λ over $\mathbb{F}_p$ as $(\mathbb{Q}_p\{F\}/(F^r - p^s), *F)$. Here, $\mathbb{Q}_p\{F\}$ is the polynomial ring. For k a perfect field we define the $E^\lambda_k := i^*_k E^\lambda$ for the canonical inclusion $i_k : \mathbb{F}_p \hookrightarrow k$.

Remark 5.4. The isocrystal E^λ_k on k may be thought of as the pair $(K(k)\{F\}/(F^r - p^s), *F)$ where $K(k)\{F\}$ is the *twisted polynomial ring,* i.e. where $Fx = \sigma(x)F$ for $x \in K(k)$.

Remark 5.5. The isocrystal E^λ_k has rank r.

Remark 5.6. Let $\lambda = \frac{s}{r}$ and suppose $\mathbb{F}_{p^r} \subset k$. Then $\mathrm{End}(E^\lambda_k)$ is the division algebra D^λ over $\mathbb{Q}_p$ with invariant λ, up to choice of normalization in local class field theory, [12, 3.2].

Proposition 5.7. *Suppose $\lambda \neq \mu$. Then $Hom_{\textbf{\textit{F-Isoc}}(k)}(E^\lambda_k, E^\mu_k) = 0$.*

Proof. This is [12, Proposition 3.3] □

Theorem 5.8. *(Dieudonné-Manin) Let k be an algebraically closed field of characteristic p. Then the category F-Isoc(k) is a semisimple abelian category with simple objects the E^λ_k. In particular, we have a direct sum decomposition of the abelian category*

$$\textbf{\textit{F-Isoc}}(k) \cong \bigoplus_{\lambda\in\mathbb{Q}_{\geq 0}} \textbf{\textit{F-Isoc}}(k)^\lambda$$

*where **F-Isoc**$(k)^\lambda$ is the full abelian subcategory of **F-Isoc**(k) generated by E^λ_k. We call objects in **F-Isoc**$(k)^\lambda$ "isoclinic of slope λ".*

Definition 5.9. Let k be a perfect field and let E be an F-isocrystal over k. Over $\bar{k}$, $E_{\bar{k}} \cong \oplus(E^\lambda_{\bar{k}})$. The *slopes of* E are the elements of the multiset $\{\lambda\}$ that occur in this decomposition.

An F-isocrystal E is called *isoclinic* (or *isopentic*) of slope λ if λ is the only slope that occurs in E. The category **F-Isoc**$(k)^\lambda$ is defined to be the full abelian subcategory of F-isocrystals that are isoclinic of slope λ. If $E^{\lambda_1}$and E^{λ_2} are simple F-isocrystals on k with slopes λ_1 and λ_2 respectively, then $E^{\lambda_1} \otimes E^{\lambda_2}$ is isoclinic of slope $\lambda_1 + \lambda_2$ and $(E^{\lambda_1})^*$ is simple of slope $-\lambda_1$.

Remark 5.10. In [12, Proposition 5.3], there is another equivalent definition of isoclinic of slope λ. An object (V, F) of **F-Isoc**(k) is said to be isoclinic with slope $\lambda = \frac{s}{r}$ if the sum of all $W(k)$-submodules M of V with $F^r(M) = p^sM$ is V.

Proposition 5.11. *Let k be a perfect field. There is a direct sum decomposition of abelian categories:*

$$\textbf{\textit{F-Isoc}}(k) \cong \bigoplus_{\lambda \in \mathbb{Q}_{\geq 0}} \textbf{\textit{F-Isoc}}(k)^{\lambda}$$

In summary, given an F-isocrystal M on k, there exists a canonical direct sum decomposition $M \cong \oplus M^{\lambda}$ into isoclinic factors. For further details, see [12, Proposition 5.3].

We now discuss extension of scalars of F-isocrystals and the inherited slope decomposition. As noted above, $\mathbf{F\text{-}Isoc}(k)$ is an abelian $\mathbb{Q}_p$-linear tensor category. Given any p-adic local field $L/\mathbb{Q}_p$ we define $\mathbf{F\text{-}Isoc}(k)_L$ to be the base-changed category. We refer to objects of this category as "F-isocrystals on k with coefficients in L". By the decomposition of abelian categories $\mathbf{F\text{-}Isoc}(k) \cong \oplus \mathbf{F\text{-}Isoc}(k)^{\lambda}$, Proposition 3.2 implies

$$\mathbf{F\text{-}Isoc}(k)_L \cong \bigoplus \left(\mathbf{F\text{-}Isoc}(k)^{\lambda}\right)_L$$

We may therefore unambiguously refer to the "abelian category of λ-isoclinic F-isocrystals with coefficients in L", which we denote by $\mathbf{F\text{-}Isoc}(k)^{\lambda}_L$. This direct sum decomposition also allows us to speak about the slope decomposition (and the slopes) of F-isocrystals with coefficients in L. The category $\mathbf{F\text{-}Isoc}(k)_L$ has a natural notion of rank. If E^{λ_1} and E^{λ_2} are simple F-isocrystals on k (with coefficients in a field L) with slopes λ_1 and λ_2 respectively, then $E^{\lambda_1} \otimes E^{\lambda_2}$ is isoclinic of slope $\lambda_1 + \lambda_2$ and $(E^{\lambda_1})^*$ is simple of slope $-\lambda_1$.

Lest this discussion seem abstract, there is the following concrete description of $\mathbf{F\text{-}Isoc}(k)_L$.

Proposition 5.12. *The category $\textbf{\textit{F-Isoc}}(k)_L$ is equivalent to the category of pairs (V, F) where V is a finite free module over $K(k) \otimes_{\mathbb{Q}_p} L$ and $F : V \to V$ is a $\sigma \otimes 1$-linear bijective map. The rank of (V, F) is the rank of V as a free $K(k) \otimes L$-module.*

Remark 5.13. Note that $K(k) \otimes L$ is *not necessarily a field*; rather, it is a direct product of fields and $\sigma \otimes Id$ permutes the factors. It is a field iff L and $K(k)$ are linearly disjoint over $\mathbb{Q}_p$. This occurs, for instance, if L is totally ramified over $\mathbb{Q}_p$ or if the maximal finite subfield of k is $\mathbb{F}_p$.

Proof. By definition, an object $((V', F'), f)$ of $\mathbf{F\text{-}Isoc}(k)_L$ consists of (V', F') an F-isocrystal on k and a $\mathbb{Q}_p$-algebra homomorphism $f : L \to End_{\mathbf{F\text{-}Isoc}(k)}(V', F')$. Recall that V' is a finite dimensional $K(k)$ vector space and F' is a σ-linear bijective map. This gives V' the structure of a finite module (not a priori free) over $K(k) \otimes_{\mathbb{Q}_p} L$. The bijection F' commutes with the action of L, hence F' is $\sigma \otimes 1$-linear. We need only prove that V' is a free $K(k) \otimes_{\mathbb{Q}_p} L$-module.

Let L° be the maximal unramified subfield of L and let M be the maximal subfield of L° contained in $K(k)$. This notion is unambiguous: M is the unramified extension of $\mathbb{Q}_p$ with residue field the intersection of the maximal finite subfield of k and the residue field of the local field L. Note that L and $K(k)$ are linearly disjoint over M, so $K(k) \otimes_M L$ is a field. Let r be the degree of the extension $M/\mathbb{Q}_p$. Then $K(k) \otimes_{\mathbb{Q}_p} L$ is the direct product $\prod_{i=1}^{r} (K(k) \otimes_M L)_i$ and $\sigma \otimes 1$ permutes the factors transitively. Because of this direct product decomposition, V' can be written as $\prod_{i=1}^{r} V'_i$ with each V'_i a $K(k) \otimes_M L$-vector space. As F' is $\sigma \otimes 1$ linear and bijective, F' transitively permutes the factors V'_i and hence the dimension of each V'_i as a $K(k) \otimes_M L$ vector space is the same. This implies that V is a free $K(k) \otimes_{\mathbb{Q}_p} L$-module. □

Definition 5.14. Fix once and for all a compatible family $(p^{\frac{1}{n}}) \in \overline{\mathbb{Q}}_p$ of roots of p. Let $\overline{\mathbb{Q}}_p(-\frac{a}{b})$ be the following rank 1 object in $\mathbf{F\text{-}Isoc}(\mathbb{F}_p)_{\overline{\mathbb{Q}}_p}$ (using the description furnished by Proposition 5.12)

$$(\overline{\mathbb{Q}}_p < v >, *p^{\frac{a}{b}})$$

where v is a basis vector. Abusing notation, given any perfect field k of characteristic p, we similarly denote the pullback to k by $\overline{\mathbb{Q}}_p(-\frac{a}{b})$. In terms of Proposition 5.12, it is given by the rank 1 module $K(k) \otimes \overline{\mathbb{Q}}_p < v >$ with $F(v) = p^{\frac{a}{b}} v$ and extended $\sigma \otimes 1$-linearly.

Corollary 5.15. *Let $M = (V, F)$ be a rank 1 object of $\textbf{\textit{F-Isoc}}(k)_L$ with unique slope $\lambda = \frac{s}{r}$. Then $r | \deg[L : \mathbb{Q}_p]$.*

Proof. Consider the object $M' = \mathrm{Res}^L_{\mathbb{Q}_p} M$. This is an isoclinic F-isocrystal on k of rank $\deg[L:\mathbb{Q}_p]$ with slope λ. We may suppose k is algebraically closed; then by Dieudonné-Manin, $M' \cong \oplus E^\lambda$. The rank of E^λ is r, so the rank of M' is divisible by r. □

6. F-isocrystals on Finite fields

We describe in a more concrete way the slopes of an object M in $\mathbf{F\text{-}Isoc}(\mathbb{F}_q)_L$ where L is a p-adic local field, using the explicit description given by Proposition 5.12. Again, this section is just to fix notation and phrase things exactly as we will need them. Our conventions are: $q = p^d$, $\mathbb{Z}_q := W(\mathbb{F}_q)$ and $\mathbb{Q}_q := \mathbb{Z}_q \otimes \mathbb{Q}$. As always, σ denotes the lift of the absolute Frobenius, here viewed as a ring automorphism of $\mathbb{Z}_q$ or $\mathbb{Q}_q$. Whenever we use the phrase "p-adic valuation", we always mean the valuation v_p normalized such that $v_p(p) = 1$.

Proposition 6.1. *Let (V, F) be an F-isocrystal on $\mathbb{F}_q$ with coefficients in L a p-adic local field. Then F^d is a $\mathbb{Q}_q \otimes L$-linear endomorphism of V. Let $P_F(t) \in \mathbb{Q}_q \otimes L[t]$ be the characteristic polynomial of F^d acting on V. Then $P_F(t) \in L[t]$.*

Proof. The ring automorphism $\sigma \otimes 1$ has order d so F^d is a linear endomorphism on the free $\mathbb{Q}_q \otimes L$-module V. The characteristic polynomial $P_F(t)$ a priori has coefficients in $\mathbb{Q}_q \otimes L$, so we must show that the coefficients of $P_F(t)$ are invariant under $\sigma \otimes 1$. Recall that the coefficients of the characteristic polynomial of an operator A are, up to sign, the traces of the exterior powers of A. As there is a notion of $\bigwedge$ for F-isocrystals it is enough to show that $\mathrm{trace}(F^d)$ is invariant under $\sigma \otimes 1$.

To do this, pick a $\mathbb{Q}_q \otimes L$ basis $\{v_i\}$ of V. Let $S := (s_{ij})$ be the "matrix" of F in this basis, i.e. $F(v_i) = \sum_j s_{ij} v_j$. Then an easy computation shows that the matrix of F^d in this basis is given by

$$({}^{\sigma^{d-1}\otimes 1}S)({}^{\sigma^{d-2}\otimes 1}S)\cdots({}^{\sigma\otimes 1}S)(S)$$

Then ${}^{\sigma\otimes 1}\mathrm{trace}(F^d) = \mathrm{trace}({}^{\sigma\otimes 1}(F^d)) = \mathrm{trace}\big((S)({}^{\sigma^{d-1}\otimes 1}S)\cdots({}^{\sigma^2\otimes 1}S)({}^{\sigma\otimes 1}S)\big) = \mathrm{trace}(F^d)$ because $\mathrm{trace}(AB) = \mathrm{trace}(BA)$. □

Proposition 6.2. *Let (V, F) be an F-isocrystal on $\mathbb{F}_q$ (with coefficients in $\mathbb{Q}_p$). Then F^d is a $\mathbb{Q}_q$-linear endomorphism of V hence has a monic characteristic polynomial $P_F(t) \in \mathbb{Q}_p[t]$ by the previous proposition. The slopes of (V, F) are $\frac{1}{d}$ times the p-adic valuations of the roots of $P_F(t)$.*

Proof. This is in Katz's paper [24], see the remark after Lemma 1.3.4 where he cites Manin. Alternatively, see Grothendieck's letter to Barsotti from 1970 at the end of [21]. (Note that Grothendieck differs in choice of normalization: he choses $v(q) = 1$, which allows him to avoid the factor of $\frac{1}{d}$ above.) For a third reference, see [36, Corollaire 3.4.2.2] where it is deduced from the elegant equivalence of (Tannakian) categories: $\mathbf{F\text{-}Isoc}(\mathbb{F}_q) \cong \mathbf{F\text{-}Isoc}(\mathbb{F}_p)_{\mathbb{Q}_q}$. □

Proposition 6.3. *Let (V, F) be an F-isocrystal on $\mathbb{F}_q$ with coefficients in L. Then F^d is a $\mathbb{Q}_q \otimes L$-linear endomorphism of V with characteristic polynomial $P_F(t) \in L[t]$. The slopes of (V, F) are $\frac{1}{d}$ times the p-adic valuations of the roots of $P_F(t)$.*

Proof. Consider the diagram

$$\bigoplus_\lambda \mathbf{F\text{-}Isoc}(k)^\lambda \cong \mathbf{F\text{-}Isoc}(k) \rightleftarrows \mathbf{F\text{-}Isoc}(k)_L \cong \bigoplus_\lambda \mathbf{F\text{-}Isoc}(k)^\lambda_L$$

where the top functor is induction and the bottom functor is restriction. By definition these functors respect the slope decomposition. Let n be the degree of $L/\mathbb{Q}_p$. Given $(V, F) \in \mathrm{Ob}(\mathbf{F\text{-}Isoc}(\mathbb{F}_q)_L)$, let $(V', F') = Res^L_{\mathbb{Q}_p}(V, F)$ be the restriction, an F-isocrystal on $\mathbb{F}_q$ with coefficients in $\mathbb{Q}_p$. On the one hand, the slopes of (V', F') are the slopes of (V, F) repeated n times. On the other hand, $P_{F'}(t)$ is the norm of $P_F(t)$ with regards to the map $\mathbb{Q}_p[t] \hookrightarrow L[t]$, so the multiset of p-adic valuations of the roots $P_{F'}(t)$ is the multiset of p-adic valuations of the roots of $P_F(t)$ repeated n times. By Proposition 6.2, we know that the p-adic valuations of the roots of $P_{F'}(t)$ are the slopes of (V', F'), so by the above discussion the same is true for (V, F). □

Example 6.4. The slope of $\overline{\mathbb{Q}}_p(-\frac{a}{b})$, an object of $\mathbf{F\text{-}Isoc}(\mathbb{F}_p)_{\overline{\mathbb{Q}}_p}$, is $\frac{a}{b}$.

We now classify rank 1 objects $(V, F) \in \mathrm{Ob}\ \mathbf{F\text{-}Isoc}(\mathbb{F}_q)_L$, again using the explicit description in Proposition 5.12. As discussed above, the eigenvalue of F^d is in L. The slogan of Proposition 6.5 is that this eigenvalue determines (V, F) up to isomorphism. This discussion will allow us to classify semi-simple F-isocrystals on $\mathbb{F}_q$.

Let v be a free generator of V over $\mathbb{Q}_q \otimes_{\mathbb{Q}_p} L$. Then $F(v) = av$ with $a \in (\mathbb{Q}_q \otimes_{\mathbb{Q}_p} L)^*$ and $F^d(v) = (\mathrm{Nm}_{\mathbb{Q}_q \otimes L/L} a)v$ where the norm is taken with respect to the cyclic Galois morphism $L \to \mathbb{Q}_q \otimes L$. Therefore, any $\lambda \in L$ that is in the image of the norm map $\mathrm{Nm} : (\mathbb{Q}_q \otimes L)^* \to L^*$ can be realized as the eigenvalue of F^d. To prove that F^d uniquely determines a rank 1 F-isocrystal, we need only prove that there is a unique rank 1 F-isocrystal with F^d the identity map.

Proposition 6.5. *Let (V, F) be a rank 1 F-isocrystal on $\mathbb{F}_q$ with coefficients in L. Suppose F^d is the identity map. Then (V, F) is isomorphic to the trivial F-isocrystal, i.e. there is a basis vector $v \in V$ such that $F(v) = v$.*

Proof. Suppose $F(v) = \lambda v$ where $\lambda \in (\mathbb{Q}_q \otimes L)^*$. Then $F^d(v) = \mathrm{Nm}_{\mathbb{Q}_q \otimes L/L}(\lambda)v$, where the Norm map is defined with respect to the Galois morphism of algebras $L \hookrightarrow Q \otimes L$. This Galois group is cyclic, generated by $\sigma \otimes 1$. By assumption, we have $\mathrm{Nm}_{\mathbb{Q}_q \otimes L/L}(\lambda) = 1$. Now, $F(av) = ({}^{\sigma\otimes 1}a)\lambda v$ and we want to find an $a \in (\mathbb{Q}_q \otimes L)^*$ such that $F(av) = av$, i.e. $\frac{a}{{}^{\sigma\otimes 1}a} = \lambda$ where λ has norm 1. This is guaranteed by Hilbert's Theorem 90 for the cyclic morphism of algebras $L \hookrightarrow Q \otimes L$. $\square$

Corollary 6.6. *Let $M = (V, F)$ be a rank-1 F-isocrystal on $\mathbb{F}_q$ with coefficients in L. If the eigenvalue of F^d lives in finite index subfield $K \subset L$ and is a norm in K with regards to the algebra homomorphism $K \to \mathbb{Q}_q \otimes K$, then M descends to K.*

Proof. Let the eigenvalue of F^d be λ and let $a \in \mathbb{Q}_q \otimes K$ have norm λ. Consider the rank 1 object E of $\mathbf{F\text{-}Isoc}(\mathbb{F}_q)_K$ given on a basis element e by $F(e) = a^{-1}e$. Then $E_L \otimes M$ has F^d being the identity map. Proposition 6.5 implies that $E_L \otimes M$ is the trivial F-isocrystal, i.e. that $M \cong (E_L)^* \cong (E^*)_L$. Therefore M descends as desired. $\square$

Example 6.7. Consider the object $\overline{\mathbb{Q}}_p(-\frac{1}{2})$ of $\mathbf{F\text{-}Isoc}(\mathbb{F}_{p^2})_{\overline{\mathbb{Q}}_p}$. The eigenvalue of F^2 is $p \in \mathbb{Q}_p$, but p is not in the image of the norm map and hence the object does not descend to an object of $\mathbf{F\text{-}Isoc}(\mathbb{F}_{p^2})$. One can also see this by virtue of the fact that no non-integral fraction can occur as the slope of a rank 1 object of $\mathbf{F\text{-}Isoc}(k)$. Note that the object does in fact descend to $\mathbf{F\text{-}Isoc}(\mathbb{F}_p)_{\mathbb{Q}_p(\sqrt{p})}$.

Remark 6.8. Consider $\overline{\mathbb{Q}}_p(-\frac{a}{b})$ as an object of $\mathbf{F\text{-}Isoc}(\mathbb{F}_q)_{\overline{\mathbb{Q}}_p}$ where $\frac{a}{b}$ is in lowest terms. It is isomorphic to its Galois twists by the group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ if and only if $b|d$ where $q = p^d$. Indeed, the isomorphism class of this F-isocrystal is determined by the eigenvalue of F^d, which is $p^{\frac{ad}{b}}$. This is in $\mathbb{Q}_p$ if and only if $b \mid d$.

Corollary 6.6 poses a natural question. Let K be a p-adic local field and let $q = p^d$. Which elements of K are in the image of the norm map?

$$\mathrm{Norm} : (\mathbb{Q}_q \otimes K)^* \to K^*$$

If $K = \mathbb{Q}_p$, then $K^* \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^*$, and the image of the norm map is exactly $p^{d\mathbb{Z}} \times \mathbb{Z}_p^*$. More generally, if $K/\mathbb{Q}_p$ is a totally ramified extension with uniformizer ϖ, then $K^* \cong \varpi^{\mathbb{Z}} \times \mathcal{O}_K^*$, $\mathbb{Q}_q \otimes K$ is a field that is unramified over K, and the image of the norm map is exactly $\varpi^{d\mathbb{Z}} \times \mathcal{O}_K^*$. On the other extreme, if $K \cong \mathbb{Q}_q$, then the following proposition shows that the norm map is surjective.

Proposition 6.9. *Let L/K be a cyclic extension of fields, with $Gal(L/K)$ generated by g and of order n. Consider the induced Galois morphism of algebras: $L \to L \otimes_K L$. The image of the norm map for this extension is surjective.*

Proof. $L \otimes_K L \cong \prod_{x \in G} L$, where the L-algebra structure is given by the first (identity) factor. Then g acts by cyclically shifting the factors and the norm of an element $(\dots, l_x, \dots)$ is just $\prod_{x \in G} l_x$. Therefore the norm map is surjective. $\square$

Example 6.10. We have the following strange consequence. Consider the object $\overline{\mathbb{Q}}_p(-\frac{1}{2})$ of $\mathbf{F\text{-}Isoc}(\mathbb{F}_{p^2})_{\overline{\mathbb{Q}}_p}$. It descends to $\mathbf{F\text{-}Isoc}(\mathbb{F}_{p^2})_{\mathbb{Q}_{p^2}}$: F^2 has unique eigenvalue p, and p is a norm for the algebra homomorphism $\mathbb{Q}_{p^2} \hookrightarrow \mathbb{Q}_{p^2} \otimes \mathbb{Q}_{p^2}$ by Proposition 6.9, so we may apply Corollary 6.6.

In general the image of the norm map is rather complicated to classify. However, in our case we can say the following.

Lemma 6.11. *Let K be a p-adic local field and let $q = p^d$. Then $\mathcal{O}_K^*$ is in the image of the norm map $(\mathbb{Q}_q \otimes K)^* \to K^*$.*

Proof. $\mathbb{Q}_q \otimes K \cong \prod K'$ where K' is an unramified extension of K and the norm of an element $(\alpha) \in \prod K'$ is the product of the individual norms of the components with respect to the unramified extension K'/K. On the other hand, the image of the norm map $K'^* \to K^*$ certainly contains $\mathcal{O}_K^*$ as K'/K is unramified. □

Corollary 6.12. *Let (V, F) be a rank-1 F-isocrystal on $\mathbb{F}_q$ with coefficients in L. If the eigenvalue of F^d lives in finite index subfield $K \subset L$ and (V, F) has slope 0, then (V, F) descends to K.*

Proof. Apply Corollary 6.6 and Lemma 6.11. □

Remark 6.13. The slope 0 a.k.a unit root part of $\mathbf{F\text{-}Isoc}(k)$ is a neutral Tannakian subcategory (equivalent to a category of p-adic representation of G_k by e.g. [9, Theorem 2.1].) One may therefore apply the "neutral rank-1 descent" Proposition 4.14.

We now discuss simple objects in $\mathbf{F\text{-}Isoc}(\mathbb{F}_q)_{\overline{\mathbb{Q}}_p}$. This is the p-adic incarnation of [32, Proposition 2.21], where Milne proves that the simple objects of $\mathbf{Mot}(\mathbb{F}_q)_{\overline{\mathbb{Q}}}$ are rank 1.

Proposition 6.14. *Let $(V, F) \in Ob(\boldsymbol{F\text{-}Isoc}(\mathbb{F}_q))_{\overline{\mathbb{Q}}_p}$ be a simple object. Then (V, F) has rank 1.*

Proof. We prove the contrapositive: if (V, F) has rank greater than 1, it is not simple. V is a free $\mathbb{Q}_q \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ module and $F : V \to V$ is a $\sigma \otimes 1$-linear bijection. The ring $\mathbb{Q}_q \otimes \overline{\mathbb{Q}}_p$ is isomorphic to $\prod(\overline{\mathbb{Q}}_p)_i$ where i runs over the $\mathbb{Q}_p$-algebra homomorphisms $i : \mathbb{Q}_q \to \overline{\mathbb{Q}}_q$ and $\sigma \otimes 1$ acts as the cyclic shift. Therefore $V \cong \prod V_i$ as well and F, being $\sigma \otimes 1$-linear, cyclically shifts the factors. F^d is a $\overline{\mathbb{Q}}_p$-linear automorphism of V_1 and hence has an eigenvector v_1 with eigenvalue λ. Let $v_i = F^i v$. Note that $v_i \in V_i$ because F cyclically shifts the factors V_i. The sub-F-isocrystal generated by v_1 is generated over $\overline{\mathbb{Q}}_p$ by $v_1, v_2, \ldots, v_{d-1}$ and hence has rank 1 as an object of $\mathbf{F\text{-}Isoc}(\mathbb{F}_q)_{\overline{\mathbb{Q}}_p}$. Therefore (V, F) is not simple. □

Remark 6.15. Proposition 6.14 is *not necessarily true* if one replaces $\mathbb{F}_q$ with an arbitrary perfect field of characteristic p. For instance, let k be the perfection of the field $\mathbb{F}_p(t)$, i.e. $k \cong \mathbb{F}_p(t^{\frac{1}{p^\infty}})$. Then the rank 2 F-isocrystal given on a basis by the matrix

$$\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

is absolutely irreducible.

Proposition 6.5 implies that an absolutely semi-simple F-isocrystal (V, F) over $\mathbb{F}_q$ with coefficients in L is determined up to isomorphism by $P_F(t)$. Now, Corollary 3.12 implies that an object (V, F) is absolutely semi-simple if and only if it is semi-simple. Therefore a semi-simple F-isocrystal on $\mathbb{F}_q$ is entirely determined by the characteristic polynomial of Frobenius.

Note that the semi-simplicity hypothesis is necessary; for instance an F-isocrystal on $\mathbb{F}_p$ is a $\mathbb{Q}_p$ vector space V and a linear endomorphism F, and one can choose two non-conjugate endomorphisms that have the same characteristic polynomial). Alternatively phrased, $P_F(t)$ determines the isomorphism class of the semi-simplification $(V, F)^{ss}$.

Proposition 6.16. *Let L/K be a finite Galois extension with group G with K a p-adic local field. Let (V, F) be an F-isocrystal on $\mathbb{F}_q$with coefficients in L and denote $({}^gV, {}^gF) := {}^g(V, F)$. Then*

$$P_{{}^gF}(t) = {}^gP_F(t)$$

Proof. First of all, V is a finite free $\mathbb{Q}_q \otimes L$ module. Let $g \in G$ and consider the object ${}^g(V, F)$. The underlying sets V and ${}^gV$ may be naturally identified, and if $v \in V$, we write ${}^gv$ for the corresponding element of ${}^gV$; here $l.({}^gv) ={}^g (g^{-1}(l).v)$. Pick a free basis $\{v_i\}$ of V and let the "matrix" of F in this basis be S. Then the "matrix" of ${}^gF$ in the basis $\{{}^gv_i\}$ is ${}^gS$. Moreover, the actions of G and σ commute, so $({}^gF)^d = {}^g(F^d)$. Therefore, $P_{{}^gF}(t) = {}^gP_F(t)$ as desired. □

Corollary 6.17. *Let L/L_0 be a Galois extension of p-adic local fields with group G. Let $E = (V, F)$ be a semi-simple object of $\boldsymbol{F}$-$\boldsymbol{Isoc}(\mathbb{F}_q)_L$. Then $P_F(t) \in L_0[t]$ implies that ${}^gE \cong E$ for all $g \in G$.*

Proof. Immediate from Proposition 6.16 and the fact that a semi-simple F-isocrystal is determined up to isomorphism by its characteristic polynomials. □

7. Coefficient Objects

For a more comprehensive introduction to this section, see the recent surveys of Kedlaya [25, 27].

Let X be a normal geometrically connected variety over $\mathbb{F}_q$ and $\mathbb{F}$ a fixed algebraic closure of $\mathbb{F}_q$. Denote by $\overline{X}$ the base change $X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\mathbb{F})$. We have the following exact sequence

$$0 \to \pi_1(\overline{X}) \to \pi_1(X) \to G(\mathbb{F}_q) \to 0$$

(suppressing the implicit geometric point required to define π_1.) The profinite group $G(\mathbb{F}_q)$ has a dense subgroup $\mathbb{Z}$ generated by the Frobenius. The inverse image of this copy of $\mathbb{Z}$ is called the *Weil Group* $W(X)$. We note that the topology on $W(X)$ is given by declaring the *not* the subspace topology from $\pi_1(X)$; rather, we demand that$\pi_1(\overline{X}) \subset W(X)$ is an open and closed subgroup. This assignment can be enchanced to a functorial assignment from the category of smooth separated schemes over $\mathbb{F}_q$ to topological groups [14, 1.1.7].

Definition 7.1. Let X be a normal geometrically connected variety over $\mathbb{F}_q$ and K an l-adic local field. A *(lisse) Weil Sheaf of rank r with coefficients in K* is a continuous representation $W(X) \to GL_r(K)$. We denote the category of Weil sheaves with coefficients in E by $\mathbf{Weil}(X)_K$.

The category of $\mathbb{Q}_l$-Weil sheaves, $\mathbf{Weil}(X)$, is naturally an $\mathbb{Q}_l$-linear neutral Tannakian category, and $\mathbf{Weil}(X)_K \cong (\mathbf{Weil}(X))_K$. We define $\mathbf{Weil}(X)_{\overline{\mathbb{Q}}_l}$ as in Remark 3.5. Alternatively $\mathbf{Weil}(X)_{\overline{\mathbb{Q}}_l}$ is the category of continuous, finite dimensional representations of $W(X)$ in $\overline{\mathbb{Q}}_l$-vector spaces where $\overline{\mathbb{Q}}_l$ is equipped with the colimit topology.

Let $x = \mathrm{Spec}(\mathbb{F}_{p^d})$ and suppose we have a map $i : x \to X$. Then given a Weil sheaf $\mathscr{L}$ on X, we get a Weil sheaf $\mathscr{L}_x$ on x. Denote by Fr the geometric Frobenius. Then $Fr^d \in W(x)$ and we define $P_x(\mathscr{L}, t)$, *the characteristic polynomial of $\mathscr{L}$ at x,* to be the characteristic polynomial of Fr^d on $\mathscr{L}_x$.

The following Lemma, a.k.a. the Brauer-Nesbitt theorem, implies that the isomorphism type of a semi-simple Weil sheaf $\mathscr{L}$ is determined by $P_x(\mathscr{L}, t)$ for all closed points $x \in |X|$.

Lemma 7.2. *(Brauer-Nesbitt) Let Γ be a group and E a characteristic 0 field. Suppose ρ_1 and ρ_2 are two semi-simple finite dimensional representations of Γ with coefficients in E. Then $\rho_1 \cong \rho_2$ if and only if the trace functions are the same.*

Now, let X/k be a scheme of finite type over a perfect field k. Berthelot has defined the absolute crystalline site on X: for a "modern" reference, see [39, TAG 07I5]. (We implicitly take the crystalline site with respect to $W(k)$ without further comment; in other words, in the formulation of the Stacks Project, $S = \mathrm{Spec}(W(k))$ with the canonical PD structure.) Let $Crys(X)$ be the category of crystals in *finite locally free $\mathcal{O}_{X/W(k)}$-modules.* By functoriality of the crystalline topos, the absolute Frobenius $Frob : X \to X$ gives a functor $Frob^* : Crys(X) \to Crys(X)$.

Definition 7.3. [39, TAG 07N0]A (non-degenerate) *F-crystal* on X is a pair (M, F) where M is a crystal in finite locally free modules over the crystalline site and $F : Frob^*M \to M$ is an injective map of crystals.

The category F-Crys(X) is a $\mathbb{Z}_p$-linear category with an internal $\otimes$ but without internal homs or duals in general. There is a object $\mathbb{Z}_p(-1)$ which given by the pair $(\mathcal{O}_{cris}, p)$. We denote by $\mathbb{Z}_p(-n)$ the nth tensor power of $\mathbb{Z}_p(-1)$.

Definition 7.4. A Dieudonné crystal is a pair (M, F, V) where (M, F) is an F-crystal in finite locally free modules and $V : M \to Frob^*M$ is a map of crystals such that $V \circ F = p$ and $F \circ V = p$.

We define the category F-Crys$^+(X)$ to be F-Crys(X) with the morphism modules tensored with $\mathbb{Q}_p$, i.e. we formally adjoin $\frac{1}{p}$ to all endomorphism rings. In F-Crys$^+(X)$ the image of the object $\mathbb{Z}_p(-1)$ is denoted $\mathbb{Q}_p(-1)$. The category **F-Isoc**(X) is given by formally inverting $\mathbb{Q}_p(-1)$ with regards to $\otimes$; the dual to $\mathbb{Q}_p(-1)$ is called the Tate object $\mathbb{Q}_p(1)$.

See Saavedro Rivano's thesis [36, Section 3] for further details. If $\mathscr{E}$ is an object of **F-Isoc**(X), we denote by $\mathscr{E}(n)$ the object $\mathscr{E} \otimes \mathbb{Q}_p(n)$. For any object $\mathscr{E}$ of **F-Isoc**(X) there is an n such that $\mathcal{E}(-n)$ is effective, i.e. is in the essential image of the functor

$$F\text{-Crys}^+(X) \to \mathbf{F\text{-}Isoc}(X)$$

F-Isoc(X) is a $\mathbb{Q}_p$-linear Tannakian category that is non-neutral in general. The category of F-isocrystals with coefficients in L, where L is a p-adic local field, is by definition the base-changed category **F-Isoc**$(X)_L$. Similarly, the category **F-Isoc**$(X)_{\overline{\mathbb{Q}}_p}$ is defined as in Remark 3.5. There is a notion of the rank of an F-isocrystal that satisfies that expected constraints given by $\otimes$ and $\oplus$. We remark that every F-crystal is automatically "convergent", see e.g. [23, Proposition 3.1]; hence many people refer to the category **F-Isoc**(X) as the category of "*convergent F-isocrystals on X*"; moreover, this nomenclature is consistent with other ways of constructing the category of convergent F-isocrystals, e.g. as in [34]. If X is a proper curve, an F-isocrystal is supposed to be the $\mathbb{Q}_p$-analog of a lisse sheaf.

As Crew notes in [10], for general smooth X over a perfect field k the p-adic analog of a lisse l-adic sheaf seems to be an overconvergent F-isocrystal with coefficients in $\overline{\mathbb{Q}}_p$. Instead of indicating a definition, we simply point the reader to the recent survey of Kedlaya [27].

Notation 7.5. The category of overconvergent F-Isocrystals on X is denoted by **F-Isoc**$^\dagger(X)$.

The category **F-Isoc**$^\dagger(X)$ is a non-neutral $\mathbb{Q}_p$-linear Tannakian category. There is a natural functor

$$\mathbf{F\text{-}Isoc}^\dagger(X) \to \mathbf{F\text{-}Isoc}(X)$$

which is fully faithful in general [26] and an equivalence of categories when X is projective.

Remark 7.6. The category **F-Isoc**$^\dagger(X)$ has finite-dimensional Hom spaces and every object has finite length.

Let $x = \text{Spec}(k)$ where k is a perfect field. Then any map $i : x \to X$ induces a pullback map

$$i_x^* : \mathbf{F\text{-}Isoc}^\dagger(X)_L \to \mathbf{F\text{-}Isoc}(k)_L$$

In particular, if $k \cong \mathbb{F}_{p^d}$, then $\mathscr{E}_x$ is just an F-isocrystal on $\mathbb{F}_{p^d}$. $\mathscr{E}_x$ is therefore equivalent to the data of a pair (V, F) where V is a free module over $\mathbb{Q}_{p^d} \otimes_{\mathbb{Q}_p} L$ and F is a $\sigma \otimes 1$-linear map. We define $P_x(\mathscr{E}, t)$, the *characteristic polynomial of $\mathscr{E}$ at x,* to be the characteristic polynomial of F^d acting on V. (See Proposition 6.3.)

Lemma 7.7. *(Abe) Let X be a smooth variety over $\mathbb{F}_q$. Let $\mathscr{E}$ and $\mathscr{E}'$ be semi-simple overconvergent F-Isocrystals on X. Suppose $P_x(\mathscr{E}, t) = P_x(\mathscr{E}', t)$ for all closed points $x \in |X|$. Then $\mathscr{E} \cong \mathscr{E}'$.*

Proof. The unique proposition of [1, Section 3] shows that ι-mixed overconvergent F-isocrystals are determined by characteristic polynomials of Frobenius elements. It is then shown in [2] that all absolutely irreducible overconvergent F-isocrystals with finite determinant satisfy the ι-mixedness hypothesis over a curve. Finally, the recent work of Abe-Esnault shows in fact that all irreducible overconvergent F-isocrystals satisfy ι-mixedness, [3, Theorem 2.6]. ∎

Notation 7.8. (Kedlaya [25]) Let X be a smooth connected variety over $\mathbb{F}_q$. A *coefficient object* is an object either of $\mathbf{Weil}(X)_{\overline{\mathbb{Q}}_l}$ or $\mathbf{F\text{-}Isoc}^\dagger(X)_{\overline{\mathbb{Q}}_p}$. We informally call the former the *étale case* and the latter the *crystalline case*. We say that a coefficient object has coefficients in K if may be descended to the appropriate category with coefficients in K.

Definition 7.9. Let X be a normal geometrically connected variety over $\mathbb{F}_q$. Let $\mathcal{F}$ be a coefficient object. We say $\mathcal{F}$ is *algebraic* if $P_x(\mathcal{F},t) \in \overline{\mathbb{Q}}[t]$ for all closed points $x \in |X|$. Let E be a number field. We say $\mathcal{F}$ is *E-algebraic* if $P_x(\mathcal{F},t) \in E[t]$ for all closed points $x \in |X|$.

Putting together the classical Brauer-Nesbitt theorem with Abe's analog for p-adic coefficients, we obtain the following Brauer-Nesbitt theorem for coefficient objects.

Theorem 7.10. *Let X be a normal geometrically connected variety over $\mathbb{F}_q$. Let $\mathcal{F}$ be a semi-simple coefficient object. $\mathcal{F}$ is determined, up to isomorphism, by $P_x(\mathcal{F},t)$ for all closed points $x \in |X|$.*

8. Compatible Systems and Companions

Definition 8.1. Let X be a normal geometrically connected variety over $\mathbb{F}_q$. Let $\mathcal{E}$ and $\mathcal{E}'$ be semi-simple algebraic coefficient objects on X with coefficient fields K and K' respectively. Fix an isomorphism $\iota : \overline{K} \to \overline{K'}$. We say $\mathcal{E}$ and $\mathcal{F}$ are *ι-companions* if they are both algebraic and ${}^\iota P_x(\mathcal{E},t) = P_x(\mathcal{E}',t)$ for all closed points $x \in |X|$.

Definition 8.2. Let X be a normal geometrically connected variety over $\mathbb{F}_q$ and let E be a number field. Then an *E-compatible system* is a system of E_λ-coefficient objects $(\mathcal{E}_\lambda)_{\lambda \nmid p}$ over places $\lambda \nmid p$ of E such that

$$P_x(\mathcal{E}_\lambda, t) \in E[t] \subset E_\lambda[t]$$

and is independent of λ. A *complete E-compatible system* $(\mathcal{E}_\lambda)$ is an E-compatible system together with, for each $\lambda | p$, an object

$$\mathcal{E}_\lambda \in \mathbf{F\text{-}Isoc}^\dagger(X)_{E_\lambda}$$

such that for every place λ of E, $P_x(\mathcal{E}_\lambda,t) \in E[t] \subset E_\lambda[t]$ is independent of λ and that $(\mathcal{E}_\lambda)$ satisfies the following completeness condition:

Condition. Consider $\mathcal{E}_\lambda$ as a $\overline{\mathbb{Q}}_\lambda$-coefficient object. Then for any $\iota : \overline{\mathbb{Q}}_\lambda \to \overline{\mathbb{Q}}_{\lambda'}$, the ι-companion to $\mathcal{E}_\lambda$ is isomorphic to an element of $(\mathcal{E}_\lambda)$.

Remark 8.3. The ι in Definition 8.1 does not reflect the topology of $\overline{\mathbb{Q}}_p$ or $\overline{\mathbb{Q}}_l$; in particular, it need not be continuous.

Deligne's conjecture 1.2.10 of Weil II [14] is the following.

Conjecture 8.4. *(Companions) Let X be a normal connected variety over a finite field k of cardinality p^f with a geometric point $\overline{x} \to X$. Let $l \neq p$ be a prime. Let $\mathscr{L}_l$ be an absolutely irreducible l-adic local system with finite determinant on X. The choice of $\overline{x}$ allows us to think of this as a representation $\rho_l : \pi_1(X,\overline{x}) \to GL(n,\overline{\mathbb{Q}}_l)$. Then*

1. *ρ_l is pure of weight 0.*
2. *There exists a number field E such that for all closed points $x \in |X|$, the polynomial $det(1 - \rho_l(F_x)t)$, has all of its coefficients in E. Here F_x is a Frobenius element of x. In particular, the eigenvalues of $\rho_l(F_x)$ are all algebraic numbers.*
3. *For each place $\lambda \nmid p$, the inverse roots α of $det(1 - \varrho_l(F_x)t)$ are λ-adic units in $\overline{E_\lambda}$.*
4. *For each $\lambda | p$, the λ-adic valuations of the inverse roots α satisfy*
$$|\frac{v(\alpha)}{v(Nx)}| \leq \frac{n}{2}$$
5. *After possibly replacing E by a finite extension, for each $\lambda \nmid p$ there exists a λ-adic local system $\rho_\lambda : \pi_1(X,\overline{x}) \to SL(n,E_\lambda)$ that is compatible with ρ_l.*

(6) *Again after possibly replace E by a finite extension, for $\lambda | p$, there exists a crystalline companion to ρ_l.*

Deligne was inspired by the work of Drinfeld on the Langlands Correspondence for $GL(2)$. In particular, if one believed that ρ_l was *"of geometric origin"*, one would strongly suspect that ρ_l gives rise to a compatible family of l'-adic representations: that is, there exist representations $\rho_{l'} : \pi_1(X, \overline{x}) \to SL(n, \overline{\mathbb{Q}}_{l'})$ compatible with ρ_l for each l' prime to p. Note that this is slightly weaker than Part 5 of the conjecture, which guarantees that there exists a number field E such that ρ_l fits into an E-compatible system. Nonetheless, this is still quite striking: an l-adic local system is, in this formulation, simply a continuous homomorphism from $\pi_1(X)$, and the topologies on $\mathbb{Q}_l$ and $\mathbb{Q}_{l'}$ are completely different.

By work of Deligne, Drinfeld, and Lafforgue, if X is a curve all such local systems are of geometric origin. Moreover, in this case Chin has proved Part 5 of the conjecture. See, for instance [8, Theorem 4.1].

Recent work of Deligne and Drinfeld prove (1), (2), (3), and (5) of the conjecture. See the article of Esnault-Kertz [19] for a precise chronology.

Crew [10] formulated Part 6 of Deligne's conjecture as follows: for every $\lambda | p$, there is an *overconvergent F-isocrystal on X with coefficients in E_λ* that is compatible with $\mathscr{L}$. Abe has recently constructed a sufficiently robust theory of p-adic cohomology to prove a p-adic Langlands correspondence and hence answer affirmatively part 6 of Deligne's conjecture when X is a curve [1, 2].

Theorem 8.5. *(Abe, Lafforgue) Let C be a smooth curve over $\mathbb{F}_q$. Then Deligne's conjecture is true.*

In fact, as a byproduct of the Abe's proof there is a slight refinement of part (6) which we record.

Theorem 8.6. *(Abe, Lafforgue) Let C be a smooth curve over $\mathbb{F}_q$. Let $l \neq p$ be a prime. For every isomorphism $\iota : \overline{\mathbb{Q}}_l \to \overline{\mathbb{Q}}_p$, there is a bijective correspondence*

$$\left\{\begin{array}{l}\text{Local systems } \mathscr{L} \text{ on } C \text{ such that}\\ \bullet\, \mathscr{L} \text{ has coefficients in } \overline{\mathbb{Q}}_l\\ \bullet\, \mathscr{L} \text{ is irreducible of rank } n\\ \bullet\, \mathscr{L} \text{ has finite determinant}\\ \text{up to isomorphism}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Overconvergent } F\text{-Isocrystals } \mathscr{E} \text{ on } C \text{ such that}\\ \bullet\, \mathscr{E} \text{ has coefficients in } \overline{\mathbb{Q}}_p\\ \bullet\, \mathscr{E} \text{ is irreducible of rank } n\\ \bullet\, \mathscr{E} \text{ has finite determinant}\\ \text{up to isomorphism}\end{array}\right\}$$

depending on ι such that $\mathscr{L}$ and $\mathscr{E}$ are ι-compatible.

Remark 8.7. This "p-companions" part of the conjecture is *not known* if $\dim C > 1$. Recent work of Abe-Esnault [3] and Kedlaya [25, Theorem 5.3] associates an l-adic object to a p-adic object. On the other hand, given a coefficient object $\mathcal{E}_\lambda$ and a non-continuous isomorphism $\iota : \overline{\mathbb{Q}}_\lambda \to \overline{\mathbb{Q}}_p$, it is unknown how to associate a crystalline ι-companion to $\mathcal{E}_\lambda$.

Theorem 8.8. *(Abe, Lafforgue) Let C be a curve over $\mathbb{F}_q$. Let $\mathcal{E}$ be an irreducible coefficient object with finite order determinant. Then there exists a number field E such that $\mathcal{E}$ is part of a complete E-compatible system.*

Remark 8.9. Part (4) of the conjecture is not tight for $n = 2$. The case to have in mind is the universal elliptic curve over $Y(1)$ over $\mathbb{F}_p$. By taking relative l-adic cohomology, one gets a weight 1 rank-2 $\mathbb{Z}_l$ local system on $Y(1)$. Twist by $\overline{\mathbb{Q}}_l(1/2)$ to get a local system which is pure of weight 0. For an ordinary $\mathbb{F}_q$ point, the ratio in question is either $-1/2$ or $1/2$ and for a supersingular point it is 0. This motivates the bound for a rank 2 local system, as proven L. Lafforgue (Theorem VII.6.(iv) of [30]).

Theorem 8.10. *(Lafforgue) Let C be a smooth connected curve over $\mathbb{F}_q$ and let $\mathscr{L}$ be an absolutely irreducible rank 2 l-adic local system on C with finite determinant. Then for all closed points $x \in |C|$, the eigenvalues α of F_x satisfy*

$$|\frac{v(\alpha)}{v(Nx)}| \leq \frac{1}{2}$$

Remark 8.11. A refined version of part (4) of Conjecture 8.4 has been resolved in general by V. Lafforgue [31, Corollaire 2.2] and more recently Drinfeld-Kedlaya [18, Theorem 1.1.5]. They prove that the "consecutive slopes do not differ by more than 1", reminiscent of Griffith's transversality. There is also forthcoming work of Kramer-Miller, providing another p-adic proof of this statement.

9. F-isocrystals and l-adic local systems on Curves

Let C be a smooth connected curve over $\mathbb{F}_q$ and let $\bar{c}$ be a geometric point. Let $\mathscr{L}$ be an irreducible l-adic local system on C with trivial determinant, which one may think of as a continuous representation $\rho_l : \pi_1(C, \bar{c}) \to SL(n, \overline{\mathbb{Q}}_l)$. Theorem 8.8 tells us that we can find a number field E such that ρ_l fits into a complete E-compatible system.

Example 9.1. *The number field E can in general be larger than the field extension of $\mathbb{Q}$ generated by the coefficients of the characteristic polynomials of all Frobenius elements F_x.* For instance, let D be a non-split quaternion algebra over $\mathbb{Q}$ that is split at ∞ and let l and p be finite primes where D splits. The Shimura curve X^D exists as a smooth complete curve (in the sense of stacks) over $\mathbb{F}_p$ and it admits a universal abelian surface $f : \mathcal{A} \to X^D$ with multiplication by $\mathcal{O}_D$. Then $R^1 f_* \mathbb{Q}_l$ is a rank 4 $\mathbb{Q}_l$ local system with an action of $D \otimes \mathbb{Q}_l$. As we assumed l was a split prime, $D \otimes \mathbb{Q}_l \cong M_{2\times 2}(\mathbb{Q}_l)$ and so we can use Morita equivalence, i.e. look at the image of

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{Q}_l)$$

to get a rank 2 local system $\mathscr{L}_l$ with coefficients in $\mathbb{Q}_l$; moreover, $\mathscr{L}_l$ has all Frobenius traces in $\mathbb{Q}$ and is pure of weight 1. The local system $\mathscr{L}_l$ has open image and is hence absolutely irreducible. The same trick shows that the p-adic companion may be defined over $\mathbb{Q}_p$; in fact the p-divisible group splits.

On the other hand, let l' be a prime where D ramifies (these exist as we assumed D was not split.) Suppose $\mathscr{L}_{l'}$ were a rank 2 $\mathbb{Q}_{l'}$-local system compatible with $\mathscr{L}_l$, necessarily absolutely irreducible because $\mathscr{L}_l$ has big image. Then $\mathscr{L}_{l'} \oplus \mathscr{L}_{l'} \cong R^1 f_* \mathbb{Q}_{l'}$ and this rank 4 local system has an action of $D_{l'} = D \otimes \mathbb{Q}_{l'}$, the unique non-split quaternion algebra over $\mathbb{Q}_{l'}$. By picking a geometric point $\bar{c}$ of X^D, we may think of $R^1 f_* \mathbb{Q}_{l'}$ as rank 4 $\mathbb{Q}_{l'}$-representation V of $\pi_1 = \pi_1(X^D, \bar{c})$ with a commuting action of $D_{l'}$. The action of $D_{l'}$ here is the regular representation, thinking of $D_{l'}$ as a rank 4 $\mathbb{Q}_{l'}$-vector space. In particular, the commutant of π_1 in $\mathrm{Mat}_{4\times 4}(\mathbb{Q}_{l'})$ contains $D_{l'}$.

Supposing $\mathscr{L}_{l'}$ exists implies $V \cong W \oplus W$, where W was a rank 2 $\mathbb{Q}_{l'}$-representation of π_1 that is absolutely irreducible. Then the commutant of π_1 in $\mathrm{Mat}_{4\times 4}(\mathbb{Q}_{l'})$ would be $\mathrm{Mat}_{2\times 2}(\mathbb{Q}_{l'})$, which cannot possibly contain $D_{l'}$. Therefore $\mathscr{L}_{l'}$ does not exist. In particular, even though *all Frobenius traces are in $\mathbb{Q}$, not all of the l-adic companions can be defined over $\mathbb{Q}_l$*. In other words, they do not form a $\mathbb{Q}$-compatible system. However, we see from this examples that only finitely many of the l-adic companions cannot be defined over $\mathbb{Q}_l$.

In line with Example 9.1, Drinfeld recently proved the following [17, Section E.10].

Theorem 9.2. *(Drinfeld) Let X be a smooth, geometrically connected variety over $\mathbb{F}_q$ and $K \subset L$ be number fields. Let $(\mathcal{E}_\lambda)$ be a complete L-compatible system with trivial determinant that is K-algebraic. Then for all but finitely many λ, $\mathcal{E}_\lambda$ is defined over K_λ.*

Using our simple descent machinery, we construct criteria for the field-of-definition of a coefficient object to be as small as possible. For instance, in Lemma 9.3 we reprove a proposition of Chin [7, Proposition 7]. We hope it shows the motivation behind our descent criterion.

Lemma 9.3. *Let C be a normal variety over $\mathbb{F}_q$. Let $\mathscr{L}$ be an absolutely irreducible rank n l-adic local system, defined over L, an l-adic local field. Suppose all Frobenius traces are in L_0, an l-adic subfield of L. Further suppose there is a point $x \in C$ such that there is an eigenvalue α of F_x that occurs with multiplicity 1 and $\alpha \in L_0$. Then $\mathscr{L}$ can be defined over L_0.*

Proof. We may suppose the field extension L/L_0 is Galois with group G. The absolute irreducibility hypothesis ensures, by Schur's Lemma (e.g. Lemma 3.13), that $L \cong \mathrm{End}(\mathscr{L}_l)$. Let χ_l be the trace of the representation of $\pi_1(C)$ associated to $\mathscr{L}$ (suppressing the implicit basepoint.) By continuity and Chebotarev's Density theorem, χ_l is a function with values in L_0. The Brauer-Nesbitt Theorem 7.2 ensures that ${}^g\mathscr{L} \cong \mathscr{L}$ for all $g \in G$.

Let $i : x \to C$ denote the inclusion and consider the restriction functor i^* from the category of L-valued local systems on C to the category L-valued local systems on x. Now, write $i^*\mathscr{L} \cong N_1 \oplus N_2$ where N_1 is the rank one local system with Frobenius eigenvalue α; the direct sum decomposition holds because α occurs with multiplicity 1 in the spectrum of F_x. There is no map $N_1 \to {}^gN_2$ for any $g \in G$, again because α occurs with multiplicity 1, and $L \cong \mathrm{End}(N_1)$. Finally, N_1 descends to L_0 as $\alpha \in L_0$. We may therefore use Lemma 4.12 to deduce that $\mathscr{L}$ (and as a corollary N_2) can be descended to L_0. □

Exercise 9.4. Let $\mathscr{L}_l$ be an absolutely irreducible rank 2 l-adic local system on C with trivial determinant and infinite monodromy. Suppose all of the Frobenius traces of $\mathscr{L}_l$ are in $\mathbb{Q}$. Using Theorem 8.10, show that there does not exist a single closed point $x \in |C|$ such that the characteristic polynomial of Fr_x is separable and splits over $\mathbb{Q}$. In other words, we cannot possibly use Lemma 9.3 in conjunction with a single closed point x to prove that the compatible system $\{\mathscr{L}_l\}$ descends to a $\mathbb{Q}$-compatible system.

Lemma 9.3 has a p-adic analog, which is genuinely new as far as we understand.

Lemma 9.5. *Let $L/\mathbb{Q}_p$ be a finite extension, C a curve over $\mathbb{F}_q$, and $\mathscr{E}$ an absolutely irreducible overconvergent F-isocrystal on C of rank n with coefficients in L. Suppose that there exists a closed point $i : x \to C$ such that $i^*\mathscr{E}$ has 0 as a slope with multiplicity 1. Suppose further that for all closed points $x \in |C|$, $P_{\mathscr{E}}(x,t) \in L_0[t]$ for some p-adic subfield $L_0 \subset L$. Then $\mathscr{E}$ can be descended to an F-isocrystal with coefficients in L_0.*

Proof. We proceed exactly as in Lemma 9.3. By enlarging L we may suppose that L/L_0 is Galois; let $G = \mathrm{Gal}(L/L_0)$. First, Abe's Lemma 7.7 implies that a semi-simple overconvergent F-isocrystal on a smooth curve is determined by the characteristic polynomials of the Frobenius elements at all closed points. In particular, it implies that for any $g \in G$, ${}^g\mathscr{E} \cong \mathscr{E}$ because the characteristic polynomials at all closed points agree by Proposition 6.16 and $\mathscr{E}$ is irreducible (and hence semi-simple.) Now let us use Lemma 4.12 for the restriction functor

$$i^* : \textbf{F-Isoc}^\dagger(C) \to \textbf{F-Isoc}(x)$$

Because 0 occurs as a slope with multiplicity 1 in $i^*\mathscr{E}$ we can write $i^*\mathscr{E} \cong N_1 \oplus N_2$ by the isoclinic decomposition, Proposition 5.11. Here N_1 has rank 1 and unique slope 0 while no slope of N_2 is 0. There are no maps between N_1 and any Galois twist of N_2 because twisting an F-isocrystal does not change the slope. The endomorphism algebra of any rank-1 object in $F\text{-Isoc}(\mathbb{F}_q)_L$ is L by Corollary 4.13. Now, $\mathrm{End}(\mathscr{E}) \cong L$ by Schur's Lemma 3.13 because $\mathscr{E}$ is absolutely irreducible.

Finally, we must argue that N_1 descends to L_0. Let $x = \mathrm{Spec}(\mathbb{F}_{p^d})$. The fact that the slope 0 occurs exactly once in $i^*\mathscr{E}$ implies that the eigenvalue α of F^d on N_1 is an element of L_0. Moreover, $\alpha \in \mathcal{O}_{L_0}^*$ because N_1 is slope 0. By Corollary 6.12, N_1 descends to L_0. The hypotheses of Lemma 4.12 are all satisfied and we may conclude that $\mathscr{E}$ (and N_2) descends to L_0. □

The following is a special case of the main theorem of a recent article of Koshikawa [28]. For completeness, we give the proof written in our PhD thesis.

Lemma 9.6. *Let C a curve over $\mathbb{F}_q$, and $\mathscr{E}$ an irreducible rank n overconvergent F-isocrystal on C with coefficients in $\overline{\mathbb{Q}}_p$ such that $\mathscr{E}$ has trivial determinant. By Theorem 8.8 there is a number field E such that $\mathscr{E}$ is part of a complete E-compatible system. In particular, for every $\lambda|p$ there is a compatible overconvergent F-isocrystal $\mathscr{E}_\lambda$ with coefficients in E_λ.*

Suppose that for all $\lambda|p$ and for all closed points $x \in |C|$, $i_x^*\mathscr{E}_\lambda$ is an isoclinic F-isocrystal on x. Then the representation has finite image: for instance, for every $\lambda \nmid p$, the associated λ-adic representation has finite image. Equivalently, the "motive" can be trivialized by a finite étale cover $C' \to C$

Proof. Part 3 of Conjecture 8.4 implies that the eigenvalues of F_x are λ-adic units for all $\lambda \nmid p$. On the other hand, for each $\lambda | p$ and for every closed point $x \in |C|$, $i_x^* \mathscr{E}_\lambda$ being isoclinic and having trivial determinant implies the slopes of $i_x^* \mathscr{E}_\lambda$ are 0 and hence that the eigenvalues of F_x are λ-adic units. As eigenvalues of F_x are algebraic numbers, this implies that they are all roots of unity. Moreover, each of these roots of unity lives in a degree n extension of E and there are only finitely many roots of unity that live in such extensions: there are only finitely many roots of unity with fixed bounded degree over $\mathbb{Q}$. Therefore there are only finitely many eigenvalues of F_x as x ranges through the closed points of C.

Now, pick $\lambda \nmid p$ and consider the associated representation $\varrho_\lambda : \pi_1(C, \bar{c}) \to SL(n, E_\lambda)$. By the above discussion, there exists some integer k such that for every closed point $x \in |C|$, the generalized eigenvalues $\varrho(F_x)$ are k^{th} roots of unity. But Frobenius elements are dense and ϱ_λ is a continuous homomorphism, so that the same is true for the entire image of ϱ_λ. The image of ϱ_λ therefore only has finitely many traces.

Burnside proved that if $G \subset GL(n, \mathbb{C})$ has finitely many traces and the associated representation is irreducible, then G is finite: see, for instance the proof of 19.A.9 on page 231 of [35] which directly implies this statement. Thus the entire image of ϱ_λ is finite, as desired. □

Remark 9.7. Koshikawa [28] calls such F-isocrystals *absolutely unit root.*

Proposition 9.8. *Let C be a curve over $\mathbb{F}_q$ and let $\mathscr{L}$ be an absolutely irreducible rank 2 l-adic local system with trivial determinant, infinite image, and all Frobenius traces in a number field E. There exists a finite extension F of E such that $\mathscr{L}$ is a part of a complete F-compatible system. Not every crystalline companion $\mathcal{E}_\lambda \in$ **F-Isoc**$(C)_{F_\lambda}$ is everywhere isoclinic by Lemma 9.6. For each such not-everywhere-isoclinic $\mathscr{E}_\lambda$ there exists*

(1) *A positive rational number $\frac{s}{r}$*
(2) *A finite field extension $\mathbb{F}_{q'}/\mathbb{F}_q$ with C' denoting the base change of C to $\mathbb{F}_{q'}$ (q' will be the least power of q divisible by p^r)*

such that the F-isocrystal $\mathscr{M} := \mathscr{E}_\lambda \otimes \overline{\mathbb{Q}}_p(-\frac{s}{r})$ on C' descends to E_λ.

The point of Proposition 9.8 is that the field of definition of $\mathscr{M}$ is E_λ, a completion of the field of traces.

Proof. Theorem 8.8 imply that there is a finite extension F/E, WLOG Galois, such that $\mathscr{L}$ fits into a complete F-compatible system on C. As we assumed $\mathscr{L}$ had infinite image, Lemma 9.6 implies that there is a place $\lambda | p$ of F together with an object $\mathscr{E}_\lambda \in \mathbf{F\text{-}Isoc}^\dagger(C)_{F_\lambda}$ that is compatible with $\mathscr{L}$ and such that the general point is not isoclinic. We abuse notation and denote the restriction of λ to E by λ again. The object $\mathscr{E}_\lambda$ is isomorphic to its twists by $\mathrm{Gal}(F_\lambda/E_\lambda)$ by Abe's Lemma 7.7 because $P_{\mathscr{E}_\lambda}(x,t) \in E[t]$ for all closed points $x \in X$. Pick a closed point $i : x \to C$ such that $i^*\mathscr{E}_\lambda$ has slopes $(-\frac{s}{r}, \frac{s}{r})$. Let q' be the smallest power of q that is divisible by p^r and let C' denote the base change of C to $\mathbb{F}_{q'}$.

Consider the twist $\mathscr{M} := \mathscr{E}_\lambda \otimes \overline{\mathbb{Q}}_p(-\frac{s}{r})$, thought of as an F-isocrystal on C' with coefficients in $\overline{\mathbb{Q}}_p$. We have enlarged q to q' where $p^r | q'$; Remark 6.8 says that $\overline{\mathbb{Q}}_p(-\frac{s}{r})$, considered as object of $\mathbf{F\text{-}Isoc}(\mathbb{F}_{q'})_{\overline{\mathbb{Q}}_p}$, is isomorphic to all of its twists by $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Therefore, $P_{\mathscr{M}}(x,t) \in E_\lambda[t]$ for all closed points x. Abe's Lemma 7.7 then implies that $\mathscr{M}$ is isomorphic to all of its Galois twists by $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_\lambda)$. At the point "$x$", the slopes are now $(0, \frac{2s}{r})$. Apply Lemma 9.5 to descend $\mathscr{M}$ to the field of traces E_λ, as desired. □

Remark 9.9. Proposition 9.8 has the following slogan: for every not-everywhere-isoclinic crystalline companion of $\mathscr{L}$, there exists a twist such that the Brauer obstruction vanishes. This is in stark contrast to the l-adic case, where there is no *a priori* reason an l-adic Brauer obstruction should vanish.

We now specialize to the case where $\mathscr{L}$ is a rank 2 l-adic local system with trivial determinant, infinite image, and having all Frobenius traces in a number field E where p splits completely. Proposition 9.8 implies that, up to extension of the ground field $\mathbb{F}_q$, we can find an F-isocrystal $\mathscr{E}$ with *coefficients in* $\mathbb{Q}_p$ that is compatible with $\mathscr{L}$ up to a twist and is not everywhere isoclinic. Moreover, by construction, there is a point x such that $\mathscr{E}_x$ is not isoclinic and the slopes are $(0, \frac{2s}{r})$. On the one hand the slope of the determinant of $\mathscr{E}_x$ is necessarily an integer because the coefficients of $\mathscr{E}$ are $\mathbb{Q}_p$. Therefore $\frac{2s}{r}$ is a positive integer. On the other hand, by Lafforgue's slope bounds (Theorem 8.10), $\frac{2s}{r} \leq 1$, so $\frac{s}{r} = \frac{1}{2}$ and $\det(\mathscr{E}) \cong \mathbb{Q}_p(-1)$. We record this analysis in the following important corollary.

Corollary 9.10. *Let C be a curve over $\mathbb{F}_q$ and let $\mathscr{L}$ be an absolutely irreducible rank 2 l-adic local system with trivial determinant, infinite image, and all Frobenius traces in $\mathbb{Q}$. Suppose q is a square. Then there exists a unique absolutely irreducible overconvergent F-isocrystal $\mathscr{E}$* with coefficients in $\mathbb{Q}_p$ *that is compatible with $\mathscr{L} \otimes \mathbb{Q}_l(-\frac{1}{2})$. By construction, $\mathscr{E}$ is generically ordinary i.e. there exists a closed point $x \in |C|$ such that $\mathscr{E}_x$ has slopes $(0, 1)$.*

Proof. Only the uniqueness needs to be proved. As q is a square, the character $\overline{\mathbb{Q}}_l(\frac{1}{2})$ in fact descends to a character $\mathbb{Q}_l(\frac{1}{2})$ (because q is a quadratic residue mod l) and the coefficients of the characteristic polynomials of the Frobenius elements on $\mathscr{L} \otimes \mathbb{Q}_l(-\frac{1}{2})$ are all in $\mathbb{Q}$. By Abe's Lemma 7.7, any absolutely irreducible F-isocrystal on C is uniquely determined by these characteristic polynomials as there is only one embedding $\mathbb{Q} \to \overline{\mathbb{Q}}_p$. □

Lemma 9.11. *Let C a complete curve over a perfect field k, and $\mathscr{E}$ an absolutely irreducible rank-2 F-isocrystal on C with coefficients in $\mathbb{Q}_p$. Suppose $\mathscr{E}$ has determinant $\mathbb{Q}_p(-1)$. If for all closed points $x \in |C|$ the restriction $\mathscr{E}_x$ is not isoclinic, then $\mathscr{E}$ is not irreducible.*

Proof. As we are assuming $\mathscr{E}$ is rank 2 and has coefficients in $\mathbb{Q}_p$, if either of the slopes of $\mathscr{E}_x$ were non-integral, the slopes would have to be $(\frac{1}{2}, \frac{1}{2})$. If the slopes were integral, they just have to be integers (a, b) that sum to 1. On other other hand, the slope bounds of 8.10 shows that in this case, the slopes must be $(0, 1)$. As we assumed $\mathscr{E}_x$ is never isoclinic, this implies that every $\mathscr{E}_x$ has slopes $(0, 1)$ for all closed points $x \in |C|$. The slope filtration [24, 2.6.2] is therefore non-trivial: the point is that if at every $x \in |C|$, the slope 0 occurs, then there is in fact a rank 1 sub-object whose slopes at all closed points x are 0. □

Remark 9.12. There is a subtle point to Lemma 9.11. Let U be a modular curve $Y(N)$ over $\mathbb{F}_p$ minus the supersingular points. The universal elliptic curve yields an (overconvergent) F-isocrystal $\mathscr{E}$ on U and the slopes do not jump. By the slope filtration, $\mathscr{E}$ is not irreducible as a convergent F-isocrystal. However, it is irreducible as an overconvergent F-isocrystal. The point is that the slope filtration does not give a sub-object which is overconvergent. However, for proper curves this issue does not arise: the slope filtration will automatically yield an overconvergent F-isocrystal. See [27, Example 4.6].

Combining Lemma 9.11 with the $\mathbb{Q}_p$ companion $\mathscr{E}$ constructed in Corollary 9.10, we see that $\mathscr{E}$ is generically ordinary and has (finitely many) supersingular points. In particular, the slopes of $\mathscr{E}_x$ for $x \in |C|$ are either $(0, 1)$ or $(\frac{1}{2}, \frac{1}{2})$.

Let C be smooth curve over $\mathbb{F}_q$ and let $\mathscr{E}$ be an F-isocrystal on C. We are interested in when $\mathscr{E}$ "comes from an F-crystal". That is, there is a functor $F\text{-Cris}(C) \to \mathbf{F\text{-}Isoc}(C)$ as in Section 7 and we would like to characterize the essential image of this functor.

Lemma 9.13. *Let C be a curve over a perfect field k and let $\mathscr{E}$ be an F-isocrystal on C. Then $\mathscr{E}$ comes from an F-crystal if and only if all of the slopes at all closed points $c \in \mid C \mid$ are positive. Furthermore, $\mathscr{E}$ comes from a Dieudonné crystal if and only if all of the slopes at all closed points $c \in \mid C \mid$ are between 0 and 1.*

Proof. This is implicit in a theorem of Katz [24, Theorem 2.6.1]. Strictly speaking, Katz proves that if all of the slopes of an F-crystal $\mathscr{M}$ are bigger than some positive number λ, then $\mathscr{M}$ is isogenous to an F-crystal divisible by p^λ. From our conventions in Section 7, $\mathscr{E} \otimes \mathbb{Q}_p(-n)$ has an underlying F-crystal $\mathscr{M}$ for all $n \gg 0$. Katz's trick shows then that $\mathscr{M}$ is isogenous to an F-crystal $\mathscr{M}'$ that is divisible by p^n. Dividing F by p^n on $\mathscr{M}'$ produces the required F-crystal.

Finally, given any F-crystal $\mathscr{M}$ with slopes no greater than 1, define $V = F^{-1} \circ p$. This makes $\mathscr{M}$ into a Dieudonné crystal. □

Remark 9.14. The BT group constructed in Lemma 9.13 is *not unique*.

Definition 9.15. Let S be a normal scheme over k, a field of characteristic p. We define the category $BT(S)$ to be the category of Barsotti-Tate ("p-divisible") groups on S.

For an introduction to p-divisible groups and their contravariant Dieudonné theory, see Grothendieck [21] or Berthelot-Breen-Messing [5]. In particular, there exist a contravariant Dieudonné functor: given a BT group $\mathscr{G}/S$ one can construct a Dieudonné crystal $\mathbb{D}(\mathscr{G})$ over S. A consequence of the main theorem of [11] is the following.

Theorem 9.16. *(de Jong) Let S be a smooth scheme over k, a perfect field of characteristic p. Then the category $BT(S)$ is anti-equivalent to the category of Dieudonné crystals on S via $\mathbb{D}$.*

Definition 9.17. Let k be a field of characteristic p and let G be a BT group over k. We say G is *ordinary* if $G[p]$ has no local-local part. We say G is *supersingular* if all of the slopes of $\mathbb{D}(G)$ are $\frac{1}{2}$.

If G is a height 2 dimension 1 BT group over k, then G is ordinary iff the slopes of $\mathbb{D}(G)$ are $(0, 1)$. On the other hand, G is supersingular if and only if $G[p]$ is local-local.

Corollary 9.18. *Let C be a complete curve over $\mathbb{F}_q$ and let $\mathscr{L}$ be an absolutely irreducible rank 2 l-adic local system with trivial determinant, infinite image, and all Frobenius traces in $\mathbb{Q}$. There is a BT group $\mathscr{G}$ on C with the following properties*

- *$\mathscr{G}$ has height 2 and dimension 1.*
- *$\mathscr{G}$ has slopes $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$ (a.k.a. $\mathscr{G}$ is generically ordinary with supersingular points)*
- *The Dieudonné crystal $\mathbb{D}(\mathscr{G})$ is compatible with $\mathscr{L}(-\frac{1}{2})$.*

$\mathscr{G}$ has the following weak uniqueness property: the F-isocrystal $\mathbb{D}(\mathscr{G})$ is unique.

Proof. The existence comes from combining Corollary 9.10, Lemma 9.11Lemma 9.13, and Theorem 9.16. The uniqueness comes from the following facts: an absolutely irreducible overconvergent F-isocrystal with trivial determinant is uniquely specified by Frobenius eigenvalues (Lemma 7.7), $\mathbb{D}(\mathscr{G})$ is absolutely irreducible because it has both ordinary and supersingular points, and there is a unique embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_p$. □

10. Kodaira-Spencer

This section discusses the deformation theory of Barsotti-Tate groups in order to refine Corollary 9.18. It will also be useful in the applications to Shimura curves. The main references are Illusie [22], Xia [43], and de Jong [11, Section 2.5].

Let S be a smooth scheme over a perfect field k of characteristic p. Given a BT group $\mathscr{G}$ over S and a closed point $x \in |S|$, there is a map of formal schemes $u_x : \hat{S_{/x}} \to Def(\mathscr{G}_x)$ to the (equal characteristic) universal deformation space of $\mathscr{G}_x$. If $\mathscr{G}$ has dimension d and codimension c, then the dimension of the universal deformation space is cd [22, Corollary 4.8 (i)]. In particular, in the case $\mathscr{G}$ has height 2 and dimension 1, $Def(\mathscr{G}_x)$ is a one-dimensional formal scheme, exactly as in the familiar elliptic modular case.

Let $\mathscr{G} \to S$ be a Barsotti-Tate group. Then $\mathbb{D}(\mathscr{G})$ is a Dieudonné crystal which we may evaluate on S to obtain a vector bundle $\mathbb{D}(\mathscr{G})_S$ of rank $c+d = \mathrm{ht}(\mathscr{G})$. Let be ω the *Hodge bundle* of $\mathscr{G}$ (equivalently, of $\mathscr{G}[p]$): if $e : S \to \mathscr{G}[p]$ is the identity section, then $\omega := e^*\Omega^1_{\mathscr{G}[p]/S}$. Let α be the dual of the Hodge bundle of the Serre dual $\mathscr{G}^t$. There is the Hodge filtration

$$0 \to \omega \to \mathbb{D}(\mathscr{G})_S \to \alpha \to 0$$

We remark that ω has rank d and α has rank c. The Kodaira-Spencer map is as follows

$$KS : T_S \to Hom_{\mathcal{O}_S}(\omega, \alpha) \cong \omega^* \otimes \alpha$$

Illusie [22, A.2.3.6] proves that KS_x is surjective if and only if the d_x is surjective, see also [11, 2.5.5]. This motivates the following definition.

Definition 10.1. Let S/k be a smooth scheme over a perfect field k of characteristic p. Let $\mathscr{G} \to S$ be a Barsotti-Tate group and $x \in |X|$ a closed point. We say that $\mathscr{G}$ is *versally deformed at x* if either of the following equivalent conditions hold

- The fiber at x of KS, $KS_s : T_{S,x} \to (\omega^* \otimes \alpha)_x$, is surjective.
- The map $u_x : \hat{S_{/x}} \to Def(\mathscr{G}_x)$ is surjective.

Definition 10.2. Let S/k be a smooth scheme over a perfect field k of characteristic p. Let $\mathscr{G} \to S$ be a Barsotti-Tate group. We say that $\mathscr{G}$ is *generically versally deformed* if, for every connected component S_i, there exists a closed point $x_i \in |S_i|$ such that $\mathscr{G}$ is versally deformed at x_i.

Remark 10.3. If $\mathscr{G} \to S$ is generically versally deformed, there exists a dense Zariski open $U \subset X$ such that $\mathscr{G}|_U \to U$ is *everywhere versally deformed.* This is because the condition "a map of vector bundles is surjective" is an open condition.

Example 10.4. Let us recall Igusa level structures. Let $Y(1) = \mathcal{M}_{1,1}$. There is a universal elliptic curve $\mathcal{E} \to Y(1)$. Let $\mathscr{G} = \mathcal{E}[p^\infty]$ be the associated p-divisible group over $Y(1)$. Here, $\mathscr{G}$ is height 2, dimension 1, and *everywhere versally deformed* on $Y(1)$. Let X be the cover of $Y(1)$ that trivializes the finite flat group scheme $\mathscr{G}[p]^{ét}$ away from the supersingular locus of $Y(1)$. X is branched exactly at the supersingular points. Pulling back $\mathscr{G}$ to X yields a BT group that is generically versally deformed but *not everywhere versally deformed* on X. See Ulmer's article [41] for an especially enlightening introduction to Igusa level structures.

Example 10.5. Let $S = \mathcal{A}_{2,1} \otimes \mathbb{F}_p$, the moduli of principally polarized abelian surfaces. Then the BT group of the universal abelian scheme, $\mathscr{G} \to S$, is *nowhere versally deformed*: the formal deformation space of a height 4 dimension 2 BT group is 4, whereas $\dim(S) = 3$. Serre-Tate theory relates this to the fact that most formal deformations of an abelian variety of dimension $\dim(A) > 1$ are not algebraizable.

Question 10.6. *Is there an example of a Barsotti-Tate group $\mathscr{G} \to S$ over a smooth (algebraic) scheme where $\dim(\mathscr{G})$, $codim(\mathscr{G}) > 1$ and $\mathscr{G}$ is generically versally deformed?*

Lemma 10.7. *(Xia's Frobenius Untwisting Lemma) Let S be a smooth scheme over a perfect field k of characteristic p. Let $\mathscr{G} \to S$ be a BT group over. Then $\mathscr{G} \to S$ has trivial KS map if and only if there exists a BT group $\mathscr{H}$ on S such that $\mathscr{H}^{(p)} \cong \mathscr{G}$.*

Proof. This is [43, Theorem 4.13]. □

Xia's Lemma 10.7 allows us to set up the following useful equivalence for characterizing when a height 2 dimension 1 BT group $\mathscr{G}$ is generically versally deformed.

Lemma 10.8. *Let C be a smooth curve over a perfect field k of characteristic p. Let $\mathscr{G}$ be a height 2, dimension 1 BT group over C. Let η be the generic point of C. Suppose $\mathscr{G}_\eta$ is ordinary. Then the following are equivalent*

1. *The KS map is 0.*
2. *There exists a BT group $\mathscr{H}$ on C such that $\mathscr{H}^{(p)} \cong \mathscr{G}$.*
3. *There exists a finite flat subgroup scheme $N \subset \mathscr{G}$ over C such that N has order p and is generically étale.*
4. *The connected-étale exact sequence $\mathscr{G}_\eta[p]^\circ \to \mathscr{G}_\eta[p] \to \mathscr{G}_\eta[p]^{ét}$ over the generic point splits.*

Proof. The equivalence of (1) and (2) is Xia's Lemma 10.7. Now, let us assume (2). Then there is a Verschiebug map

$$V_{\mathscr{H}} : \mathscr{G} \cong \mathscr{H}^{(p)} \to \mathscr{H}$$

whose kernel is generically étale and has order p because we assumed $\mathscr{G}$ (and hence $\mathscr{H}$) were generically ordinary. Therefore (3) is satisfied. Conversely, given (3), N is p-torsion (Deligne proved that all finite flat commutative group schemes are annihilated by their order [40].) Therefore we have a factorization:

$$\begin{tikzcd} \mathscr{G} \arrow[rr, "\times p"] \arrow[dr] & & \mathscr{G}/N \\ & \mathscr{G} \arrow[ur] & \end{tikzcd}$$

Set $\mathscr{H} = \mathscr{G}/N$. As we assumed N was generically étale and $\mathscr{G}$ was generically ordinary, the map $\mathscr{G} \to \mathscr{G}/N$ may be identified with Verschiebung: $\mathscr{H}^{(p)} \to \mathscr{H}$. In particular, $\mathscr{H}^{(p)} \cong \mathscr{G}$ as desired.

Let us again assume (3). Then $N_\eta \subset \mathscr{G}_\eta[p]$ projects isomorphically onto $\mathscr{G}_\eta[p]^{ét}$. Therefore the connected-étale sequence over the generic point splits. To prove the converse, simply take the Zariski closure of the section to $\mathscr{G}_\eta[p] \to \mathscr{G}_\eta[p]^{ét}$ inside of $\mathscr{G}[p]$ to get N (this will be a finite flat group scheme because C is a smooth curve.) □

Lemma 10.9. *Let C be a smooth curve over a perfect field k of characteristic p. Suppose $\mathscr{G}$ and $\mathscr{G}'$ are height 2, dimension 1 BT groups over C that are generically versally deformed and generically ordinary. Suppose further that their Dieudonné Isocrystals are isomorphic: $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q} \cong \mathbb{D}(\mathscr{G}') \otimes \mathbb{Q}$. Then $\mathscr{G}$ and $\mathscr{G}'$ are isomorphic.*

Proof. The isocrystals being isomorphic implies that their is an isogeny $\mathbb{D}(\mathscr{G}) \to \mathbb{D}(\mathscr{G}')$. De Jong's Theorem 9.16 then implies that their is an associated isogeny $\phi : \mathscr{G} \to \mathscr{G}'$. By "dividing by p", we may ensure that ϕ does not restrict to 0 on $\mathscr{G}[p]$. Now suppose for contradiction that ϕ is not an isomorphism, i.e. that it has a kernel. Then $\phi|_{\mathscr{G}[p]}$ is also has a nontrivial kernel.

We have the following diagram of connected-generically étale sequences.

$$\begin{tikzcd} \mathscr{G}[p]^\circ \arrow[r] \arrow[d] & \mathscr{G}'[p]^\circ \arrow[d] \\ \mathscr{G}[p] \arrow[r] \arrow[d] & \mathscr{G}'[p] \arrow[d] \\ \mathscr{G}[p]^{ét} \arrow[r] & \mathscr{G}'[p]^{ét} \end{tikzcd}$$

As we have assumed $\mathscr{G}$ is generically versally deformed, the kernel of $\phi|_{\mathscr{G}[p]}$ cannot be generically étale by (3) of Lemma 10.8. Thus the kernel must be the connected group scheme $\mathscr{G}[p]^\circ$ because the order of $\mathscr{G}[p]$ is p^2. We therefore get a nonzero map $\mathscr{G}[p]^{et} \to \mathscr{G}'[p]$. Now $\mathscr{G}[p]^{ét}$ has order p and is generically étale by definition, so by (3) of Lemma 10.8, $\mathscr{G}'$ is not generically versally deformed. Thus the assumption that ϕ had a kernel yields that $\mathscr{G}'$ is not generically versally deformed, contradicting our original hypothesis; ϕ must therefore be an isomorphism. □

Theorem 10.10. *Let C be a smooth, geometrically irreducible, complete curve over $\mathbb{F}_q$. Suppose q is a square. There is a natural bijection between the following two sets.*

$$\left\{\begin{array}{l} \overline{\mathbb{Q}}_l\text{-local systems } \mathscr{L} \text{ on } C \text{ such that} \\ \bullet\ \mathscr{L} \text{ is irreducible of rank 2} \\ \bullet\ \mathscr{L} \text{ has trivial determinant} \\ \bullet \text{ The Frobenius traces are in } \mathbb{Q} \\ \bullet\ \mathscr{L} \text{ has infinite image,} \\ \text{up to isomorphism} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} p\text{-divisible groups } \mathscr{G} \text{ on } C \text{ such that} \\ \bullet\ \mathscr{G} \text{ has height 2 and dimension 1} \\ \bullet\ \mathscr{G} \text{ is generically versally deformed} \\ \bullet\ \mathscr{G} \text{ has all Frobenius traces in } \mathbb{Q} \\ \bullet\ \mathscr{G} \text{ has ordinary and supersingular points,} \\ \text{up to isomorphism} \end{array}\right\}$$

such that if $\mathscr{L}$ corresponds to $\mathcal{G}$, then $\mathscr{L} \otimes \mathbb{Q}_l(-1/2)$ is compatible with the F-isocrystal $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q}$.

Proof. Given such an $\mathscr{L}$, we can make a BT group $\mathscr{G}$ as in Corollary 9.18. Xia's Lemma 10.7 ensures that we can modify $\mathscr{G}$ to be generically versally deformed by Frobenius "untwisting"; this process terminates because there are both supersingular and ordinary points, so the map to the universal deformation space cannot be identically 0. This BT group is unique up to (*non-unique*) isomorphism by Lemma 10.9.

To construct the map in the opposition direction, just reverse the procedure. Given such a $\mathscr{G}$, first form Dieudonné isocrystal $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q}$. This is an absolutely irreducible F-isocrystal because there are both ordinary and supersingular points. Twisting by $\overline{\mathbb{Q}}_p(1/2)$ yields an irreducible object of $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(X)_{\overline{\mathbb{Q}}_p}$ that has trivial determinant. By Theorem 8.6, for any $l \neq p$ there is a compatible $\mathscr{L}_l$ that is absolutely irreducible and has all Frobenius traces in $\mathbb{Q}$. This $\mathscr{L}_l$ is unique: there is only one embedding of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_l$ and the Brauer-Nesbitt Theorem 7.2 ensures that $\mathscr{L}_l$ is uniquely determined by the Frobenius traces. Finally, $\mathscr{L}_l$ has infinite image: as $\mathscr{G}$ had both ordinary and supersingular points, it cannot be trivialized on a finite étale cover. □

Remark 10.11. Theorem 10.10 has the following strange corollary. Let $\mathscr{L}$ and C be as in the theorem. Then there is a natural effective divisor on C associated to $\mathscr{L}$: the points where the associated $\mathscr{G}$ is not versally deformed, together with their multiplicity. This divisor is trivial if and only if $\mathscr{G} \to C$ is everywhere versally deformed. We wonder if this has an interpretation on the level of cuspidal automorphic representations.

11. Algebraization and Finite Monodromy

In Theorem 10.10, the finiteness of the number of such local systems (a theorem whose only known proof goes through the Langlands correspondence) implies the finiteness of such BT groups. In general, BT groups on varieties are far from being "algebraic": for instance, over $\mathbb{F}_p$ there are uncountably many BT groups of height 2 and dimension 1 as one can see from Dieudonné theory. However, here they are constructed rather indirectly from a motive via the Langlands correspondence. All examples of such local systems that we can construct involve abelian schemes and we are very interested in the following question.

Question 11.1. *Let X be a smooth projective variety over $\mathbb{F}_q$ and let $\mathscr{G} \to X$ be a height 2, dimension 1 p-divisible group with ordinary and supersingular points. Is there an embedding as follows, where $A \to X$ is an abelian scheme?*

$$\begin{tikzcd} \mathscr{G} \arrow[r] \arrow[dr] & A \arrow[d] \\ & X \end{tikzcd}$$

Remark 11.2. We explain the hypotheses Question 11.1. That X is complete ensures that the convergent F-isocrystal $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q}$ is automatically overconvergent. The existence of both ordinary and supersingular points ensures that the Dieudonné isocrystal $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q}$ is absolutely irreducible. That $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q}$ is overconvergent and absolutely irreducible ensures that the Frobenius traces are algebraic numbers by either Abe-Esnault [3, 4.1,4.2] or Kedlaya [25, Theorem 5.3].

The following would be implied by an affirmative answer to Question 11.1. We will see the utility of the condition "everywhere versally deformed" in Section 12.

Conjecture 11.3. *Let X be a complete curve over $\mathbb{F}_q$ and let $\mathscr{G} \to X$ be a p-divisible group with ordinary and supersingular points. Does there exist a variety Y and a height 2 dimension 1 p-divisible group $\mathscr{H}$ on Y that is* ***everywhere*** *versally deformed such that $\mathscr{G}$ is pulled back from $\mathscr{H}$?*

$$\begin{tikzcd} \mathscr{G} \arrow[r] \arrow[d] & \mathscr{H} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Remark 11.4. Johan de Jong and Ching-Li Chai explained to us that there are examples of (fake) Hilbert modular surfaces to show that one cannot hope for Y to be a curve in Conjecture 11.3.

To motivate the next conjecture, recall that the modular curves are not complete curves. On the other hand, the universal local systems on Shimura curves parameterizing fake elliptic curves cannot all be defined over $\mathbb{Q}_l$, as we saw in Example 9.1. In other words, they do not form a $\mathbb{Q}$-compatible system.

Conjecture 11.5. *Let X be a complete curve over $\mathbb{F}_q$. Suppose $\{\mathscr{L}_l\}_{l\neq p}$ is a $\mathbb{Q}$-compatible system of absolutely irreducible rank 2 local systems with trivial determinant. Then they have finite monodromy.*

Using our techniques we can prove the related but elementary Theorem 11.6: it is a straightforward application of the slope bounds, Theorem 8.10. Note that, in the context of Conjecture 11.5, the hypothesis of Theorem 11.6 is stronger exactly one way: namely, one assumes that the p-adic companion of $\{\mathscr{L}_l\}_{l\neq p}$ has coefficients in $\mathbb{Q}_p$. Note also that Theorem 11.6 does not assume that X is complete.

Theorem 11.6. *Let X be a curve over $\mathbb{F}_q$. Let $\mathcal{E} \in \textbf{F-Isoc}^{\dagger}(X)$ be an overconvergent F-isocrystal on X with coefficients in $\mathbb{Q}_p$ that is rank 2, absolutely irreducible, and has finite determinant. Suppose further that the field of traces of $\mathcal{E}$ is $\mathbb{Q}$. Then $\mathcal{E}$ has finite monodromy.*

Proof. We claim that $\mathscr{E}$ is isoclinic at every closed point $x \in |X|$. Indeed, by the Slope Bounds Theorem 8.10, the slopes of $\mathscr{E}_x$ differ by at most 1, forbidding slopes of the form $(-a, a)$ for $0 \neq a \in \mathbb{Z}$. As the coefficients of $\mathscr{E}$ are $\mathbb{Q}_p$, any fractional slope must appear more than once. (In fact, a slope of $\frac{s}{r}$ has to appear a multiple of r times by the slope decomposition.)

By Abe's Lemma 7.7, absolutely irreducible overconvergent F-isocrystals are determined the characteristic polynomials of Frobenius elements. As there is a *unique* embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, this implies that any p-adic companion to $\mathscr{E}$ is isomorphic to $\mathscr{E}$ itself. Therefore, we may conclude by Lemma 9.6. $\square$

Remark 11.7. Note that Theorem 11.6 uses Lemma 9.6 which critically uses Deligne's Companions Conjecture 8.4. In particular, we use that $\mathscr{E}$ lives in a (complete) compatible system.

Corollary 11.8. *Let X be a curve over $\mathbb{F}_q$. Let $\mathcal{E} \in \textbf{F-Isoc}^{\dagger}(X)$ be an overconvergent F-isocrystal on X (with coefficients in $\mathbb{Q}_p$) that is rank 2, absolutely irreducible, and has determinant $\mathbb{Q}_p(i)$ for $i \in 2\mathbb{Z}$. Suppose further that the field of traces of $\mathcal{E}$ is $\mathbb{Q}$. Then $\mathcal{E}$ has finite monodromy.*

Proof. As i is even, $\mathcal{E}(\frac{i}{2}) \in \textbf{F-Isoc}^{\dagger}(X)$, i.e. $\mathcal{E}(\frac{i}{2})$ has coefficients in $\mathbb{Q}_p$. Apply Theorem 11.6. $\square$

12. Application to Shimura Curves

In this section, we indicate a criterion for an étale correspondence of projective hyperbolic curves over $\mathbb{F}_q$ to be the reduction modulo p of some Shimura curves over $\mathbb{C}$. Our goal was to find a criterion that was as "group theoretic" as possible.

Definition 12.1. Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves over k. We say it *has no core* if $k(X) \cap k(Y)$ has transcendence degree 0 over k.

For much more on the theory of correspondences without a core, see our article [29]. In general, correspondences of curves do not have cores. However, in the case of étale correspondences over fields of characteristic 0, we have the following remarkable theorem of Mochizuki.

Theorem 12.2. *(Mochizuki [33]) If $X \leftarrow Z \rightarrow Y$ is an étale correspondence of hyperbolic curves without a core over a field k of characteristic 0, then $X, Y,$ and Z are all Shimura (arithmetic) curves. In particular, all of the curves and maps can be defined over $\overline{\mathbb{Q}}$.*

Remark 12.3. Hecke correspondences of modular/Shimura curves furnish examples of étale correspondences without a core.

Our strategy will therefore be to find an additional structure on an étale correspondence without a core such that the whole picture *canonically lifts* from $\mathbb{F}_q$ to characteristic 0 and apply Mochizuki's theorem. To make this strategy work, we need two inputs.

Lemma 12.4. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of projective hyperbolic curves over $\mathbb{F}$ without a core. If the correspondence lifts to $W(\mathbb{F})$, then the lifted correspondence is étale and has no core. In particular, X, Z, and Y are the reductions modulo p of Shimura curves.*

Proof. That the lifted correspondence is étale follows from open-ness of the étale locus. It has no core by [29, Lemma 4.13]. Apply Mochizuki's theorem to the generic fiber. □

Xia proved the following theorem, which may be thought of as "Serre-Tate canonical lift" in families.

Theorem 12.5. *Let k be an algebraically closed field of characteristic p and let X be a smooth proper hyperbolic curve over k. Let $\mathscr{G} \to X$ be a height 2, dimension 1 BT group over X. If $\mathscr{G}$ is everywhere versally deformed on X, then there is a unique curve $\tilde{X}$ over $W(k)$ which is a lift of X and admits a lift $\tilde{\mathscr{G}}$ of $\mathscr{G}$. Furthermore, the lift $\tilde{\mathscr{G}}$ is unique.*

Proof. This is [43, Theorem 1.2]. □

Remark 12.6. Note that the hypothesis of Theorem 12.5 implies that $\mathscr{G} \to X$ is generically ordinary. This is why we call it an analog to the Serre-Tate canonical lift.

Example 12.7. Let D be a non-split quaternion algebra over $\mathbb{Q}$ that is split at ∞ and let p be a finite prime where D splits. The Shimura curve X^D parametrizing fake elliptic curves for $\mathcal{O}_D$ exists as a smooth complete curve (in the sense of stacks) over $\mathbb{Z}[\frac{1}{2d}]$ and hence over $\mathbb{F}_p$. (See Definition 1.1.) Abusing notation, we denote by X^D the Shimura curve over $\mathbb{F}_p$. It admits a universal abelian surface $f : \mathcal{A} \to X^D$ with multiplication by $\mathcal{O}_D$. As $D \otimes \mathbb{Q}_p \cong M_{2\times 2}(\mathbb{Q}_p)$, one can use Morita equivalence (as in Example 9.1), i.e. apply the idempotent

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

on the height 4 dimension 1 BT group $\mathcal{A}[p^\infty]$ to get a height 2 dimension 1 BT group $\mathscr{G}$, and in fact $\mathcal{A}[p^\infty] \cong \mathscr{G} \oplus \mathscr{G}$. Here $\mathscr{G}$ is *everywhere versally deformed* on X^D. Moreover, $\mathscr{G}$ has both supersingular and ordinary points. For similar discussion, see Section 5.1 of [20].

On the other hand, as in Example 10.4, let X^D_{Ig} be the Igusa cover which trivializes $\mathscr{G}[p]^{et}$ on the ordinary locus. This cover is branched precisely over the supersingular locus of $\mathscr{G}$ on X. Pulling back $\mathscr{G}$ to X^D_{Ig} yields a BT group which is *generically versally deformed* but not everywhere versally deformed.

Corollary 12.8. *Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be an étale correspondence of projective hyperbolic curves without a core over a perfect field k. Let $\mathscr{G} \to X$ be a BT group of height 2 and dimension 1 that is everywhere versally deformed. Suppose further that $f^*\mathscr{G} \cong g^*\mathscr{G}$. Then X and Z are the reduction modulo p of Shimura curves.*

Proof. Any lift $\tilde{X}$ of X naturally induces a lifts $\tilde{Z}_f$ and $\tilde{Z}_g$ of Z because f and g are étale and the goal is to find a lift $\tilde{X}$ such that $\tilde{Z}_f \cong \tilde{Z}_g$. Note that $f^*\mathscr{G}$ and $g^*\mathscr{G}$ are everywhere versally deformed on Z because f and g are étale. By Theorem 12.5, the pairs $(X, \mathscr{G})$, $(Z, f^*\mathscr{G})$, and $(Z, g^*\mathscr{G})$ canonically lift to $W(k)$. As $f^*\mathscr{G} \cong g^*\mathscr{G}$, the lifts of $(Z, f^*\mathscr{G})$ and $(Z, g^*\mathscr{G})$ are isomorphic and we get an étale correspondence of curves over $W(k)$:

$$\begin{tikzcd} & \tilde{Z} \arrow[dl, "\tilde{f}"'] \arrow[dr, "\tilde{g}"] & \\ \tilde{X} & & \tilde{X} \end{tikzcd}$$

Finally by Lemma 12.4, X and Z are the reductions modulo p of Shimura curves. □

Example 12.9. Let $X = X^D$ be a Shimura curve parametrizing fake elliptic curves with multiplication by $\mathcal{O}_D$, as in Definition 1.1, with D having discriminant d. Let N be prime to d and let $Z = X^D_0(N)$, i.e. a moduli space of pairs $(A_1 \to A_2)$ of fake elliptic curves equipped with a "cyclic isogeny of fake-degree N" [4, Section 2.2]. There exist Hecke correspondences

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

Moreover, as in Example 9.1, as long as $(p, dN) = 1$, this correspondence has good reduction at p and as in Example 12.7 the universal p-divisible group splits as $\mathscr{G} \oplus \mathscr{G}$ on X. In this example, $\mathscr{G} \to X$ is everywhere versally deformed and $\pi_1^*\mathscr{G} \cong \pi_2^*\mathscr{G}$. In particular, there are examples where the conditions of Corollary 12.8 are met.

Theorem 12.10. *Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be an étale correspondence of smooth, geometrically connected, complete curves without a core over $\mathbb{F}_q$ with q a square. Let $\mathscr{L}$ be a $\overline{\mathbb{Q}}_l$-local system on X as in Theorem 10.10 such that $f^*\mathscr{L} \cong g^*\mathscr{L}$ as local systems on Z. Suppose the $\mathscr{G} \to X$ constructed via Theorem 10.10 is everywhere versally deformed. Then X and Z are the reductions modulo p of Shimura curves.*

Proof. The uniqueness statement in Theorem 10.10 immediately implies that $f^*\mathscr{G} \cong g^*\mathscr{G}$. Apply Corollary 12.8. □

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