DEFORMATION THEORY OF PERIODIC HIGGS-DE RHAM FLOWS

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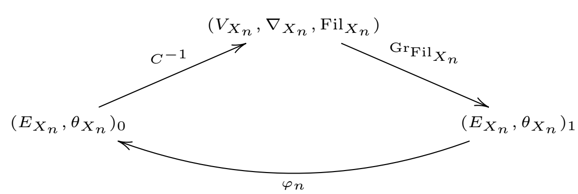
Abstract. In this note we study the deformation theory of periodic (logarithmic) Higgs-de Rham flows. Under suitable numerical assumptions, this is equivalent to the deformation theory of torsion (logarithmic) Fontaine-Faltings modules. As an application, we formulate an *ordinarity* condition, which provides a sufficient condition for a p^n-torsion crystalline representation to deform to a p^{n+1}-torsion crystalline representation.

1. Introduction

Let k be a perfect field of odd characteristic $p > 0$, let $W := W(k)$ denote the ring of Witt vectors, and let $K := \mathrm{Frac}(W)$ be the field of fractions. Let X/W be a smooth scheme and let $D \subset X$ to be a relative simple normal crossings divisor; the pair (X, D) is a called *smooth pair over* W. For any positive integer $n > 0$, denote by (X_n, D_n) the reduction of (X, D) modulo p^n. In this article, we typically consider objects on X that are allowed logarithmic poles along D.

Suppose X/W is projective and consider an 1-periodic (logarithmic) Higgs-de Rham flow HDF_{X_n} over X_n, for an positive integer $n > 0$,

(1.1)



Our goal is to understand the conditions/obstructions to lifting HDF_{X_n} to a 1-periodic Higgs-de Rham flow $HDF_{X_{n+1}}$. We now state our main results and applications.

Theorem 1.1 (Theorem 6.2). *Let $(X, D)/W$ be a smooth pair with X/W projective. Let HDF_{X_n} be a periodic Higgs-de Rham flow over (X_n, D_n). For a given lifting $(E, \theta)_{X_{n+1}}$ of the initial graded Higgs bundle over (X_{n+1}, D_{n+1}), there is at most one Higgs-de Rham flow with initial term $(E, \theta)_{X_{n+1}}$ that lifts HDF_{X_n} up to isomorphism.*

The following theorem gives a precise condition for the following. Let $(X_n, D_n) \subset (X_{n+1}, D_{n+1})$ be a thickening of smooth pairs over $W_n(k) \subset W_{n+1}(k)$, with X_n/W_n projective. Suppose we have a logarithmic Higgs bundle (E_n, θ_n) on (X_n, D_n) that initiates a 1-periodic Higgs-de Rham flow. When does there exist a lift $(\widetilde{E}_{n+1}, \widetilde{\theta}_{n+1})$ of (E_n, θ_n) together with a lift (X_{n+2}, D_{n+2}) of (X_{n+1}, D_{n+1}) over $W_{n+2}(k)$ such that $(\widetilde{E}_{n+1}, \widetilde{\theta}_{n+1})$ initiates a 1-periodic Higgs-de Rham flow with respect to the thickening $(X_{n+1}, D_{n+1}) \subset (X_{n+2}, D_{n+2})$? In the following, $\mathbb{K}$ and H are explicitly defined in 6.2, and IC stands for the *parametrized* inverse Cartier transform.

Theorem 1.2 (Theorem 6.4). *Let $(X_n, D_n)/W_n$ be a smooth pair with X_n/W_n projective. Suppose $\mathbb{K}$ is not empty and the projection $H \to \mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1, \theta_1))\big)$ is surjective. Then there exists some finite* *extension* k'/k *and* $((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(X_{n+2},D_{n+2}))\in\mathbb{K}_{k'}$ *such that*

$$\mathrm{Gr}\circ\mathrm{IC}((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(X_{n+2},D_{n+2}))=(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}).$$

In other words, $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})$ *is a* 1*-periodic Higgs bundle under the lifting* $(X_{n+1},D_{n+1})\subset(X_{n+2},D_{n+2})$.

We briefly explain the main ingredients to the proofs of the main results.
Suppose there exists a graded Higgs bundle $(E_{X_{n+1}},\theta_{X_{n+1}})_0$ which lifts $(E_{X_n},\theta_{X_n})_0$. Taking the n-truncated inverse Cartier transform, one gets a de Rham bundle

$$(V_{X_{n+1}},\nabla_{X_{n+1}})=C^{-1}\big((E_{X_{n+1}},\theta_{X_{n+1}})_0,(V_{X_n},\nabla_{X_n},\mathrm{Fil}_{X_n}),\varphi_n\big).$$

Suppose there exists a Hodge filtration $\mathrm{Fil}_{X_{n+1}}$ on $(V_{X_{n+1}},\nabla_{X_{n+1}})$. Then the associated grading Higgs bundle

$$(E_{X_{n+1}},\theta_{X_{n+1}})_1=\mathrm{Gr}(V_{X_{n+1}},\nabla_{X_{n+1}},\mathrm{Fil}_{X_{n+1}})$$

lifts $(E_{X_n},\theta_{X_n})_1$. Via the isomorphism φ_n, both $(E_{X_{n+1}},\theta_{X_{n+1}})_1$ and $(E_{X_{n+1}},\theta_{X_{n+1}})_0$ lift $(E_{X_n},\theta_{X_n})_0$; therefore, their difference defines an obstruction to lifting the periodicity map φ_n.
In conclusion, there are three obstructions to lifting a 1-periodic Higgs-de Rham flow:

(1) the obstruction to lift the initial Higgs bundle;
(2) the obstruction to lift the Hodge filtration; and
(3) the obstruction to lift the periodicity map φ_n.

We will study each of these in turn, using pure deformation theory. These deformation questions are general and have nothing to do with the ring of Witt vectors; we state our assumptions and results below.

Setup 1. Let S be a noetherian scheme, let $S\hookrightarrow\widetilde{S}$ be a square-zero thickening, i.e., the ideal sheaf $\mathfrak{a}$ of S satisfies $\mathfrak{a}^2=0$. Let $(\widetilde{X},\widetilde{D})$ be a smooth pair over $\widetilde{S}$; that is, $\widetilde{X}$ is a smooth $\widetilde{S}$-scheme and $\widetilde{D}\subset\widetilde{X}$ is a $\widetilde{S}$-flat relative simple normal crossings divisor with $D\times_S\widetilde{S}$ reduced. Set $(X,D):=(\widetilde{X}\times_{\widetilde{S}}S,\widetilde{D}\times_{\widetilde{S}}S)$. Then $\mathcal{I}=\mathcal{O}_{\widetilde{X}}\cdot\mathfrak{a}$ is the ideal of definition of $X\hookrightarrow\widetilde{X}$, and $\mathcal{I}^2=0$.
Set $\Omega^1_{\widetilde{X}/\widetilde{S}}$ to be the sheaf of relative 1-forms and $\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})$ to be the sheaf of relative 1-forms with logarithmic poles along $\widetilde{D}$. (This latter sheaf $\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})$ may be also defined as the sheaf of 1-forms on the f.s. log scheme induced from the pair $(\widetilde{X},\widetilde{S})$) Both of these sheaves are finite and locally free.

In the context of Setup 1, we show the following results on lifting (filtered) de Rham bundles and graded Higgs bundles. As explained above these results will be used to characterize when a Higgs-de Rham flow modulo p^n lifts to a Higgs-de Rham flow modulo p^{n+1}. Note that Setup 1 does not assume that X/S is projective.

Theorem 1.3 (Theorem 3.1)**.** *Notation as in Setup 1 and let* (V,∇,Fil) *be a (logarithmic) filtered de Rham bundle over* $(X,D)/S$*. Then*

(1). *the obstruction to lifting* (V,∇,Fil) *to a filtered de Rham bundle over* $(\widetilde{X},\widetilde{D})/\widetilde{S}$ *lies in* $\mathbb{H}^2\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$*;*
(2). *if* (V,∇,Fil) *has lifting* $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$*, then the set of liftings is a torsor for* $\mathbb{H}^1\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$*;*
(3). *the infinitesimal automorphism group of* $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ *over* (V,∇,Fil) *is* $\mathbb{H}^0\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$.

Theorem 1.4 (Theorem 3.9)**.** *Notation as in Setup 1 and let* (V,∇,Fil) *be a filtered de Rham bundle over* $(X,D)/S$*. Let* $(\widetilde{V},\widetilde{\nabla})$ *be lifting of the underlying de Rham bundle* (V,∇) *over* $(\widetilde{X},\widetilde{D})/\widetilde{S}$*. Then*

1). *the obstruction to lifting* Fil *to a Hodge filtration on* $(\widetilde{V},\widetilde{\nabla})$ *lies in* $\mathbb{H}^1(\mathscr{C})$*;*
2). *if* Fil *has a lifting, then the set of liftings is a torsor for* $\mathbb{H}^0(\mathscr{C})$.

Theorem 1.5 (Theorem 4.1)**.** *Notation as in Setup 1 and let* (E,θ,Gr) *be a graded Higgs bundle over* $(X,D)/S$*. Then*

1). *the obstruction to lifting* (E,θ,Gr) *to a graded Higgs bundle over* $(\widetilde{X},\widetilde{D})/\widetilde{S}$ *lies in* $\mathbb{H}^2\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$*;*
2). *if* (E,θ,Gr) *has a graded lifting* $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$*, then the set of liftings is a torsor for* $\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$*;*
3). *the infinitesimal automorphism group of* $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$ *over* (E,θ,Gr) *is* $\mathbb{H}^0\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$.

Theorem 1.6 (Theorem 5.2)**.** *Notation as in Setup 1 and let* (V,∇,Fil) *be a filtered de Rham bundle on* $(X,D)/S$*. Suppose that the Hodge-de Rham spectral sequence (5.1) attached to* (V,∇,Fil) *degenerates at* E_1*.*

(1). *If* (V,∇) *is liftable, then* (V,∇,Fil) *is also liftable.*
(2). *Let* $(\widetilde{V},\widetilde{\nabla})$ *be a lifting of the underlying de Rham bundle of* (V,∇,Fil)*. Then for two liftings* $\widetilde{\mathrm{Fil}}$ *and* $\widetilde{\mathrm{Fil}}'$ *of the Hodge filtration* Fil *there exists an isomorphism*

$$(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}) \xrightarrow{\simeq} (\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}')$$

of filtered de Rham bundles on $(\widetilde{X},\widetilde{D})/\widetilde{S}$.

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2. Preliminaries

2.1. Filtered (logarithmic) de Rham bundles

Filtered vector bundles. Let X be a scheme and let V be a vector bundle on X. Recall that a *descending filtration* of V is a decreasing sequence of subbundles

$$V \supset \cdots \supset \mathrm{Fil}^{-2}V \supset \mathrm{Fil}^{-1}V \supset \mathrm{Fil}^{0}V \supset \mathrm{Fil}^{1}V \supset \mathrm{Fil}^{2}V \supset \cdots.$$

The condition that $\mathrm{Fil}^i \subset \mathrm{Fil}^{i-1}$ is a subbundle means that it is locally a direct summand. This filtration is called separated and exhaustive if for sufficiently large integer $N >> 0$

$$\mathrm{Fil}^{-N}V = V \quad \text{and} \quad \mathrm{Fil}^{N+1}V = 0.$$

In this note, all filtrations are assume to be descending, separated and exhaustive. The pair (V,Fil) is called a *filtered vector bundle over* X.
Let (V,Fil) be a filtered vector bundle with $\mathrm{Fil}^\ell V$ free over $\mathcal{O}_X$ for each $\ell\in\mathbb{Z}$. Then a basis $\{e_1,e_2\cdots,e_r\}$ of V is called *adapted* basis of (V,Fil) if $\mathrm{Fil}^\ell V$ is generated by a subset of $\{e_1,e_2\cdots,e_r\}$ for each $\ell\in\mathbb{Z}$.

Lemma 2.1. *Let* (V,Fil) *be a filtered vector bundle on a scheme* X*. Then there exists an open affine covering* $\{U_i\}_{i\in I}$ *of* X *such that* $\mathrm{Fil}^\ell V|_{U_i}$ *is free* $\mathcal{O}_{U_i}$*-module for each* $\ell\in\mathbb{Z}$ *and each* $i\in I$*.*

Proof. There are only finitely many vector bundles Fil^ℓ, so we may find an open cover that trivializes them all. □

Dual filtration. Let V be a vector bundle on a scheme X. The dual vector bundle $V^\vee$ is defined by

$$V^\vee(U) := \mathrm{Hom}_{\mathcal{O}_X(U)}(V(U),\mathcal{O}_X(U)), \qquad \text{for any open subset } U \text{ of } X.$$

Let (V,Fil) be a filtered vector bundle over a scheme X. The *dual filtration* $\mathrm{Fil}^\vee$ on the dual vector bundle $V^\vee$ is defined by

$$\mathrm{Fil}^{\vee,\ell}(V^\vee) := \Big(V/\mathrm{Fil}^{1-\ell}V\Big)^\vee \subset V^\vee, \tag{2.1}$$

Filtration on **Hom.** Let (V_1,Fil_1) and (V_2,Fil_2) be two filtered vector bundle over X. There is a natural filtration $\mathrm{Fil}_{(\mathrm{Fil}_1,\mathrm{Fil}_2)}$ on the Hom vector bundle $\mathscr{H}om(V_1,V_2)=V_1^\vee\otimes V_2$ given by

$$\mathrm{Fil}^\ell_{(\mathrm{Fil}_1,\mathrm{Fil}_2)}\mathscr{H}om(V_1,V_2)=\sum_{\ell_1}(V_1/\mathrm{Fil}_1^{\ell_1}V_1)^\vee\otimes \mathrm{Fil}_2^{\ell_1+\ell-1}V_2. \tag{2.2}$$

Recall that $\mathrm{Hom}(V_1,V_2):=\Gamma(X,\mathscr{H}om(V_1,V_2))$. The following lemma sum up some properties about the filtration on Hom vector bundles

Lemma 2.2. *Let (V_1,Fil_1), (V_2,Fil_2) and (V_3,Fil_3) be three filtered vector bundles over a scheme X.*

(1). *For any $f\in\mathrm{Hom}(V_1,V_2)$. Then $f\in\mathrm{Fil}^\ell_{(\mathrm{Fil}_1,\mathrm{Fil}_2)}\mathrm{Hom}(V_1,V_2)$ if and only if $f(\mathrm{Fil}_1^{\ell'}V_1)\subset\mathrm{Fil}_2^{\ell'+\ell}V_2$ holds for all ℓ'.*

(2). *The composition operator induces a map $\circ:\mathrm{Hom}(V_2,V_3)\times\mathrm{Hom}(V_1,V_2)\to\mathrm{Hom}(V_1,V_3)$. Under this composition map, the image of $\mathrm{Fil}^{\ell_1}_{(\mathrm{Fil}_2,\mathrm{Fil}_3)}\mathrm{Hom}(V_2,V_3)\times\mathrm{Fil}^{\ell_2}_{(\mathrm{Fil}_1,\mathrm{Fil}_2)}\mathrm{Hom}(V_1,V_2)$ is contained in $\mathrm{Fil}^{\ell_1+\ell_2}_{(\mathrm{Fil}_1,\mathrm{Fil}_3)}\mathrm{Hom}(V_1,V_3)$.*

(3). *Assume $V_1=V_2=:V$ and $\mathrm{rank}(\mathrm{Fil}_1^\ell V)=\mathrm{rank}(\mathrm{Fil}_2^\ell V)$ for all ℓ. Then*

$$\mathrm{Fil}_1=\mathrm{Fil}_2 \text{ if and only if } \mathrm{id}_V\in\mathrm{Fil}^0_{(\mathrm{Fil}_1,\mathrm{Fil}_2)}\mathrm{End}(V).$$

Proof. The first statement can be easily checked locally by choosing adapted local basis of $V_1^\vee$ and V_2. The second statement follows the first one. For the third one, it follows the fact that if $V''\subseteq V'\subseteq V$ are sub bundles with $\mathrm{rank}V'=\mathrm{rank}V''$, then $V''=V'$. □

de Rham bundle. Let X be an S-scheme with relative flat normal crossing divisor D. A (logarithmic) connection on a vector bundle V over $(X,D)/S$ is an $\mathcal{O}_S$-linear map $\nabla\colon V\to V\otimes_{\mathcal{O}_X}\Omega^1_{X/S}(\log D)$ satisfying the Leibniz rule $\nabla(rv)=v\otimes \mathrm{d}r+r\nabla(v)$ for any local section $r\in\mathcal{O}_X$ and $v\in V$. Give a connection, there are canonical maps

$$\nabla\colon V\otimes\Omega^i_{X/S}(\log D)\to V\otimes\Omega^i_{X/S}(\log D)$$

given by $s\otimes\omega\mapsto\nabla(s)\wedge\omega+s\otimes \mathrm{d}\omega$. The curvature $\nabla\circ\nabla$ is $\mathcal{O}_X$-linear and contained in $\mathscr{E}nd(V)\otimes_{\mathcal{O}_X}\Omega^2_{X/S}(\log D)$. The connection is called *integrable* and (V,∇) is called a *(logarithmic) de Rham bundle* if the curvature vanishes. For a de Rham bundle, one has a natural *de Rham complex*:

$$\mathrm{DR}(V,\nabla):\qquad 0\to V\xrightarrow{\nabla}V\otimes\Omega^1_{X/S}(\log D)\xrightarrow{\nabla}V\otimes\Omega^2_{X/S}(\log D)\xrightarrow{\nabla}V\otimes\Omega^3_{X/S}(\log D)\to\cdots$$

The hypercohomology of $\mathrm{DR}(V,\nabla)$ are called the *de Rham cohomology* of the de Rham bundle (V,∇) and are denoted as follows

$$H^i_{dR}(X,(V,\nabla)):=\mathbb{H}^i(X,\mathrm{DR}(V,\nabla)).$$

Connection on Hom vector bundles. Let (V_1,∇_1) and (V_2,∇_2) be two vector bundles with a connections over X/S. Then there is a natural connection $\nabla_{(\nabla_1,\nabla_2)}$ on the vector bundle $\mathscr{H}om_{\mathcal{O}_X}(V_1,V_2)$ given by

$$\nabla_{(\nabla_1,\nabla_2)}(f):=\nabla_2\circ f-f\circ\nabla_1 \tag{2.3}$$

for any local section $f\in\mathscr{H}om(V_1,V_2)$. Let's note that

- for any $f\in\mathscr{H}om(V_1,V_2)$, $\nabla_{(\nabla_1,\nabla_2)}(f)=0$ if and only if $f\in\mathscr{H}om((V_1,\nabla_1),(V_2,\nabla_2))$, i.e. f is parallel;
- for any $\omega\in\mathscr{H}om(V_1,V_2)\otimes\Omega^i_{X/S}(\log D)$,

$$\nabla_{(\nabla_1,\nabla_2)}(\omega)=\nabla_2\circ\omega+(-1)^{i+1}\omega\circ\nabla_1.$$

Lemma 2.3. *Let (V_1,∇_1) and (V_2,∇_2) be two de Rham bundles over X/S. Then*

$$\mathscr{H}om((V_1,\nabla_1),(V_2,\nabla_2)):=\big(\mathscr{H}om(V_1,V_2),\nabla_{(\nabla_1,\nabla_2)}\big)$$

is also a de Rham bundle. In particular, $\nabla^{\mathrm{End}}_{\nabla_1}:=\nabla_{(\nabla_1,\nabla_1)}$ is an flat connection on $\mathscr{E}nd(V_1)$.

Proof. Note that $\left(\nabla_{(\nabla_1,\nabla_2)}\right)^2(f) = (\nabla_2^2)\circ f - f\circ(\nabla_1^2)$. The lemma follows. □

Filtered de Rham bundle. Let X be an S-scheme with relative flat normal crossing divisor D. Let (V,∇,Fil) be a triple such that (V,Fil) is a filtered vector bundle and (V,∇) be a de Rham bundle over $(X,D)/S$. Then the triple (V,∇,Fil) is called *filtered de Rham bundle* if ∇ and Fil satisfy the Griffith transversality. i.e.

$$\nabla(\mathrm{Fil}^\ell V)\subset \mathrm{Fil}^{\ell-1}V\otimes\Omega^1_{X/S}(\log D).$$

Thus we have sub complexes of $\mathrm{DR}(V,\nabla)$

$$\mathrm{Fil}^\ell\mathrm{DR}(V,\nabla):\qquad 0\to\mathrm{Fil}^\ell V\xrightarrow{\nabla}\mathrm{Fil}^{\ell-1}V\otimes\Omega^1_{X/S}(\log D)\xrightarrow{\nabla}\mathrm{Fil}^{\ell-2}V\otimes\Omega^2_{X/S}(\log D)\xrightarrow{\nabla}\cdots$$

These sub complexes define a filtration on $\mathrm{DR}(V,\nabla)$

$$\cdots\subset\mathrm{Fil}^{\ell+1}\mathrm{DR}(V,\nabla)\subset\mathrm{Fil}^{\ell}\mathrm{DR}(V,\nabla)\subset\mathrm{Fil}^{\ell-1}\mathrm{DR}(V,\nabla)\subset\cdots\subset\mathrm{DR}(V,\nabla)\tag{2.4}$$

Lemma 2.4. *Let $(V_1,\nabla_1,\mathrm{Fil}_1)$ and $(V_2,\nabla_2,\mathrm{Fil}_2)$ be two filtered de Rham bundles over $(X,D)/S$. Then*

$$\mathscr{H}om((V_1,\nabla_1,\mathrm{Fil}_1),(V_2,\nabla_2,\mathrm{Fil}_2)) := \left(\mathscr{H}om(V_1,V_2),\nabla_{(\nabla_1,\nabla_2)},\mathrm{Fil}_{(\mathrm{Fil}_1,\mathrm{Fil}_2)}\right)$$

is also a filtered de Rham bundle.

Proof. We only need to check that the connection and the filtration satisfy Griffith transversality. This follows ii) of Lemma 2.2. □

Remark. Let (V,∇,Fil) be a filtered de Rham bundle. By abuse of notation, we often write $\nabla_{(\nabla,\nabla)}$ as ∇^{End} and $\mathrm{Fil}_{(\mathrm{Fil},\mathrm{Fil})}$ as Fil for short.

Lemma 2.5. *Let (V,Fil) be a filtered vector bundle over X. Suppose ∇ and ∇' are two connections on V over $(X,D)/S$ with $\nabla-\nabla'\in\mathrm{Fil}^{-1}_{(\mathrm{Fil},\mathrm{Fil})}\mathrm{Hom}(V,V\otimes\Omega^1_{X/S}(\log D))$. Then ∇ satisfies Griffith's transversality if and only if ∇' does. In other words, (V,∇,Fil) is a filtered de Rham bundle if and only if (V,∇',Fil) is.*

Proof. We only need to show that if ∇ satisfies Griffith's transversality then ∇' also does. Let v be any local section in $\mathrm{Fil}^\ell V$. By the Griffith's transversality, $\nabla(v)\in\mathrm{Fil}^{\ell-1}V\otimes\Omega^1_{X/S}(\log D)$. By Lemma 2.2, $\nabla-\nabla'\in\mathrm{Fil}^{-1}_{(\mathrm{Fil},\mathrm{Fil})}\mathrm{Hom}(V,V\otimes\Omega^1_{X/S}(\log D))$ implies $\nabla(v)-\nabla'(v)\in\mathrm{Fil}^{\ell-1}V\otimes\Omega^1_{X/S}(\log D)$. Thus

$$\nabla'(v)=\nabla(v)-\Big(\nabla(v)-\nabla'(v)\Big)\in\mathrm{Fil}^{\ell-1}V\otimes\Omega^1_{X/S}(\log D).$$

By the arbitrary choice of the local section v, ∇' satisfies Griffith's transversality. □

2.2. Graded (logarithmic) Higgs bundles

Graded vector bundles. Let X be a scheme. Let E be a vector bundle on X. Let $\{\mathrm{Gr}^\ell E\}_{\ell\in\mathbb{Z}}$ be subbundles of V. The pair (E,Gr) is called *graded vector bundle* over X if the natural map $\oplus_{\ell\in\mathbb{Z}}\mathrm{Gr}^\ell E\cong E$ is an isomorphism.

Lemma 2.6. *For any graded vector bundle (E,Gr) on X there exists an open affine covering $\{U_i\}_{i\in I}$ of X such that $\mathrm{Gr}^\ell\mid_{U_i}$ is a finite free $\mathcal{O}_{U_i}$-module for each $i\in I$ and $\ell\in\mathbb{Z}$.*

Proof. There are only finitely many non-zero Gr^ℓ, so we may choose an open affine covering that trivializes them all. □

Let (E,Gr) be a graded vector bundle over a scheme X. There is a natural grading structure $\mathrm{Gr}^\vee$ on the dual vector bundle $E^\vee$ given by

$$\mathrm{Gr}^{\vee\ell}(E^\vee)=(\mathrm{Gr}^{-\ell}E)^\vee.\tag{2.5}$$

For two graded vector bundles (E_1, Gr_1) and (E_2, Gr_2) on X, there is a natural grading structure $\mathrm{Gr}_{(\mathrm{Gr}_1,\mathrm{Gr}_2)}$ on the Hom vector bundle $\mathscr{H}om(E_1, E_2) = E_1^\vee \otimes E_2$ given by

$$\mathrm{Gr}^{\ell}_{(\mathrm{Gr}_1,\mathrm{Gr}_2)}\mathscr{H}om(E_1,E_2) = \bigoplus_{\ell_1\in\mathbb{Z}} (\mathrm{Gr}^{\ell_1}V_1)^\vee \otimes_R \mathrm{Gr}^{\ell+\ell_1}V_2. \tag{2.6}$$

Higgs bundles. Let X be a smooth S-scheme with relative flat normal crossing divisor D. Let E be vector bundle over X. Let $\theta\colon E \to E\otimes_{\mathcal{O}_X} \Omega^1_{X/S}(\log D)$ be an $\mathcal{O}_X$-linear morphism. The pair (E,θ) is called *(logarithmic) Higgs bundle* over $(X,D)/S$ if θ is integrable. i.e. $\theta\wedge\theta = 0$. For a Higgs bundle, one has a natural *Higgs complex*

$$\mathrm{DR}(E,\theta): \qquad 0 \to E \xrightarrow{\theta} E\otimes\Omega^1_{X/S}(\log D) \xrightarrow{\theta} E\otimes\Omega^2_{X/S}(\log D) \xrightarrow{\theta} E\otimes\Omega^3_{X/S}(\log D) \to \cdots$$

The hypercohomology group of $\mathrm{DR}(E,\theta)$ is called *Higgs cohomology* of the Higgs bundle (E,θ), which is denoted by

$$H^i_{Higgs}(X,(E,\theta)) := \mathbb{H}^i(X, \mathrm{DR}(E,\theta)).$$

Graded Higgs bundles. Let X be an S-scheme with relative flat normal crossing divisor D. A *graded (logarithmic) Higgs bundle* over $(X,D)/S$ is a Higgs bundle (E,θ) together with a grading structure Gr on E satisfying

$$\theta(\mathrm{Gr}^{\ell}E) \subset \mathrm{Gr}^{\ell-1}E\otimes_{\mathcal{O}_X}\Omega^1_{X/S}(\log D).$$

Thus we have subcomplexes of $\mathrm{DR}(E,\theta)$

$$\mathrm{Gr}^{\ell}\mathrm{DR}(E,\theta): \qquad 0\to \mathrm{Gr}^{\ell}E \xrightarrow{\theta} \mathrm{Gr}^{\ell-1}E\otimes\Omega^1_{X/S}(\log D) \xrightarrow{\theta} \mathrm{Gr}^{\ell-2}E\otimes\Omega^2_{X/S}(\log D) \xrightarrow{\theta} \cdots$$

These subcomplexes define a graded structure on the complex $\mathrm{DR}(E,\theta)$

$$\mathrm{DR}(E,\theta) = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{Gr}^{\ell}\mathrm{DR}(E,\theta). \tag{2.7}$$

The following is the main example we will be concerned with.

Example 2.7. *Let (V,∇,Fil) be a filtered de Rham bundle over V. Denote $E = \bigoplus_{\ell\in\mathbb{Z}}\mathrm{Fil}^{\ell}V/\mathrm{Fil}^{\ell+1}V$ and $\mathrm{Gr}^{\ell}E = \mathrm{Fil}^{\ell}V/\mathrm{Fil}^{\ell+1}V$. By Griffith's transversality, the connection induces an $\mathcal{O}_X$-linear map $\theta\colon \mathrm{Gr}^{\ell}E \to \mathrm{Gr}^{\ell-1}E\otimes_{\mathcal{O}_X}\Omega^1_{X/S}(\log D)$ for each $\ell\in\mathbb{Z}$. Then (E,θ,Gr) is a graded Higgs bundle. Moreover we have*

$$\mathrm{Gr}^{\ell}\mathrm{DR}(E,\theta) = \mathrm{Fil}^{\ell}\mathrm{DR}(V,\nabla)/\mathrm{Fil}^{\ell+1}\mathrm{DR}(V,\nabla).$$

2.3. Cech resolution

Double complex. Recall that a double complex is an array M^{pq} together with morphisms

$$\begin{cases} \mathrm{d}\colon M^{pq}\to M^{p,q+1} \\ \mathrm{d}'\colon M^{pq}\to M^{p+1,q} \end{cases}$$

satisfying $\mathrm{d}\circ\mathrm{d}'+\mathrm{d}'\circ\mathrm{d}=0$.

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow{\scriptstyle \mathrm{d}'} & & \downarrow{\scriptstyle \mathrm{d}'} & & \\ \cdots & \xrightarrow{\mathrm{d}} & M^{p,q} & \xrightarrow{\mathrm{d}} & M^{p,q+1} & \xrightarrow{\mathrm{d}} & \cdots \\ & & \downarrow{\scriptstyle \mathrm{d}'} & & \downarrow{\scriptstyle \mathrm{d}'} & & \\ \cdots & \xrightarrow{\mathrm{d}} & M^{p+1,q} & \xrightarrow{\mathrm{d}} & M^{p+1,q+1} & \xrightarrow{\mathrm{d}} & \cdots \\ & & \downarrow{\scriptstyle \mathrm{d}'} & & \downarrow{\scriptstyle \mathrm{d}'} & & \\ & & \vdots & & \vdots & & \end{array} \tag{2.8}$$

The *total complex* is

$$\cdots \xrightarrow{\mathrm{d}^{\mathrm{Tot}}} \bigoplus_{p+q=n} M^{p,q} \xrightarrow{\mathrm{d}^{\mathrm{Tot}}} \bigoplus_{p+q=n+1} M^{p,q} \xrightarrow{\mathrm{d}^{\mathrm{Tot}}} \bigoplus_{p+q=n+2} M^{p,q} \xrightarrow{\mathrm{d}^{\mathrm{Tot}}} \cdots \tag{2.9}$$

where $\mathrm{d}^{\mathrm{Tot}}=\mathrm{d}+\mathrm{d}'$. i.e. $\mathrm{d}^{\mathrm{Tot}}((m^{p,q})_{p+q=n}) := (\mathrm{d}(m^{p,q-1})+\mathrm{d}'(m^{p-1,q}))_{p+q=n+1}$.
One example of a double complex comes from the *Cech complex* attached to a complex of sheaves and an open covering. Let X be a topological space. Let

$$(\mathcal{F}^\bullet,\nabla): \qquad 0\to\mathcal{F}^0 \xrightarrow{\nabla} \mathcal{F}^1 \xrightarrow{\nabla} \mathcal{F}^2 \xrightarrow{\nabla} \mathcal{F}^3 \xrightarrow{\nabla} \mathcal{F}^4 \xrightarrow{\nabla} \cdots$$

be a complex of sheaves of abelian groups on X. Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an open covering of X. For any $\iota=(i_1,\cdots,i_s)\in I^s$, denote $U_\iota=\cap_{\ell=1}^s U_{i_\ell}$ and

$$C^p(\mathcal{U},\mathcal{F}^q)=\prod_{\iota\in I^{p+1}}\mathcal{F}^q(U_\iota).$$

Then the *Cech resolution* of the complex $(\mathcal{F}^\bullet,\nabla)$ with respect to the open covering $\mathcal{U}$ of X is the total complex of the following double complex

$$\begin{array}{ccccccccc} C^0(\mathcal{U},\mathcal{F}^0) & \xrightarrow{\nabla} & C^0(\mathcal{U},\mathcal{F}^1) & \xrightarrow{\nabla} & C^0(\mathcal{U},\mathcal{F}^2) & \xrightarrow{\nabla} & C^0(\mathcal{U},\mathcal{F}^3) & \xrightarrow{\nabla} & \cdots \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \\ C^1(\mathcal{U},\mathcal{F}^0) & \xrightarrow{-\nabla} & C^1(\mathcal{U},\mathcal{F}^1) & \xrightarrow{-\nabla} & C^1(\mathcal{U},\mathcal{F}^2) & \xrightarrow{-\nabla} & C^1(\mathcal{U},\mathcal{F}^3) & \xrightarrow{-\nabla} & \cdots \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \\ C^2(\mathcal{U},\mathcal{F}^0) & \xrightarrow{\nabla} & C^2(\mathcal{U},\mathcal{F}^1) & \xrightarrow{\nabla} & C^2(\mathcal{U},\mathcal{F}^2) & \xrightarrow{\nabla} & C^2(\mathcal{U},\mathcal{F}^3) & \xrightarrow{\nabla} & \cdots \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \\ C^3(\mathcal{U},\mathcal{F}^0) & \xrightarrow{-\nabla} & C^3(\mathcal{U},\mathcal{F}^1) & \xrightarrow{-\nabla} & C^3(\mathcal{U},\mathcal{F}^2) & \xrightarrow{-\nabla} & C^3(\mathcal{U},\mathcal{F}^3) & \xrightarrow{-\nabla} & \cdots \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \\ \vdots & & \vdots & & \vdots & & \vdots & & \end{array} \tag{2.10}$$

where δ is defined by restrictions.

$$\delta((s_\iota)_{\iota\in I^m}) = \left(\sum_{t=0}^{p+1}(-1)^t s_{(i_0,\cdots,\hat{i}_t,\cdots,i_{p+1})}\mid_{U_{(i_0,i_1,\cdots,i_{p+1})}}\right)_{(i_0,i_1,\cdots,i_{p+1})\in I^{m+1}}$$

Lemma 2.8. (1). *An element $(-a,b,c)$ in $C^2(\mathcal{U},\mathcal{F}^0)\times C^1(\mathcal{U},\mathcal{F}^1)\times C^0(\mathcal{U},\mathcal{F}^2)$ is a 2-cocycle if and only if*

$$0=\delta(a), \nabla(a)=\delta(b), \nabla(b)=\delta(c) \text{ and } \nabla(c)=0;$$

(2). *an element (a,b) in $C^1(\mathcal{U},\mathcal{F}^0)\times C^0(\mathcal{U},\mathcal{F}^1)$ 1-cocycle if and only if*

$$0=\delta(a), \nabla(a)=\delta(b) \text{ and } \nabla(b)=0.$$

2.4. Complexes associated to a filtered de Rham bundle.

Let S, $\widetilde{S}$, $\mathfrak{a}$, $\widetilde{X}$, $\widetilde{D}$, X, D, $\mathcal{I}$ be as in Setup 1. Let $\widetilde{V}$ be a vector bundle on $\widetilde{X}$. We denote $V=\widetilde{V}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{O}_X$ and $V_{\mathcal{I}}=V\otimes_{\mathcal{O}_X}\mathcal{I}$.

Canonical isomorphism.

Lemma 2.9. *Let $\widetilde{V}$ be a vector bundle on $\widetilde{X}$. Then*

(1). *there is an canonical isomorphism of sheaves over the underlying space of X and $\widetilde{X}$*

$$V_{\mathcal{I}}\xrightarrow{\sim}\mathcal{I}\cdot\widetilde{V}\subset\widetilde{V}$$

which maps $v\otimes r$ to $r\widetilde{v}$, for any $v\in V$, $r\in\mathcal{I}$ and any lift $\widetilde{v}$ of $v\in V$ in $\widetilde{V}$.

(2). *if $\widetilde{v}\in\widetilde{V}$ is a local section, then $\widetilde{v}$ is a section of the image of the sheaf $V_{\mathcal{I}}$ if and only if $\widetilde{v}\equiv 0 \pmod{\mathcal{I}}$.*

Remark. The isomorphism in (1) of Lemma 2.9 is well defined, i.e., it does not depend on the choice of $\widetilde{v}$.

Natural connection on $\mathcal{I}$. Consider the restriction of $\mathrm{d}\colon\mathcal{O}_{\widetilde{X}}\to\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})$ to $\mathcal{I}$

$$\mathrm{d}\colon\mathcal{I}\to\mathcal{I}\cdot\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})\simeq\mathcal{I}\otimes\Omega^1_{X/S}(\log D).$$

One checks that the pair $(\mathcal{I},\mathrm{d})$ is a de Rham sheaf over $(X,D)/S$. Note that it is not a de Rham bundle as $\mathcal{I}$ is not a vector bundle on X.

de Rham bundle twisted by $(\mathcal{I},\mathrm{d})$**.** Let (V,∇) be a de Rham bundle over $(X,D)/S$. Taking tensor product, one gets a tensor de Rham sheaf

$$(V,\nabla)_{\mathcal{I}}:=(V,\nabla)\otimes(\mathcal{I},\mathrm{d})$$

over $(X,D)/S$ with connection $\nabla_{\mathcal{I}}\colon V_{\mathcal{I}}\to V_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)$ defined by $\nabla_{\mathcal{I}}(v\otimes r):=\nabla(v)\otimes r+v\otimes\mathrm{d}r$.

The complex $\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$**.** Let (V,∇) be a de Rham bundle over $(X,D)/S$. By Lemma 2.3, $\mathscr{E}nd(V,\nabla)$ is also a de Rham bundle over $(X,D)/S$. Consider

$$\mathscr{E}nd(V,\nabla)_{\mathcal{I}}:=\mathscr{E}nd(V,\nabla)\otimes(\mathcal{I},\mathrm{d}),$$

which is a de Rham sheaf over $(X,D)/S$. Then its de Rham complex $\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$ is

$$0\to\mathscr{E}nd(V)_{\mathcal{I}}\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^2_{X/S}(\log D)\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\cdots \tag{2.11}$$

where $\overline{\nabla}^{\mathrm{End}}=\left(\nabla_{(\nabla,\nabla)}\right)_{\mathcal{I}}$ in terms of our earlier notation in Equation 2.3.

Lemma 2.10. *Suppose there are three de Rham bundles $(\widetilde{V}_1,\widetilde{\nabla}_1)$, $(\widetilde{V}_2,\widetilde{\nabla}_2)$ and $(\widetilde{V}_3,\widetilde{\nabla}_3)$ over $(\widetilde{X},\widetilde{D})/\widetilde{S}$ which lift (V,∇), and there are morphisms $f_{12}\colon\widetilde{V}_1\to\widetilde{V}_2$ and $f_{23}\colon\widetilde{V}_2\to\widetilde{V}_3$ lifting the identity map on V. Then*

(1). *there are short exact sequence of de Rham sheaves over* $\widetilde{X}$

$$0 \to \mathscr{E}nd(V,\nabla)_{\mathcal{I}} \xrightarrow{\iota_{(\widetilde{V}_i,\widetilde{V}_j)}} \mathscr{H}om((\widetilde{V}_i,\widetilde{\nabla}_i),(\widetilde{V}_j,\widetilde{\nabla}_j)) \longrightarrow \mathscr{E}nd(V,\nabla) \to 0.$$

In particular, $\iota^{-1}_{(\widetilde{V}_i,\widetilde{V}_j)} \circ \nabla_{(\widetilde{\nabla}_i,\widetilde{\nabla}_j)} = \overline{\nabla}^{\mathrm{End}} \circ \iota^{-1}_{(\widetilde{V}_i,\widetilde{V}_j)}$ *on* $\mathcal{I}\cdot\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_i) = \iota_{(\widetilde{V}_i,\widetilde{V}_j)}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$.

(2). *One has* $\iota_{(\widetilde{V}_1,\widetilde{V}_3)} \circ \iota^{-1}_{(\widetilde{V}_2,\widetilde{V}_3)} = (-\circ f_{12})$ *on* $\mathcal{I}\cdot\mathscr{H}om(\widetilde{V}_1,\widetilde{V}_2)$ *and* $\iota_{(\widetilde{V}_1,\widetilde{V}_3)} \circ \iota^{-1}_{(\widetilde{V}_1,\widetilde{V}_2)} = (f_{23}\circ -)$ *on* $\mathcal{I}\cdot\mathscr{H}om(\widetilde{V}_1,\widetilde{V}_2)$*. i.e. the following diagram communicates*

$$\begin{array}{ccc}
\mathscr{E}nd(V)_{\mathcal{I}} & \xrightarrow{\iota_{(\widetilde{V}_1,\widetilde{V}_2)}} & \mathscr{H}om(\widetilde{V}_1,\widetilde{V}_2) \\
\| & & \downarrow{\scriptstyle f_{23}\circ -} \\
\mathscr{E}nd(V)_{\mathcal{I}} & \xrightarrow{\iota_{(\widetilde{V}_1,\widetilde{V}_3)}} & \mathscr{H}om(\widetilde{V}_1,\widetilde{V}_3) \\
\| & & \uparrow{\scriptstyle -\circ f_{12}} \\
\mathscr{E}nd(V)_{\mathcal{I}} & \xrightarrow{\iota_{(\widetilde{V}_2,\widetilde{V}_3)}} & \mathscr{H}om(\widetilde{V}_2,\widetilde{V}_3)
\end{array}$$

Proof. Lemma CanoIsom implies (1) and the map $\iota_{(\widetilde{V}_i,\widetilde{V}_j)}$ satisfies

$$\iota_{(\widetilde{V}_i,\widetilde{V}_j)}(g\otimes r) = r\cdot\widetilde{g}$$

where g is a local section of $\mathscr{E}nd(V)$, r is a local section of $\mathcal{I}$ and $\widetilde{g}$ is a local section of $\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_j)$ which lifts g.
Let $\widetilde{g}_{23}$ be a lifting of g in $\mathscr{H}om(\widetilde{V}_2,\widetilde{V}_3)$. Then $\widetilde{g}_{23}\circ f_{12}$ is an element in $\mathscr{H}om(\widetilde{V}_2,\widetilde{V}_3)$ which also lifts f. By the construction of $\iota_{(\widetilde{V}_i,\widetilde{V}_j)}$, one has

$$\iota_{(\widetilde{V}_1,\widetilde{V}_3)}(g\otimes r) = r\cdot(\widetilde{g}_{23}\circ f_{12}) = \iota_{(\widetilde{V}_2,\widetilde{V}_3)}(g\otimes r)\circ f_{12}.$$

Similarly one has

$$\iota_{(\widetilde{V}_1,\widetilde{V}_3)}(g\otimes r) = r\cdot(f_{23}\circ\widetilde{g}_{12}) = f_{23}\circ\iota_{(\widetilde{V}_1,\widetilde{V}_2)}(g\otimes r).$$

Thus (2) follows. □

Filtered de Rham bundle twisted by $(\mathcal{I},\mathrm{d},\mathrm{Fil}_{\mathrm{tri}})$**.** Suppose (V,∇,Fil) is a filtered de Rham bundle. Taking the tensor product with $(\mathcal{I},\mathrm{d},\mathrm{Fil}_{\mathrm{tri}})$, one obtains a filtered de Rham sheaf on $(X,D)/S$.

$$(V,\nabla,\mathrm{Fil})_{\mathcal{I}} := (V,\nabla,\mathrm{Fil})\otimes(\mathcal{I},\mathrm{d},\mathrm{Fil}_{\mathrm{tri}})$$

where the underlying de Rham bundle is $(V,\nabla)_{\mathcal{I}}$ and the filtration $\mathrm{Fil}_{\mathcal{I}}$ is defined by $\mathrm{Fil}^{\ell}_{\mathcal{I}}(V_{\mathcal{I}}) := (\mathrm{Fil}^{\ell}V)_{\mathcal{I}}$. For simplifying the notations, we will write $\mathrm{Fil}^{\ell}_{\mathcal{I}}(V_{\mathcal{I}})$ as $\mathrm{Fil}^{\ell}V_{\mathcal{I}}$.
The filtration on the complex $\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$**.** For a filtered de Rham bundle (V,∇,Fil) over $(X,D)/S$, The filtration on

$$\mathscr{E}nd(V,\nabla,\mathrm{Fil})_{\mathcal{I}} := \mathscr{E}nd(V,\nabla,\mathrm{Fil})\otimes(\mathcal{I},\mathrm{d},\mathrm{Fil}_{\mathrm{tri}}).$$

induces a filtration on $\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$

$$\cdots\subset\mathrm{Fil}^{\ell+1}\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\subset\mathrm{Fil}^{\ell}\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\subset\mathrm{Fil}^{\ell-1}\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\subset\cdots\subset\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$$

where $\mathrm{Fil}^{\ell}\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$ is defined as

$$0\to\mathrm{Fil}^{\ell}\mathscr{E}nd(V)_{\mathcal{I}}\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathrm{Fil}^{\ell-1}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathrm{Fil}^{\ell-2}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^2_{X/S}(\log D)\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\cdots \tag{2.12}$$

Lemma 2.11. *Suppose* $(\widetilde{V}_1, \widetilde{\nabla}_1, \widetilde{\mathrm{Fil}}_1)$ *and* $(\widetilde{V}_2, \widetilde{\nabla}_2, \widetilde{\mathrm{Fil}}_2)$ *are de Rham bundles over* $(\widetilde{X}, \widetilde{D})/\widetilde{S}$ *which lift* $(V, \nabla, \mathrm{Fil})$. *For each* $\ell \in \mathbb{Z}$, *one has an exact sequence over* $\widetilde{X}$

$$0 \to \mathrm{Fil}^\ell \mathscr{E}nd(V)_{\mathcal{I}} \xrightarrow{\iota_{(\widetilde{V}_1,\widetilde{V}_2)}} \mathrm{Fil}^\ell_{(\widetilde{\mathrm{Fil}}_1,\widetilde{\mathrm{Fil}}_2)} \mathscr{H}om(\widetilde{V}_1, \widetilde{V}_2) \longrightarrow \mathrm{Fil}^\ell \mathscr{E}nd(V) \to 0.$$

Proof. This follows Lemma 2.9. □

The complex $\mathscr{C}$***.*** We define a complex $\mathscr{C}$ which will govern the deformation theory. Let $(V, \nabla, \mathrm{Fil})$ be a filtered de Rham bundle over $(X, D)/S$. Denote by $\mathscr{C}$ the cokernel of the natural injection $\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}}) \hookrightarrow \mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})$. i.e. $\mathscr{C} := \frac{\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})}{\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})}$

(2.13)
$$0 \to \frac{\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})}{\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})} \xrightarrow{\nabla^{\mathrm{End}}} \frac{\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})}{\mathrm{Fil}^{-1}\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})} \otimes \Omega^1_{X/S}(\log D) \xrightarrow{\nabla^{\mathrm{End}}} \frac{\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})}{\mathrm{Fil}^{-2}\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})} \otimes \Omega^2_{X/S}(\log D) \xrightarrow{\nabla^{\mathrm{End}}} \cdots$$

2.5. Complexes associated to a graded Higgs bundle.

Let S, $\widetilde{S}$, $\mathfrak{a}$, $\widetilde{X}$, $\widetilde{D}$, X, D, $\mathcal{I}$ be as in Setup 1. One has a natural Higgs sheaf $(\mathcal{I}, 0)$, i.e., the sheaf $\mathcal{I}$ together with the trivial Higgs field.

The complex $\mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}})$***.*** Let (E, θ) be a Higgs bundle over $(X, D)/S$. Then $\mathscr{E}nd(E, \theta) := (\mathscr{E}nd(E), \theta^{\mathrm{End}})$ is also Higgs bundle over $(X, D)/S$, where $\theta^{\mathrm{End}}(f) = \theta \circ f - f \circ \theta$. Consider the Higgs sheaf

$$\mathscr{E}nd(E, \theta)_{\mathcal{I}} := \mathscr{E}nd(E, \theta) \otimes (\mathcal{I}, 0),$$

over $(X, D)/S$, where $\overline{\theta}^{\mathrm{End}}(f \otimes r) = \theta^{\mathrm{End}}(f) \otimes r$. Then its Higgs complex $\mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}})$ is

$$0 \xrightarrow{\overline{\theta}^{\mathrm{End}}} \mathscr{E}nd(E)_{\mathcal{I}} \otimes \Omega^1_{X/S}(\log D) \xrightarrow{\overline{\theta}^{\mathrm{End}}} \mathscr{E}nd(E)_{\mathcal{I}} \otimes \Omega^2_{X/S}(\log D) \xrightarrow{\overline{\theta}^{\mathrm{End}}} \cdots \tag{2.14}$$

The grading structure on the complex $\mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}})$***.*** Suppose (E, θ, Gr) is a graded Higgs bundle. Then Gr induces a natural graded structure on $\mathscr{E}nd(E, \theta)_{\mathcal{I}}$ given by

$$\mathrm{Gr}^\ell(\mathscr{E}nd(E)_{\mathcal{I}}) := (\mathrm{Gr}^\ell \mathscr{E}nd(E))_{\mathcal{I}}$$

This grading structure induces a grading structure on the complex $\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})$

$$\mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}}) = \bigoplus_{l \in \mathbb{Z}} \mathrm{Gr}^\ell \mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}}),$$

where $\mathrm{Gr}^\ell\mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}})$ is defined as

$$0 \to \mathrm{Gr}^\ell \mathscr{E}nd(E)_{\mathcal{I}} \xrightarrow{\overline{\theta}^{\mathrm{End}}} \mathrm{Gr}^{\ell-1} \mathscr{E}nd(E)_{\mathcal{I}} \otimes \Omega^1_{X/S}(\log D) \xrightarrow{\overline{\theta}^{\mathrm{End}}} \mathrm{Gr}^{\ell-2} \mathscr{E}nd(E)_{\mathcal{I}} \otimes \Omega^2_{X/S}(\log D) \xrightarrow{\overline{\theta}^{\mathrm{End}}} \cdots \tag{2.15}$$

Lemma 2.12. *Let* $(V, \nabla, \mathrm{Fil})$ *be a filtered de Rham bundle over* V. *Let* (E, θ, Gr) *be the graded Higgs bundle defined in Example 2.7. Then*

$$\mathrm{Gr}^\ell \mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}}) = \mathrm{Fil}^\ell \mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}})/\mathrm{Fil}^{\ell+1}\mathrm{DR}(\mathscr{E}nd(V, \nabla)_{\mathcal{I}}).$$

In particular, the complex $\mathscr{C}$ *in 2.13 is an successive extension of* $\mathrm{Gr}^{-1}\mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}}), \mathrm{Gr}^{-2}\mathrm{DR}(\mathscr{E}nd(E, \theta)_{\mathcal{I}}), \cdots$.

2.6. Local Liftings

In this subsection, we show that a filtered vector bundle with a Griffith's tranverse filtration locally lifts. We will later use this fact to construct a tangent-obstruction theory for filtered de Rham bundles. Let S, $\widetilde{S}$, $\mathfrak{a}$, $\widetilde{X}$, $\widetilde{D}$, X, D, $\mathcal{I}$ be given as in Setup 1.

Lemma 2.13. *Let* $(V, \nabla, \mathrm{Fil})$ *be a filtered vector bundle over* X *with a connection satisfying Griffith's transversality. Suppose that* $\mathrm{Fil}^\ell V$ *is free over* $\mathcal{O}_X$ *for all* $\ell \in \mathbb{Z}$. *Then*

(1). *there exists lifting* $\widetilde{V}$ *over* $\widetilde{X}$ *of* V*;*
(2). *for any given lifting* $\widetilde{V}$ *over* $\widetilde{X}$ *of* V*, there exists a filtration* $\widetilde{\mathrm{Fil}}$ *on* $\widetilde{V}$ *which lifts* Fil*;*
(3). *for any given lifting* $(\widetilde{V},\widetilde{\mathrm{Fil}})$ *over* $\widetilde{X}$ *of the filtered vector bundle* (V,Fil)*, there exists a (not necessarily integrable) connection* $\widetilde{\nabla}$ *on* $\widetilde{V}$ *over* $(\widetilde{X},\widetilde{D})/\widetilde{S}$ *which lifts* ∇ *and satisfies Griffith's transversality with respect to* $\widetilde{\mathrm{Fil}}$.
(4). *Let* $(\widetilde{V},\widetilde{\mathrm{Fil}})$ *and* $(\widetilde{V}',\widetilde{\mathrm{Fil}}')$ *be two liftings of* (V,Fil) *over* $\widetilde{X}$*. Then there exists an isomorphism* $f\colon(\widetilde{V},\widetilde{\mathrm{Fil}})\to(\widetilde{V}',\widetilde{\mathrm{Fil}}')$ *which lifts the identity map* id_V*.*

Proof. Part (1) follows the freeness of V. For the part (2), by the freeness of $\mathrm{Fil}^\ell V$, there exists an adapted basis $\{e_1,\cdots,e_r\}$ of (V,Fil). i.e.

$$V=\bigoplus_{j=1}^r\mathcal{O}_X\cdot e_i\quad\text{and}\quad \mathrm{Fil}^\ell V=\bigoplus_{j:e_j\in\mathrm{Fil}^\ell V}\mathcal{O}_X\cdot e_j.$$

Fix a lifting $\widetilde{e}_j$ of e_j in $\widetilde{V}$ for any $j=1,\cdots,r$. Then the filtration on $\widetilde{V}$ defined by

$$\widetilde{\mathrm{Fil}}^\ell\widetilde{V}=\bigoplus_{j:e_j\in\mathrm{Fil}^\ell V}\mathcal{O}_{\widetilde{X}}\cdot\widetilde{e}_j$$

is a lifting of Fil on $\widetilde{V}$.
For part (3), choose an adopted basis $\{\widetilde{e}_1,\cdots,\widetilde{e}_r\}$ of $\widetilde{V}$ and consider the connection $\widetilde{\mathrm{d}}$ on $\widetilde{V}$ defined by $\widetilde{\mathrm{d}}(\widetilde{e}_j)=0$. It obviously satisfies Griffith's transversality. Denote $\mathrm{d}=\widetilde{\mathrm{d}}\mid_V$. Since ∇ also satisfies Griffith's transversality and $\nabla-\mathrm{d}$ is $\mathcal{O}_X$-linear, $\nabla-\mathrm{d}\in\mathrm{Fil}^{-1}\mathrm{Hom}(V,V\otimes\Omega^1_{X/S}(\log D))$ by Lemma 2.2. By Lemma 2.11, the map

$$\mathrm{Fil}^{-1}\mathrm{Hom}(\widetilde{V},\widetilde{V}\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D}))\twoheadrightarrow\mathrm{Fil}^{-1}\mathrm{Hom}(V,V\otimes\Omega^1_{X/S}(\log D))$$

is surjective. Hence there exists an element $\widetilde{\omega}$ in $\mathrm{Fil}^{-1}\mathrm{Hom}(\widetilde{V},\widetilde{V}\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D}))$ which lifts $\nabla-\mathrm{d}$. Consider the connection $\widetilde{\nabla}:=\widetilde{\mathrm{d}}+\widetilde{\omega}$ on $\widetilde{V}$, which is a lifting of ∇. By Lemma 2.5, $\widetilde{\nabla}$ satisfies Griffith's transversality.
We construct an f in the part (4) as following. Since $\mathrm{Fil}^\ell V$ is free, $\widetilde{\mathrm{Fil}}^\ell\widetilde{V}$ and $\widetilde{\mathrm{Fil}}'^\ell\widetilde{V}'$ are also free with the same rank. We may lift $\{e_1,\cdots,e_r\}$ to an adapted basis $\{\widetilde{e}_1,\cdots,\widetilde{e}_r\}$ of $(\widetilde{V},\widetilde{\mathrm{Fil}})$ and to an adopted basis $\{\widetilde{e}'_1,\cdots,\widetilde{e}'_r\}$ of $(\widetilde{V}',\widetilde{\mathrm{Fil}}')$. The isomorphism $f\colon\widetilde{V}\to\widetilde{V}'$ sending $\widetilde{e}_i$ to $\widetilde{e}'_i$ preserves the filtrations, which is what we need. □

3. Deformations of filtered de Rham bundles and Hodge filtrations

In this section we will study the deformation theory of filtered de Rham bundles and Hodge filtrations. [1]
The main results are Theorem 3.1 and Theorem 3.9.
Let S, $\widetilde{S}$, $\mathfrak{a}$, $\widetilde{X}$, $\widetilde{D}$, X, D, $\mathcal{I}$ be given as in Setup 1.

3.1. Deforming a filtered de Rham bundle

In this subsection, we study the deformation theory of a filtered de Rham bundle. For a filtered de Rham bundle (V,∇,Fil) over $(X,D)/S$, recall the complex of sheaves $\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$ defined in 2.12:

$$0\to\mathrm{Fil}^0\mathscr{E}nd(V)_{\mathcal{I}}\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathrm{Fil}^{-1}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathrm{Fil}^{-2}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\cdots$$

This complex yields a tangent-obstruction theory for a filtered de Rham bundle.

[1] Note that every de Rham bundle admits a natural filtration: the *trivial filtration*. Therefore the deformation theory of filtered de Rham bundles generalizes the deformation theory of de Rham bundles.

Theorem 3.1. *Let (V,∇,Fil) be a (logarithmic) filtered de Rham bundle over $(X,D)/S$. Then*

(1). *the obstruction to lifting (V,∇,Fil) to a filtered de Rham bundle over $(\widetilde{X},\widetilde{D})/\widetilde{S}$ lies in $\mathbb{H}^2\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$;*

(2). *if (V,∇,Fil) has lifting $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$, then the lifting set is an $\mathbb{H}^1\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$-torsor;*

(3). *the infinitesimal automorphism group of $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ over (V,∇,Fil) is $\mathbb{H}^0\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$.*

In particular, when Fil is the trivial filtration, Theorem 3.1 implies the classical result about deforming integrable connections.

Proposition 3.2. *Let (V,∇) be a (logarithmic) de Rham bundle over $(X,D)/S$. Then*

1). *the obstruction to lifting (V,∇) to a de Rham bundle over $(\widetilde{X},\widetilde{D})/\widetilde{S}$ lies in $\mathbb{H}^2\big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$;*

2). *if (V,∇) has lifting $(\widetilde{V},\widetilde{\nabla})$, then the lifting set is an $\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$-torsor;*

3). *the infinitesimal automorphism group of $(\widetilde{V},\widetilde{\nabla})$ over (V,∇) is $\mathbb{H}^0\big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$.*

The following easily follows from the earlier sections and is the key we need to define an obstruction class.

Lemma 3.3. *Let (V,∇,Fil) be a filtered vector bundle together with an connection over $(X,D)/S$ satisfying Griffith transversality.*

(1). *there exists an open affine covering $\{U_i\}_{i\in I}$ of X such that $\mathrm{Fil}^\ell V(U_i)$ is free over $\mathcal{O}_X(U_i)$ for each $i\in I$ and each $\ell\in\mathbb{Z}$;*

(2). *the restriction $(V,\mathrm{Fil})\mid_{U_i}$ extends to a filtered vector bundle $(\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i)$ over $\widetilde{U}_i$ with $\widetilde{\mathrm{Fil}}_i^\ell\widetilde{V}_i(\widetilde{U}_i)$ free over $\mathcal{O}_{\widetilde{X}}(\widetilde{U}_i)$ for each $i\in I$ and each $\ell\in\mathbb{Z}$;*

(3). *for any two $i,j\in I$, there exists a local isomorphism $f_{ij}\colon(\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i)\mid_{\widetilde{U}_{ij}}\to(\widetilde{V}_j,\widetilde{\mathrm{Fil}}_j)\mid_{\widetilde{U}_{ij}}$ lifts the identity $\mathrm{id}_{V|_{U_{ij}}}$;*

(4). *the $\nabla\mid_{U_i}$ extends to a connection $\widetilde{\nabla}_i$ on $\widetilde{V}_i$ which satisfies Griffith transversality with respect to the filtration $\widetilde{\mathrm{Fil}}_i$.*

We call a quadruple $\big(\{U_i\}_i,\{\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i\}_i,\{f_{ij}\}_{ij},\{\widetilde{\nabla}_i\}_i\big)$ a system of local data *for (V,∇,Fil).*

Proof of Lemma 3.3. The lemma follows Lemma 2.1 and Lemma 2.13. □

Remark. In (4) of Lemma 3.3, we do not require the local extension $\widetilde{\nabla}\mid_{\widetilde{U}_i}$ to be integrable.

Let (V,∇,Fil) be a filtered de Rham bundle and $\big(\{U_i\}_i,\{\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i\}_i,\{f_{ij}\}_{ij},\{\widetilde{\nabla}_i\}_i\big)$ be the local data outputted by Lemma 3.3. Since f_{ij} preserves local filtration and $\widetilde{\nabla}_i$ satisfies Griffith's transversality, by Lemma 2.2 the element

$$\widetilde{c}(V,\nabla,\mathrm{Fil}):=\Big(-(f_{jk}\circ f_{ij}-f_{ik})_{(i,j,k)},(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)_{(i,j)},(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)_i\Big)\tag{3.1}$$

is contained in

$$\prod_{(i,j,k)\in I^3}\Big(\widetilde{\mathrm{Fil}}^0_{ik}\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_k)\Big)(\widetilde{U}_{ijk})\times\prod_{(i,j)\in I^2}\Big(\widetilde{\mathrm{Fil}}^{-1}_{ij}\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_j)\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})\Big)(\widetilde{U}_{ij})\times\prod_{i\in I}\Big(\widetilde{\mathrm{Fil}}^{-2}_{ii}\mathscr{E}nd(\widetilde{V}_i)\otimes\Omega^2_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})\Big)(\widetilde{U}$$

where $\widetilde{\mathrm{Fil}}_{ij}=\mathrm{Fil}_{(\widetilde{\mathrm{Fil}}_i,\widetilde{\mathrm{Fil}}_j)}$ is the filtration on the vector bundle $\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_j)$ over $\widetilde{U}_{ij}$. Since (V,∇,Fil) is a globally defined filtered de Rham bundle, $c(V,\nabla,\mathrm{Fil})=0\mod\mathcal{I}$. One gets an element

$$c(V,\nabla,\mathrm{Fil}):=\Big(-\big(\iota_{ik}^{-1}(f_{jk}\circ f_{ij}-f_{ik})\big)_{(i,j,k)},\big(\iota_{ij}^{-1}((\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\big)_{(i,j)},\big(\iota_{ii}^{-1}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)\big)_i\Big)$$

in

$$\prod_{(i,j,k)\in I^3}\mathrm{Fil}^0\mathscr{E}nd(V)_{\mathcal{I}}(U_{ijk})\times\prod_{(i,j)\in I^2}\Big(\mathrm{Fil}^{-1}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\Big)(U_{ij})\times\prod_{i\in I}\Big(\mathrm{Fil}^{-2}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^2_{X/S}(\log D)\Big)(U_i).$$

where the notations $\iota_{ij}:=\iota_{\widetilde{V}_i,\widetilde{V}_j}$ are defined in Lemma 2.10.

Lemma 3.4. *Let (V,∇,Fil) be a filtered de Rham bundle and $\big(\{U_i\}_i,\{(\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i)\}_i,\{f_{ij}\}_{ij},\{\widetilde{\nabla}_i\}_i\big)$ be a system of local data, whose existence is guaranteed by Lemma 3.3.*

(1). *The element $c(V,\nabla,\mathrm{Fil})$ defined in (3.1) is a 2-cocycle in the Cech resolution of the complex $\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$ with respect to the open covering $\mathcal{U}=\{U_i\}_{i\in I}$ of X.*

(2). *The class $[c(V,\nabla,\mathrm{Fil})]\in\mathbb{H}^2(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}))$ does not depend on the choice of the system of local data from Lemma 3.3.*

Proof. We denote $\mathcal{F}^k=\mathrm{Fil}^{-k}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^k_{X/S}(\log D)$ for all $k\geq 0$. Then the complex $\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$ defined in 2.12 may be rewritten as

$$0\to\mathcal{F}^0\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathcal{F}^1\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathcal{F}^2\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\mathcal{F}^3\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\cdots$$

For any $\iota=(i_1,\cdots,i_s)\in I^s$, set $U_\iota=\cap_{\ell=1}^s U_{i_\ell}$ and

$$C^p(\mathcal{U},\mathcal{F}^k)=\prod_{\iota\in I^{p+1}}\mathcal{F}^k(U_\iota).$$

Recall that the Cech resolution of the complex $\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})$ with respect to the open covering $\mathcal{U}$ of X is the total complex of the following double complex

$$\begin{array}{ccccccccc}
C^0(\mathcal{U},\mathcal{F}^0) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^0(\mathcal{U},\mathcal{F}^1) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^0(\mathcal{U},\mathcal{F}^2) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^0(\mathcal{U},\mathcal{F}^3) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & \cdots\\
\downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
C^1(\mathcal{U},\mathcal{F}^0) & \xrightarrow{-\overline{\nabla}^{\mathrm{End}}} & C^1(\mathcal{U},\mathcal{F}^1) & \xrightarrow{-\overline{\nabla}^{\mathrm{End}}} & C^1(\mathcal{U},\mathcal{F}^2) & \xrightarrow{-\overline{\nabla}^{\mathrm{End}}} & C^1(\mathcal{U},\mathcal{F}^3) & \xrightarrow{-\overline{\nabla}^{\mathrm{End}}} & \cdots\\
\downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
C^2(\mathcal{U},\mathcal{F}^0) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^2(\mathcal{U},\mathcal{F}^1) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^2(\mathcal{U},\mathcal{F}^2) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^2(\mathcal{U},\mathcal{F}^3) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & \cdots\\
\downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
C^3(\mathcal{U},\mathcal{F}^0) & \xrightarrow{-\overline{\nabla}^{\mathrm{End}}} & C^3(\mathcal{U},\mathcal{F}^1) & \xrightarrow{-\overline{\nabla}^{\mathrm{End}}} & C^3(\mathcal{U},\mathcal{F}^2) & \xrightarrow{-\overline{\nabla}^{\mathrm{End}}} & C^3(\mathcal{U},\mathcal{F}^3) & \xrightarrow{-\overline{\nabla}^{\mathrm{End}}} & \cdots\\
\downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}\tag{3.2}$$

By lemma 2.8, to show that

$$c(V,\nabla,\mathrm{Fil})=\Big(-\big(\iota_{ik}^{-1}(f_{jk}\circ f_{ij}-f_{ik})\big)_{(i,j,k)},\big(\iota_{ij}^{-1}((\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\big)_{(i,j)},\big(\iota_{ii}^{-1}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)\big)_i\Big)$$

forms a cocycle, one needs to check

$$\begin{cases}
0 & = \delta\Big(\big(\iota_{ik}^{-1}(f_{jk}\circ f_{ij}-f_{ik})\big)_{(i,j,k)}\Big) & \in C^3(\mathcal{U},\mathcal{F}^0)\\
\overline{\nabla}^{\mathrm{End}}\Big(\big(\iota_{ik}^{-1}(f_{jk}\circ f_{ij}-f_{ik})\big)_{(i,j,k)}\Big) & = \delta\Big(\big(\iota_{ij}^{-1}((\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\big)_{(i,j)}\Big) & \in C^2(\mathcal{U},\mathcal{F}^1)\\
\overline{\nabla}^{\mathrm{End}}\Big(\big(\iota_{ij}^{-1}((\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\big)_{(i,j)}\Big) & = \delta\Big(\big(\iota_{ii}^{-1}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)\big)_i\Big) & \in C^1(\mathcal{U},\mathcal{F}^2)\\
\overline{\nabla}^{\mathrm{End}}\Big(\big(\iota_{ii}^{-1}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)\big)_i\Big) & = 0 & \in C^0(\mathcal{U},\mathcal{F}^3)
\end{cases}$$

This is equivalent to check

1a). $\iota_{jl}^{-1}(f_{kl}\circ f_{jk}-f_{jl})-\iota_{il}^{-1}(f_{kl}\circ f_{ik}-f_{il})+\iota_{il}^{-1}(f_{jl}\circ f_{ij}-f_{il})-\iota_{jk}^{-1}(f_{jk}\circ f_{ij}-f_{ik})=0$ for each $(i,j,k,l)\in I^4$.

1b). $\iota_{ik}^{-1}(\widetilde{\nabla}_{ik}(f_{jk}\circ f_{ij}-f_{ik}))=\iota_{jk}^{-1}(\widetilde{\nabla}_k\circ f_{jk}-f_{jk}\circ\widetilde{\nabla}_j)-\iota_{ik}^{-1}(\widetilde{\nabla}_k\circ f_{ik}-f_{ik}\circ\widetilde{\nabla}_i)+\iota_{ij}^{-1}(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)$ for each $(i,j,k)\in I^3$.

1c). $\iota_{ij}^{-1}(\widetilde{\nabla}_{ij}(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i))=\iota_{jj}^{-1}(\widetilde{\nabla}_j\circ\widetilde{\nabla}_j)-\iota_{ii}^{-1}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)$ for each $(i,j)\in I^2$; and

1d). $\widetilde{\nabla}_{ii}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)=0$.

where $\widetilde{\nabla}_{ij}:=\nabla_{(\widetilde{\nabla}_i,\widetilde{\nabla}_j)}$ be the connection on $\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_j)\mid_{\widetilde{U}_{ij}}$.
1a) follows Lemma 2.10 and

$$(f_{kl}\circ f_{jk}-f_{jl})\circ f_{ij}-(f_{kl}\circ f_{ik}-f_{il})+(f_{jl}\circ f_{ij}-f_{il})-f_{kl}\circ(f_{jk}\circ f_{ij}-f_{ik})=0$$

This is because that Lemma 2.10 implies $-\circ f_{ij}=\iota_{ik}\circ\iota_{ij}^{-1}$ and $f_{jk}\circ -=\iota_{ik}\circ\iota_{jk}^{-1}$. 1a) follows the injectivity of ι_{il} and

$$\begin{aligned}0&=(f_{kl}\circ f_{jk}-f_{jl})\circ f_{ij}-(f_{kl}\circ f_{ik}-f_{il})+(f_{jl}\circ f_{ij}-f_{il})-f_{kl}\circ(f_{jk}\circ f_{ij}-f_{ik})\\ &\overset{2.10}{=}\iota_{il}\circ\iota_{jl}^{-1}(f_{kl}\circ f_{jk}-f_{jl})-\iota_{il}\circ\iota_{il}^{-1}(f_{kl}\circ f_{ik}-f_{il})+\iota_{il}\circ\iota_{il}^{-1}(f_{jl}\circ f_{ij}-f_{il})-\iota_{il}\circ\iota_{ik}^{-1}(f_{jk}\circ f_{ij}-f_{ik})\\ &=\iota_{il}\circ\Big(\iota_{jl}^{-1}(f_{kl}\circ f_{jk}-f_{jl})-\iota_{il}^{-1}(f_{kl}\circ f_{ik}-f_{il})+\iota_{il}^{-1}(f_{jl}\circ f_{ij}-f_{il})-\iota_{ik}^{-1}(f_{jk}\circ f_{ij}-f_{ik})\Big)\end{aligned}$$

1b) follows Lemma 2.10 and

$$\begin{aligned}\widetilde{\nabla}_{ik}(f_{jk}\circ f_{ij}-f_{ik})&:=\widetilde{\nabla}_k\circ(f_{jk}\circ f_{ij}-f_{ik})-(f_{jk}\circ f_{ij}-f_{ik})\circ\widetilde{\nabla}_i\\ &=(\widetilde{\nabla}_k\circ f_{jk}-f_{jk}\circ\widetilde{\nabla}_j)\circ f_{ij}-(\widetilde{\nabla}_k\circ f_{ik}-f_{ik}\circ\widetilde{\nabla}_i)+f_{jk}\circ(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\end{aligned}$$

1c) follows Lemma 2.10 and

$$\begin{aligned}\widetilde{\nabla}_{ij}(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)&:=\widetilde{\nabla}_j\circ(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)-(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\circ\widetilde{\nabla}_i\\ &=\widetilde{\nabla}_j^2\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i^2\end{aligned}$$

1d) follow that $\widetilde{\nabla}_{ii}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)=\widetilde{\nabla}_i\circ(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)-(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)\circ\widetilde{\nabla}_i=0$.
Let $\big(\{U_i\}_{i\in J},\{(\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i)\}_{i\in J},\{f_{ij}\}_{i,j\in J},\{\widetilde{\nabla}_i\}_{i\in J}\big)$ be another system of local data. Consider the union covering $\{U_i\}_{i\in K}$, where $K=I\sqcup J$. Using the natural embedding $\sigma_1\colon I\to K$ and $\sigma_2\colon J\to K$, the coverings $\{U_i\}_{i\in I}$ and $\{U_i\}_i\in J$ are each refinements of $\{U_i\}_{i\in K}$. Thus one has following diagram

$$H^2(\mathcal{U}_I,(\mathcal{F}^{\cdot},\nabla^{\mathrm{End}}))\xleftarrow{\sigma_1^*}H^2(\mathcal{U}_K,(\mathcal{F}^{\cdot},\nabla^{\mathrm{End}}))\xrightarrow{\sigma_2^*}H^2(\mathcal{U}_J,(\mathcal{F}^{\cdot},\nabla^{\mathrm{End}}))\tag{3.3}$$

As in Lemma 3.3, we extend the two systems of local data $\big(\{U_i\}_{i\in I},\{(\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i)\}_{i\in I},\{f_{ij}\}_{i,j\in I},\{\widetilde{\nabla}_i\}_{i\in I}\big)$ and $\big(\{U_i\}_{i\in J},\{(\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i)\}_{i\in J},\{f_{ij}\}_{i,j\in J},\{\widetilde{\nabla}_i\}_{i\in J}\big)$ to a system of local data $\big(\{U_i\}_{i\in K},\{(\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i)\}_{i\in K},\{f_{ij}\}_{i,j\in K},\{\widetilde{\nabla}_i\}_{i\in K}\big)$. Then $\sigma_1^*(c_K(V,\nabla,\mathrm{Fil}))=c_I(V,\nabla,\mathrm{Fil})$ and $\sigma_2^*(c_K(V,\nabla,\mathrm{Fil}))=c_J(V,\nabla,\mathrm{Fil})$. Thus $c_I(V,\nabla,\mathrm{Fil})$ and $c_J(V,\nabla,\mathrm{Fil})$ defines the same element in $\mathbb{H}^2(\mathcal{F}^{\cdot},\overline{\nabla}^{\mathrm{End}}):=\varinjlim_{\mathcal{U}}H^2(\mathcal{U},(\mathcal{F}^{\cdot},\overline{\nabla}^{\mathrm{End}}))$. □

Lemma 3.5. *Let $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ and $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})'$ be two liftings of a filtered de Rham bundle (V,∇,Fil). Then*

(1). *there exists an open affine covering $\{U_i\}_{i\in I}$ such that $\widetilde{\mathrm{Fil}}^{\ell}\widetilde{V}(\widetilde{U}_i)$ and $\widetilde{\mathrm{Fil}}^{\ell}\widetilde{V}'(\widetilde{U}_i)$ are free over $\mathcal{O}_{\widetilde{X}}(\widetilde{U}_i)$ for all $i\in I$ and $\ell\in\mathbb{Z}$, and*

(2). *there exists a local isomorphism $f_i\colon(\widetilde{V},\widetilde{\mathrm{Fil}})\mid_{\widetilde{U}_i}\to(\widetilde{V},\widetilde{\mathrm{Fil}})'\mid_{\widetilde{U}_i}$, which lifts the identity $\mathrm{id}_{V|_{U_i}}$.*

Proof. The Lemma follows Lemma 2.1 and Lemma 2.13. □

Suppose there are two liftings $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ and $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})'$ of a filtered de Rham bundle (V,∇,Fil) and $\big(\{U_i\}_i,\{f_i\}_i\big)$ be a system of local data given as in Lemma 3.5. Denote $\iota=\iota_{\widetilde{V},\widetilde{V}'}$, which is defined in

Lemma 2.10. By similar reason as $c(V,\nabla,\mathrm{Fil})$, one constructs an element

$$b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})') := \Big(\big(\iota^{-1}(f_j-f_i)\big)_{(i,j)},\big(\iota^{-1}(\widetilde{\nabla}'\circ f_i - f_i\circ\widetilde{\nabla})\big)_i\Big) \tag{3.4}$$

in

$$\prod_{(i,j)\in I^2}\mathrm{Fil}^0\mathscr{E}nd(V)_{\mathcal{I}}\mid_{U_{ij}}\times\prod_{i\in I}\Big(\mathrm{Fil}^{-1}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\Big)\mid_{U_i}.$$

Lemma 3.6. *Let $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ and $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})'$ be two liftings of a filtered de Rham bundle (V,∇,Fil). Then*

(1). *the element $b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})')$ defined in (3.4) is a 1-cocycle in the Cech resolution of the complex $\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla))_{\mathcal{I}}$ with respect to the open covering $\{U_i\}_{i\in I}$ of X.*

(2). *The class $[b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})')]\in\mathbb{H}^1(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla))_{\mathcal{I}})$ does not depend on the choice of $\big(\{U_i\}_i,\{f_i\}_i\big)$.*

Proof. To show $b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})') := \Big(\big(\iota^{-1}(f_j-f_i)\big)_{(i,j)},\big(\iota^{-1}(\widetilde{\nabla}'\circ f_i - f_i\circ\widetilde{\nabla})\big)_i\Big)$ is a 1-cocycle, we only need to check

$$\begin{cases} 0 & = \delta\Big(\big(\iota^{-1}(f_j-f_i)\big)_{(i,j)}\Big) & \in C^2(\mathcal{U},\mathcal{F}^0)\\ \overline{\nabla}^{\mathrm{End}}\Big(\big(\iota^{-1}(f_j-f_i)\big)_{(i,j)}\Big) & = \delta\Big(\big(\iota^{-1}(\widetilde{\nabla}'\circ f_i-f_i\circ\widetilde{\nabla})\big)_i\Big) & \in C^1(\mathcal{U},\mathcal{F}^1)\\ \overline{\nabla}^{\mathrm{End}}\Big(\big(\iota^{-1}(\widetilde{\nabla}'\circ f_i-f_i\circ\widetilde{\nabla})\big)_i\Big) & = 0 & \in C^0(\mathcal{U},\mathcal{F}^2)\end{cases}$$

This is equivalent to check

1a). $(f_k-f_j)\mid_{U_{ijk}}-(f_k-f_i)\mid_{U_{ijk}}+(f_j-f_i)\mid_{U_{ijk}}=0$ for each $(i,j,k)\in I^3$.

1b). $\nabla_{(\widetilde{\nabla},\widetilde{\nabla}')}(f_j-f_i)=(\widetilde{\nabla}'\circ f_j-f_j\circ\widetilde{\nabla})\mid_{U_{ij}}-(\widetilde{\nabla}'\circ f_i-f_i\circ\widetilde{\nabla})\mid_{U_{ij}}$ for each $(i,j)\in I^2$.

1c). $\nabla_{(\widetilde{\nabla},\widetilde{\nabla}')}(\widetilde{\nabla}'\circ f_i-f_i\circ\widetilde{\nabla})=0$ for each $i\in I$.

1a) and 1b) are trivial. 1c) follows from the integrability of $\widetilde{\nabla}'$ and $\widetilde{\nabla}$.
Follow the same method as in the proof of (2) of Lemma 3.4, one gets the proof of (2). □

Lemma 3.7. *Let $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ be a lifting of a filtered de Rham bundle (V,∇,Fil). Denote $\iota=\iota_{(\widetilde{V},\widetilde{V})}$.*

(1). *For any $\epsilon\in\mathbb{H}^0\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big)$, $\mathrm{id}_{\widetilde{V}}+\iota(\epsilon)$ is an automorphism of $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ which lifts id_V.*

(2). *Let f be an automorphism of $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ which lifts id_V. Then*

$$\iota^{-1}(f-\mathrm{id}_{\widetilde{V}})\in\mathbb{H}^0\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big).$$

Proof. Note that

$$\begin{aligned}\mathbb{H}^0\big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\big) &= \Big\{\epsilon\in\Gamma\left(X,\mathrm{Fil}^0\mathscr{E}nd(V)_{\mathcal{I}}\right)\mid\overline{\nabla}^{\mathrm{End}}(\epsilon)=0.\Big\}\\ &= \Big\{\iota(f-\mathrm{id})\mid f\in\Gamma(X,\mathrm{Fil}^0_{(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}})}\mathscr{E}nd(\widetilde{V}))\text{ with }f\equiv\mathrm{id}\pmod{\mathcal{I}}\text{ and }\nabla_{(\widetilde{\nabla},\widetilde{\nabla})}(f)=0.\Big\}\end{aligned}$$

The Lemma follows that fact that for any $f\colon\widetilde{V}\to\widetilde{V}$ with $f\equiv\mathrm{id}_{\widetilde{V}}\pmod{\mathcal{I}}$

- f preserves the filtration if and only if $\eta\in\mathrm{Fil}^0_{(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}})}\mathscr{E}nd(\widetilde{V})$ (Lemma 2.2);
- f is parallel if and only if $\nabla_{(\widetilde{\nabla},\widetilde{\nabla})}(\eta)=0$. □

Proof of Proposition 3.1. According Lemma 3.4, Lemma 3.6 and Lemma 3.7 one only need to show that

(a) if the filtered de Rham bundle (V,∇,Fil) has a lifting $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$, then $[c(V,\nabla,\mathrm{Fil})]=0$;
(b) if $[c(V,\nabla,\mathrm{Fil})]=0$, then the filtered de Rham bundle (V,∇,Fil) is liftable;

(c) Let $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ be a lifting of the filtered de Rham bundle (V,∇,Fil). For any element ϵ in $\mathbb{H}^1(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}))$, there exists another lifting $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})'$ such that $\epsilon=[b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})')]$.

The proof of (a) is trivial: if we choose a system of local data $\left(\{U_i\}_i,\{(\widetilde{V}_i,\widetilde{\mathrm{Fil}}_i)\}_i,\{f_{ij}\}_{ij},\{\widetilde{\nabla}_i\}_i\right)$ such that $(\widetilde{V}_i,\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i)=(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})\mid_{\widetilde{U}_i}$ and $f_{ij}=\mathrm{id}_{\widetilde{V}|_{\widetilde{U}_{ij}}}$, then $c(V,\nabla,\mathrm{Fil})=0$.

Proof of (b). Since $c(V,\nabla,\mathrm{Fil})=\Big(-\big(\iota_{ik}^{-1}(f_{jk}\circ f_{ij}-f_{ik})\big)_{(i,j,k)},\big(\iota_{ij}^{-1}((\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\big)_{(i,j)},\big(\iota_{ii}^{-1}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)\big)_i\Big)$ is a coboundary, there exists an element $\Big(\big(\iota_{ij}^{-1}(\Delta f_{ij})\big)_{(i,j)},\big(\iota_{ii}^{-1}(\Delta\omega_i)\big)_i\Big)$ in

$$\prod_{(i,j)\in I^2}\big(\mathrm{Fil}^0\mathscr{E}nd(V)\big)_{\mathcal{I}}\mid_{U_{ij}}\times\prod_{i\in I}\Big(\big(\mathrm{Fil}^{-1}\mathscr{E}nd(V)\big)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\Big)\mid_{U_i}.$$

such that

$$\left\{\begin{array}{rcl}-\big(\iota_{ik}^{-1}(f_{jk}\circ f_{ij}-f_{ik})\big)_{(i,j,k)}&=&\delta\Big(\big(\iota_{ij}^{-1}(\Delta f_{ij})\big)_{(i,j)}\Big)\\ \big(\iota_{ij}^{-1}((\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\big)_{(i,j)}&=&-\overline{\nabla}^{\mathrm{End}}\Big(\big(\iota_{ij}^{-1}(\Delta f_{ij})\big)_{(i,j)}\Big)+\delta\left(\big(\iota_{ii}^{-1}(\Delta\omega_i)\big)_i\right)\\ \big(\iota_{ii}^{-1}(\widetilde{\nabla}_i\circ\widetilde{\nabla}_i)\big)_i&=&\overline{\nabla}^{\mathrm{End}}\left(\big(\iota_{ii}^{-1}(\Delta\omega_i)\big)_i\right)\end{array}\right.$$

This is equivalent to

$$\left\{\begin{array}{lll}f_{jk}\circ f_{ij}-f_{ik}+(\Delta f_{jk}\circ f_{ij}-\Delta f_{ik}+f_{jk}\circ\Delta f_{ij})=0&\text{for each}&(i,j,k)\in I^3\\ \widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i+(\widetilde{\nabla}_j\circ\Delta f_{ij}-\Delta f_{ij}\circ\widetilde{\nabla}_i)-(\Delta\omega_j\circ f_{ij}-f_{ij}\circ\Delta\omega_i)=0&\text{for each}&(i,j)\in I^2\\ \widetilde{\nabla}_i\circ\widetilde{\nabla}_i-(\widetilde{\nabla}_i\circ\Delta\omega_i+\Delta\omega_i\circ\widetilde{\nabla}_i)=0&\text{for each}&i\in I\end{array}\right.$$

Denote $\widetilde{\nabla}_i'=\widetilde{\nabla}_i-\Delta\omega_i$ and $f_{ij}'=f_{ij}+\Delta f_{ij}$. Then it is equivalent to

$$\left\{\begin{array}{lll}f_{jk}'\circ f_{ij}'=f_{ik}'&\text{for each}&(i,j,k)\in I^3\\ \widetilde{\nabla}_j'\circ f_{ij}'=f_{ij}'\circ\widetilde{\nabla}_i'&\text{for each}&(i,j)\in I^2\\ \widetilde{\nabla}_i'\circ\widetilde{\nabla}_i'=0&\text{for each}&i\in I\end{array}\right.$$

This implies that local data $(\widetilde{V}_i,\widetilde{\nabla}_i)_i$ can be glued via $(f_{ij}')_{ij}$ into a de Rham bundle $(\widetilde{V},\widetilde{\nabla})$ over $(\widetilde{X},\widetilde{D})/\widetilde{S}$. Since $f_{ij}\in\widetilde{\mathrm{Fil}}^0_{ij}\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_j)$ and

$$\Delta f_{ij}\in\mathrm{Fil}^0\mathscr{E}nd(V)_{\mathcal{I}}(U_{ij})\subset\widetilde{\mathrm{Fil}}^0_{ij}\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_j),$$

f_{ij}' is also contained in $\widetilde{\mathrm{Fil}}^0_{ij}\mathscr{H}om(\widetilde{V}_i,\widetilde{V}_j)$. By Lemma 2.2 f_{ij}''s preserve local filtrations $\widetilde{\mathrm{Fil}}_i$'s. Thus local filtrations $(\widetilde{\mathrm{Fil}}_i)_i$ can be glued into a filtration $\widetilde{\mathrm{Fil}}$ on $\widetilde{V}$. Since $(\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i)$ satisfying Griffith transversality and

$$\Delta\omega_i\in\Big(\mathrm{Fil}^{-1}\mathscr{E}nd(V)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\Big)(U_i)\subset\widetilde{\mathrm{Fil}}^{-1}_{ii}\mathscr{E}nd(\widetilde{V}_i)\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})(\widetilde{U}_i),$$

$(\widetilde{\nabla}_i',\widetilde{\mathrm{Fil}}_i)$ is also satisfying Griffith transversality by Lemma 2.5. In conclusion, we get a filtered de Rham bundle $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ which lifts (V,∇,Fil).

Proof of (c). Denote $\iota=\iota_{(\widetilde{V},\widetilde{V})}$. Let $\Big(\big(\iota^{-1}(\Delta f_{ij})\big)_{(i,j)},\big(\iota^{-1}(\Delta\omega_i)\big)_i\Big)$ be a representation of the 1-class ϵ with respect to a covering $\{\widetilde{U}_i\}_{i\in I}$ of $\widetilde{X}$, where

$$\Delta f_{ij}\in\mathcal{I}\cdot\mathrm{Fil}^0_{(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}})}\mathscr{E}nd(\widetilde{V})\text{ and }\Delta\omega_i\in\mathcal{I}\cdot\mathrm{Fil}^{-1}_{(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}})}\mathscr{E}nd(\widetilde{V})\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D}).$$

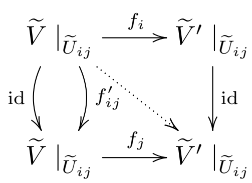
Denote $(\widetilde{V}_i,\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i):=(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})\mid_{\widetilde{U}_i}$ and $f_{ij}=\mathrm{id}_{\widetilde{V}|_{\widetilde{U}_{ij}}}$. Then

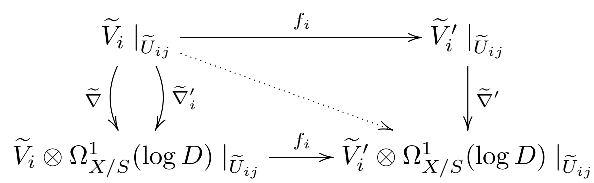
$$\begin{cases} f_{jk}\circ f_{ij}-f_{ik} & =0= \quad \Delta f_{jk}\circ f_{ij}-\Delta f_{ik}+f_{jk}\circ\Delta f_{ij}, & \text{for each} \quad (i,j,k)\in I^3;\\ -(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i) & =0= \quad -(\widetilde{\nabla}_j\circ\Delta f_{ij}-\Delta f_{ij}\circ\widetilde{\nabla}_i)+(\Delta\omega_j\circ f_{ij}-f_{ij}\circ\Delta\omega_i) & \text{for each} \quad (i,j)\in I^2;\\ -\widetilde{\nabla}_i\circ\widetilde{\nabla}_i & =0= \quad \widetilde{\nabla}_i\circ\Delta\omega_i+\Delta\omega_i\circ\widetilde{\nabla}_i, & \text{for each} \quad i\in I.\end{cases}$$

Similar as in the proof of (b), denote $\widetilde{\nabla}'_i=\widetilde{\nabla}+\Delta\omega_i$ and $f'_{ij}=\mathrm{id}-\Delta f_{ij}$ one glues $(\widetilde{V}_i,\widetilde{\nabla}'_i,\widetilde{\mathrm{Fil}}_i)$ via f'_{ij} into a filtered de Rham bundle $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})'$ on $\widetilde{X}$. The similar method in the proof of (2) in Lemma 3.4 show that up to isomorphism this filtered de Rham bundle does not depend on the choice of the representation of ϵ. In the following we show that

$$b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})')=\Big(\big(\iota^{-1}(\Delta f_{ij})\big)_{(i,j)},\big(\iota^{-1}(\Delta\omega_i)\big)_i\Big).$$

Since $(\widetilde{V}',\widetilde{\nabla}')$ is glued from $(\widetilde{V}\mid_{\widetilde{U}_i},\widetilde{\nabla}'_i)$ via local isomorphism f'_{ij}, there exists local isomoprhism $f_i\colon\widetilde{V}_{\widetilde{U}_i}\to\widetilde{V}'_{\widetilde{U}_i}$ which lifts $\mathrm{id}_{V|_{U_i}}$ such that $f_i=f_j\circ f'_{ij}$ and $\widetilde{\nabla}'\circ f_i=f_i\circ\widetilde{\nabla}'_i$.





Then $f_j\mid_{\widetilde{U}_{ij}}-f_i\mid_{\widetilde{U}_{ij}}=f_j\circ(\mathrm{id}-f'_{ij})=f_j\circ\Delta f_{ij}$ and $\widetilde{\nabla}'\circ f_i-f_i\circ\widetilde{\nabla}=f_i\circ(\widetilde{\nabla}'_i-\widetilde{\nabla})=f_i\circ\Delta\omega_i$. Thus $b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})')=\Big(\big(\iota^{-1}_{(\widetilde{V},\widetilde{V}')}(f_j\circ\Delta f_{ij})\big)_{(i,j)},\big(\iota^{-1}_{(\widetilde{V},\widetilde{V}')}(f_i\circ\Delta\omega_i)\big)_i\Big)=\Big(\big(\iota^{-1}(\Delta f_{ij})\big)_{(i,j)},\big(\iota^{-1}(\Delta\omega_i)\big)_i\Big)$. □

Notation 3.8. For any lifting $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ and any element ε in $\mathbb{H}^1(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}))$, denote the $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})'$ constructed in the proof of (c) by

$$(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})+\varepsilon.$$

By definition, one has

$$[b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})+\varepsilon)]=\varepsilon.$$

For any lifting $(\widetilde{V},\widetilde{\nabla})$ and any element ε in $\mathbb{H}^1(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}))$ one has similar notation $(\widetilde{V},\widetilde{\nabla})+\varepsilon$ and

$$[b((\widetilde{V},\widetilde{\nabla}),(\widetilde{V},\widetilde{\nabla})+\varepsilon)]=\varepsilon.$$

3.2. Deformation of the Hodge filtration

In this section we will study the deformation theory of the Hodge filtration. For a filtered de Rham bundle over $(X,D)/S$, recall the complex $\mathscr{C}$ defined in (2.13) is

(3.5)

$$0\to\frac{(\mathscr{E}nd(V))_{\mathcal{I}}}{\big(\mathrm{Fil}^0\mathscr{E}nd(V)\big)_{\mathcal{I}}}\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\frac{(\mathscr{E}nd(V))_{\mathcal{I}}}{\big(\mathrm{Fil}^{-1}\mathscr{E}nd(V)\big)_{\mathcal{I}}}\otimes\Omega^1_{X/S}(\log D)\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\frac{(\mathscr{E}nd(V))_{\mathcal{I}}}{\big(\mathrm{Fil}^{-2}\mathscr{E}nd(V)\big)_{\mathcal{I}}}\otimes\Omega^2_{X/S}(\log D)\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\cdots$$

Theorem 3.9. *Let (V,∇,Fil) be a filtered de Rham bundle over $(X,D)/S$. Let $(\widetilde{V},\widetilde{\nabla})$ be lifting of the underlying de Rham bundle (V,∇) over $(\widetilde{X},\widetilde{D})/\widetilde{S}$. Then*

1). *the obstruction to lifting* Fil *to a Hodge filtration on $(\widetilde{V},\widetilde{\nabla})$ lies in $\mathbb{H}^1(\mathscr{C})$;*
2). *if* Fil *has a lifting, then the lifting set is an $\mathbb{H}^0(\mathscr{C})$-torsor;*

In the rest of this subsection, we will always denote $\iota=\iota_{(\widetilde{V},\widetilde{V})}$.

Lemma 3.10. *Let (V,∇,Fil) be a filtered de Rham bundle over $(X,D)/S$. Let $\widetilde{V}$ be lifting of vector bundle V over $(\widetilde{X},\widetilde{D})/\widetilde{S}$. Then*

(1). *there exists an affine open covering $\{U_i\}_{i\in I}$ such that $\mathrm{Fil}^\ell V(U_i)$ is free over $\mathcal{O}_X(U_i)$ for all $i\in I$ and $\ell\in\mathbb{Z}$, and*

(2). *there exists local liftings $(\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i)$ of (∇,Fil) over $\widetilde{U}_i$ such that $(\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i)$ satisfies Griffith transversality for each $i\in I$, and*

(3). *there exists local liftings $f_{ij}\colon (\widetilde{V}\mid_{\widetilde{U}_i},\widetilde{\mathrm{Fil}}_i)\mid_{\widetilde{U}_{ij}}\to(\widetilde{V}\mid_{\widetilde{U}_j},\widetilde{\mathrm{Fil}}_j)\mid_{\widetilde{U}_{ij}}$ of $\mathrm{id}_{(V,\mathrm{Fil})}$ over $\widetilde{U}_{ij}$.*

Overloading notation, we call such a triple $(\{U_i\}_i,\{(\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i)\}_i,\{f_{ij}\}_{ij})$, whose existence is guaranteed by the lemma, a system of local data.

Remark. The connection $\widetilde{\nabla}_i$ needs not to be integrable.

Proof. The Lemma follows Lemma 2.1 and Lemma 2.13. □

Let (V,∇,Fil) be a filtered de Rham bundle over $(X,D)/S$. Let $(\widetilde{V},\widetilde{\nabla})$ be lifting of the underlying de Rham bundle (V,∇) over $(\widetilde{X},\widetilde{D})/\widetilde{S}$. Let $(\{U_i\}_i,\{(\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i)\}_i,\{f_{ij}\}_{ij})$ be a system of local data given as in Lemma 3.10. Then one defines an element

$$\widetilde{c}(\mathrm{Fil})=\Big((\mathrm{id}-f_{ij})_{(i,j)},(\widetilde{\nabla}_i-\widetilde{\nabla})_i\Big) \tag{3.6}$$

in

$$\prod_{(i,j)\in I^2}\left(\frac{\mathscr{E}nd(\widetilde{V})}{\widetilde{\mathrm{Fil}}^0_{ij}\mathscr{E}nd(\widetilde{V})}\right)(\widetilde{U}_{ij})\times\prod_{i\in I}\left(\frac{\mathscr{E}nd(\widetilde{V})}{\widetilde{\mathrm{Fil}}^{-1}_{ii}\mathscr{E}nd(\widetilde{V})}\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})\right)(\widetilde{U}_i).$$

Since $c(\mathrm{Fil})\equiv 0 \pmod{\mathcal{I}}$, one gets an element

$$c(\mathrm{Fil})=\Big(\big(\iota^{-1}(\mathrm{id}-f_{ij})\big)_{(i,j)},\big(\iota^{-1}(\widetilde{\nabla}_i-\widetilde{\nabla})\big)_i\Big) \tag{3.7}$$

in

$$\prod_{(i,j)\in I^2}\left(\frac{\mathscr{E}nd(V)_{\mathcal{I}}}{\mathrm{Fil}^0\mathscr{E}nd(V)_{\mathcal{I}}}\right)(U_{ij})\times\prod_{i\in I}\left(\frac{\mathscr{E}nd(V)_{\mathcal{I}}}{\mathrm{Fil}^{-1}\mathscr{E}nd(V)_{\mathcal{I}}}\otimes\Omega^1_{X/S}(\log D)\right)(U_i).$$

Lemma 3.11. *Let (V,∇,Fil) be a filtered de Rham bundle with $(\widetilde{V},\widetilde{\nabla})$ a lifting its underlying de Rham bundle. Let $(\{U_i\}_i,\{(\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i)\}_i,\{f_{ij}\}_{ij})$ be a system of local data given as in Lemma 3.10.*

(1). *The element $c(\mathrm{Fil})$ defined in (3.7) is a 1-cocycle in the Cech resolution of the complex (2.13) with respect to the open covering $\{U_i\}_{i\in I}$ of X.*

(2). *The class $[c(\mathrm{Fil})]\in\mathbb{H}^1(\mathscr{C})$ does not depend on the choice of $(\{U_i\}_i,\{(\widetilde{\nabla}_i,\widetilde{\mathrm{Fil}}_i)\}_i,\{f_{ij}\}_{ij})$.*

Proof. The proof of (2) follow the same method of proof of (2) in Lemma 3.4. For the (1), we denote $\overline{\mathcal{E}}^k=\frac{\mathscr{E}nd(V)_{\mathcal{I}}}{\mathrm{Fil}^{-k}\mathscr{E}nd(V)_{\mathcal{I}}}\otimes\Omega^k_{X/S}(\log D)$. Then the complex $\mathscr{C}$ may be rewritten as

$$0\to\overline{\mathcal{E}}^0\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\overline{\mathcal{E}}^1\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\overline{\mathcal{E}}^2\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\overline{\mathcal{E}}^3\xrightarrow{\overline{\nabla}^{\mathrm{End}}}\cdots$$

for all $k\geq 0$. For any $\tau=(i_1,\cdots,i_s)\in I^s$, denote $U_{(i_1,\cdots,i_s)}=\cap_{\ell=1}^s U_{i_\ell}$ and

$$C^p(\mathcal{U},\overline{\mathcal{E}}^k)=\prod_{\tau\in I^{p+1}}\overline{\mathcal{E}}^k(U_\tau).$$

Then the Cech resolution of the complex $\mathscr{C}$ with respect to the open covering $\mathcal{U}$ of X is the total complex of the following double complex

(3.8)
$$\begin{array}{ccccccc}
C^0(\mathcal{U},\overline{\mathcal{E}}^0) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^0(\mathcal{U},\overline{\mathcal{E}}^1) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^0(\mathcal{U},\overline{\mathcal{E}}^2) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & \cdots \\
\downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
C^1(\mathcal{U},\overline{\mathcal{E}}^0) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^1(\mathcal{U},\overline{\mathcal{E}}^1) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^1(\mathcal{U},\overline{\mathcal{E}}^2) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & \cdots \\
\downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
C^2(\mathcal{U},\overline{\mathcal{E}}^0) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^2(\mathcal{U},\overline{\mathcal{E}}^1) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & C^2(\mathcal{U},\overline{\mathcal{E}}^2) & \xrightarrow{\overline{\nabla}^{\mathrm{End}}} & \cdots \\
\downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

To show that
$$c(\mathrm{Fil}) = \Big(\big(\iota^{-1}(\mathrm{id}-f_{ij})\big)_{(i,j)}, \big(\iota^{-1}(\widetilde{\nabla}_i-\widetilde{\nabla})\big)_i\Big) \in C^1(\mathcal{U},\overline{\mathcal{E}}^0)\times C^0(\mathcal{U},\overline{\mathcal{E}}^1)$$
forms a cocycle, one needs to check
$$\begin{cases} 0 & = \delta\Big(\big(\iota^{-1}(\mathrm{id}-f_{ij})\big)_{(i,j)}\Big) & \in C^2(\mathcal{U},\overline{\mathcal{E}}^0) \\ \overline{\nabla}^{\mathrm{End}}\Big(\big(\iota^{-1}(\mathrm{id}-f_{ij})\big)_{(i,j)}\Big) & = \delta\Big(\big(\iota^{-1}(\widetilde{\nabla}_i-\widetilde{\nabla})\big)_i\Big) & \in C^1(\mathcal{U},\overline{\mathcal{E}}^1) \\ \overline{\nabla}^{\mathrm{End}}\Big(\big(\iota^{-1}(\widetilde{\nabla}_i-\widetilde{\nabla})\big)_i\Big) & = 0 & \in C^0(\mathcal{U},\overline{\mathcal{E}}^2) \end{cases}$$

This is equivalent to checking

1a). $(\mathrm{id}-f_{jk})-(\mathrm{id}-f_{ik})+(\mathrm{id}-f_{ij})\equiv 0 \pmod{\mathrm{Fil}^0_{ik}\mathscr{E}nd(\widetilde{V})(\widetilde{U}_{ijk})}$ for each $(i,j,k)\in I^3$.
1b). $\widetilde{\nabla}_{ij}(\mathrm{id}-f_{ij})\equiv(\widetilde{\nabla}_j-\widetilde{\nabla})-(\widetilde{\nabla}_i-\widetilde{\nabla}) \pmod{\big(\mathrm{Fil}^{-1}_{ij}\mathscr{E}nd(\widetilde{V})\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})\big)(\widetilde{U}_{ij})}$ for each $(i,j)\in I^2$,
1c). $\widetilde{\nabla}_{ii}(\widetilde{\nabla}_i-\widetilde{\nabla})\equiv 0 \pmod{\big(\mathrm{Fil}^{-2}_{ii}\mathscr{E}nd(\widetilde{V})\otimes\Omega^2_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})\big)(\widetilde{U}_i)}$ for each $i\in I$.

Denote $\alpha_{ij}=\mathrm{id}_{\widetilde{V}}-f_{ij}$. Then
$$f_{ik}-f_{jk}\circ f_{ij}=\alpha_{jk}-\alpha_{ik}+\alpha_{ij}-\alpha_{jk}\circ\alpha_{ij}.$$
Since $\alpha_{ij}\equiv 0\equiv\alpha_{jk} \pmod{\mathcal{I}}$ and $\mathcal{I}^2=0$, $\alpha_{jk}\circ\alpha_{ij}=0$. By definition $f_{ij}\in\mathrm{Fil}^0_{ij}\mathscr{E}nd(\widetilde{V})$, thus $f_{ik}-f_{jk}\circ f_{ij}\in\mathrm{Fil}^0_{ik}\mathscr{E}nd(\widetilde{V})$ by Lemma 2.2 and 1a) follows.
Note that
$$\begin{aligned}\widetilde{\nabla}_{ij}(\mathrm{id}-f_{ij}) =& \widetilde{\nabla}_j\circ(\mathrm{id}-f_{ij})-(\mathrm{id}-f_{ij})\circ\widetilde{\nabla}_i \\ =&(\widetilde{\nabla}_j-\widetilde{\nabla})-(\widetilde{\nabla}_i-\widetilde{\nabla})-(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\end{aligned}$$
By lemma 2.2, $(\widetilde{\nabla}_j\circ f_{ij}-f_{ij}\circ\widetilde{\nabla}_i)\in\mathrm{Fil}^{-1}_{ij}\mathscr{E}nd(\widetilde{V})\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})$ and 1b) follows.
Note that
$$\begin{aligned}\widetilde{\nabla}_{ii}(\widetilde{\nabla}_i-\widetilde{\nabla}\mid_{\widetilde{U}_i}) =& \widetilde{\nabla}_i\circ(\widetilde{\nabla}_i-\widetilde{\nabla}\mid_{\widetilde{U}_i})+(\widetilde{\nabla}_i-\widetilde{\nabla}\mid_{\widetilde{U}_i})\circ\widetilde{\nabla}_i \\ =&(\widetilde{\nabla}_i-\widetilde{\nabla})^2+\widetilde{\nabla}_i^2-\widetilde{\nabla}^2\end{aligned}$$
Since both $\widetilde{\nabla}_i-\widetilde{\nabla}\equiv 0 \pmod{\mathcal{I}}$, $(\widetilde{\nabla}_i-\widetilde{\nabla})^2=0$. Since $\widetilde{\nabla}$ is integrable, $\widetilde{\nabla}^2=0$. Thus 1c) follows Lemma 2.2. □

Lemma 3.12. *Let (V,∇,Fil) be a filtered de Rham bundle over $(X,D)/S$. Let $(\widetilde{V},\widetilde{\nabla})$ be a lifting of the underlying de Rham bundle (V,∇) over $(\widetilde{X},\widetilde{D})/\widetilde{S}$. Suppose there are two Hodge filtrations $\widetilde{\mathrm{Fil}}$ and $\widetilde{\mathrm{Fil}}'$ on $(\widetilde{V},\widetilde{\nabla})$ which lift the* Fil. *Then*

(1). *there exists an open affine covering $\{U_i\}_{i\in I}$ such that $\widetilde{\mathrm{Fil}}^\ell\widetilde{V}(\widetilde{U}_i)$ and $\widetilde{\mathrm{Fil}}'^\ell\widetilde{V}(\widetilde{U}_i)$ are free over $\mathcal{O}_{\widetilde{X}}(\widetilde{U}_i)$ for all $i\in I$ and $\ell\in\mathbb{Z}$, and*

(2). *there exists a (local) isomorphism $f_i\colon (\widetilde{V},\widetilde{\mathrm{Fil}})\mid_{\widetilde{U}_i}\to(\widetilde{V},\widetilde{\mathrm{Fil}}')\mid_{\widetilde{U}_i}$ lifting the identity $\mathrm{id}_{V|_{U_i}}$ for each $i\in I$.*

Overloading notation, we call a pair $(\{U_i\}_{i\in I},\{f_i\}_i)$, whose existence is guaranteed by the lemma, a system of local data.

Proof. The lemma follows Lemma 2.1 and Lemma 2.13. □

Let (V,∇,Fil) be a filtered de Rham bundle over $(X,D)/S$. Let $(\widetilde{V},\widetilde{\nabla})$ be lifting of the underlying de Rham bundle (V,∇) over $(\widetilde{X},\widetilde{D})/\widetilde{S}$. Suppose there are two Hodge filtrations $\widetilde{\mathrm{Fil}}$ and $\widetilde{\mathrm{Fil}}'$ on $(\widetilde{V},\widetilde{\nabla})$ which lift the Fil. Let $(\{U_i\}_{i\in I},\{f_i\}_i)$ be a system of local data given as in Lemma 3.12. Then

$$\widetilde{b}(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')=\Big(\mathrm{id}-f_i\Big)_i\in\prod_{i\in I}\left(\frac{\mathscr{E}nd(\widetilde{V})}{\mathrm{Fil}^0_{(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')}\mathscr{E}nd(\widetilde{V})}\right)(\widetilde{U}_i). \tag{3.9}$$

Since $\widetilde{b}(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')\equiv 0 \pmod{\mathcal{I}}$, one gets an element

$$b(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')=\Big(\iota^{-1}(\mathrm{id}-f_i)\Big)_i\in\prod_{i\in I}\left(\frac{\mathscr{E}nd(V)}{\mathrm{Fil}^0\mathscr{E}nd(V)}\right)(U_i). \tag{3.10}$$

where $\iota=\iota_{(\widetilde{V},\widetilde{V})}$.

Lemma 3.13. *Let $(\widetilde{V},\widetilde{\nabla})$ be a lifting of the underlying de Rham bundle of a filtered de Rham bundle (V,∇,Fil). Let $\widetilde{\mathrm{Fil}}$ and $\widetilde{\mathrm{Fil}}'$ be two Hodge filtrations on $(\widetilde{V},\widetilde{\nabla})$ which lift the* Fil. *Then*

(1). *The element $b(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')$ defined in (3.9) is a 0-cocycle in the Cech resolution of the complex (2.13) with respect to the open covering $\{U_i\}_{i\in I}$ of X.*

(2). *The class $[b(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')]\in\mathbb{H}^0(\mathscr{C})$ does not depend on the choice of $(\{U_i\}_{i\in I},\{f_i\}_i)$.*

Proof. The proof of (2) follows the same method of proof of (2) in Lemma 3.4. For (1), to proof

$$b(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')=\Big(\iota^{-1}(\mathrm{id}-f_i)\Big)_i$$

forms a 1-cocycle, we only need to check

$$\delta\left(\Big(\iota^{-1}(\mathrm{id}-f_i)\Big)_i\right)=0\in C^1(\mathcal{U},\overline{\mathcal{E}}^0)\quad\text{and}\quad\overline{\nabla}^{\mathrm{End}}\left(\Big(\iota^{-1}(\mathrm{id}-f_i)\Big)_i\right)=0\in C^0(\mathcal{U},\overline{\mathcal{E}}^1).$$

This is equivalent to check

1a). $(\mathrm{id}-f_j)-(\mathrm{id}-f_i)\in\mathrm{Fil}^0_{(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')}\mathscr{E}nd(\widetilde{V})(\widetilde{U}_{ij})$.

1b). $\widetilde{\nabla}_i\circ(\mathrm{id}-f_i)-(\mathrm{id}-f_i)\circ\widetilde{\nabla}_i\in\Big(\mathrm{Fil}^{-1}_{(\widetilde{\mathrm{Fil}},\widetilde{\mathrm{Fil}}')}\mathscr{E}nd(\widetilde{V})\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})\Big)(\widetilde{U}_i)$.

This follows Lemma 2.2. □

Proof of Theorem 3.9. According Lemma 3.11 and Lemma 3.13 one only need to show that

(a) if the Hodge filtration Fil has a lifting $\widetilde{\mathrm{Fil}}$, then $[c(\mathrm{Fil})]=0$;
(b) if $[c(\mathrm{Fil})]=0$, then the Hodge filtration Fil is liftable;

(c) Let $\widetilde{\mathrm{Fil}}$ be a lifting the Hodge filtration Fil. For any element ϵ in $\mathbb{H}^0(\mathscr{C})$, there exists another lifting $\widetilde{\mathrm{Fil}}'$ such that $\epsilon = b(\widetilde{\mathrm{Fil}}, \widetilde{\mathrm{Fil}}')$.

The proof of (a) is trivial: if we simply choose the system of local data $\big(\{U_i\}_i, \{f_i\}_i\big)$ with $f_i = \mathrm{id}_{\widetilde{V}|_{\widetilde{U}_i}}$, then $c(\mathrm{Fil}) = 0$.

Proof of (b). Since $c(\mathrm{Fil}) = \Big(\big(\iota^{-1}(\mathrm{id} - f_{ij})\big)_{(i,j)}, \big(\iota^{-1}(\widetilde{\nabla}_i - \widetilde{\nabla})\big)_i\Big)$ is a coboundary, there exists an element

$$\big(\iota^{-1}(\Delta f_i)\big)_i \in \prod_{i\in I} \left(\frac{\mathscr{E}nd(V)_{\mathcal{I}}}{\mathrm{Fil}^0_{ii}\mathscr{E}nd(V)_{\mathcal{I}}}\right)(U_i)$$

with $\Delta f_i \in \mathcal{I}\cdot \mathscr{E}nd(\widetilde{V})$ such that

(b1) $\mathrm{id} - f_{ij} \equiv \Delta f_j - \Delta f_i \mod \Big(\mathrm{Fil}^0_{ij}\mathscr{E}nd(\widetilde{V})\Big)(\widetilde{U}_{ij})$.

(b2) $\widetilde{\nabla}_i - \widetilde{\nabla} \equiv \widetilde{\nabla}\circ \Delta f_i - \Delta f_i \circ \widetilde{\nabla} \mod \Big(\mathrm{Fil}^{-1}_{ii}\mathscr{E}nd(\widetilde{V})\otimes \Omega^1_{\widetilde{X}/\widetilde{S}}(\log \widetilde{D})\Big)(\widetilde{U}_i)$.

Consider the new $\widetilde{\mathrm{Fil}}'_i = (\mathrm{id}_{\widetilde{V}|_{\widetilde{U}_i}} - \Delta f_i)^*(\widetilde{F}il_i)$ on $\widetilde{V}_i$. Then $\mathrm{id}_{\widetilde{V}|_{\widetilde{U}_i}} - \Delta f_i \in \mathrm{Fil}^0_{\mathrm{Fil}'_i,\mathrm{Fil}_i}\mathscr{E}nd(\widetilde{V})$

$$\begin{array}{ccc}
\Big(\widetilde{V}\mid_{\widetilde{U}_{ij}}, \widetilde{\mathrm{Fil}}'_i\Big) & \overset{\mathrm{id}}{\underset{f_{ij}+\Delta f_j - \Delta f_i}{\rightrightarrows}} & \Big(\widetilde{V}\mid_{\widetilde{U}_{ij}}, \widetilde{\mathrm{Fil}}'_j\Big) \\
{\scriptstyle \mathrm{id}-\Delta f_i}\downarrow & & \downarrow{\scriptstyle \mathrm{id}-\Delta f_j} \\
\Big(\widetilde{V}\mid_{\widetilde{U}_{ij}}, \widetilde{\mathrm{Fil}}_i\Big) & \xrightarrow{\quad f_{ij}\quad} & \Big(\widetilde{V}\mid_{\widetilde{U}_{ij}}, \widetilde{\mathrm{Fil}}_j\Big)
\end{array}$$

Thus $f_{ij} + \Delta f_j - \Delta f_i = (\mathrm{id} - \Delta f_j)^{-1} \circ f_{ij} \circ (\mathrm{id} - \Delta f_i) \in \mathrm{Fil}^0_{\mathrm{Fil}'_i,\mathrm{Fil}'_j}\mathscr{E}nd(\widetilde{V})$ by Lemma 2.2. Thus id $\in \mathrm{Fil}^0_{\mathrm{Fil}'_i,\mathrm{Fil}'_j}\mathscr{E}nd(\widetilde{V})$ by (b1), and $\widetilde{\mathrm{Fil}}'_i$, $\widetilde{\mathrm{Fil}}'_j$ coincide over $\widetilde{U}_{ij}$ by Lemma 2.2. Hence one may glue local filtrations $\{\widetilde{\mathrm{Fil}}'_i\}$ into a filtration $\widetilde{\mathrm{Fil}}$ of $\widetilde{V}$.

Since $(\widetilde{\nabla}_i, \widetilde{\mathrm{Fil}}_i)$ satisfies Griffith tranversality, $((\mathrm{id}_{\widetilde{V}|_{\widetilde{U}_i}} - \Delta f_i)^*\widetilde{\nabla}_i, \widetilde{\mathrm{Fil}}'_i)$ also satisfies Griffith transversality. Since $(\mathrm{id}_{\widetilde{V}|_{\widetilde{U}_i}} - \Delta f_i)^*\widetilde{\nabla}_i = \widetilde{\nabla}_i - \widetilde{\nabla}\circ\Delta f_i + \Delta f_i \circ \widetilde{\nabla}$, (b2) implies that $(\widetilde{\nabla}\mid_{\widetilde{U}_i}, \widetilde{\mathrm{Fil}}'_i)$ also satisfies Griffith transversality by Lemma 2.5. Thus $(\widetilde{V}, \widetilde{\nabla}, \widetilde{\mathrm{Fil}})$ is a filtered de Rham bundle.

Proof of (c). Denote $\iota = \iota_{\widetilde{V},\widetilde{V}}$, $(\widetilde{\nabla}_i, \widetilde{\mathrm{Fil}}_i) = (\widetilde{\nabla}, \widetilde{\mathrm{Fil}})\mid_{\widetilde{U}_i}$ and $f_{ij} = \mathrm{id}_{\widetilde{V}|_{\widetilde{U}_{ij}}}$. Let

$$\big(\iota^{-1}(\Delta f_i)\big)_i \in \prod_{i\in I} \left(\frac{\mathscr{E}nd(V)_{\mathcal{I}}}{\mathrm{Fil}^0_{ii}\mathscr{E}nd(V)_{\mathcal{I}}}\right)(U_i)$$

be a representation of ϵ with $\Delta f_i \in \mathcal{I}\cdot\mathscr{E}nd(\widetilde{V})$. Then

(c1) $\mathrm{id} - f_{ij} \equiv 0 \equiv \Delta f_j - \Delta f_i \mod \Big(\mathrm{Fil}^0_{ij}\mathscr{E}nd(\widetilde{V})\Big)(\widetilde{U}_{ij})$.

(c2) $\widetilde{\nabla}_i - \widetilde{\nabla} \equiv 0 \equiv \widetilde{\nabla}\circ\Delta f_i - \Delta f_i \circ \widetilde{\nabla} \mod \Big(\mathrm{Fil}^{-1}_{ii}\mathscr{E}nd(\widetilde{V})\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})\Big)(\widetilde{U}_i)$.

By the proof of (b), one constructs a new filtration $\widetilde{\mathrm{Fil}}'$ such that $(\widetilde{V}, \widetilde{\nabla}, \widetilde{\mathrm{Fil}}')$ forms a filtered de Rham bundle with local isomorphisms

$$\mathrm{id} - \Delta f_i \colon (\widetilde{V}, \widetilde{\mathrm{Fil}}')\mid_{\widetilde{U}_i} \to (\widetilde{V}, \widetilde{\mathrm{Fil}})\mid_{\widetilde{U}_i}.$$

Thus $b(\mathrm{Fil}, \mathrm{Fil}') = \big(\iota^{-1}(\mathrm{id} - (\mathrm{id} - \Delta f_i))\big)_i = \big(\iota^{-1}(\Delta f_i)\big)_i$. $\square$

4. Deformations of (graded) Higgs bundles

In this section we will study the deformation theory of graded Higgs bundles. The main result is Theorem 4.1. Let S, $\widetilde{S}$, $\mathfrak{a}$, $\widetilde{X}$, $\widetilde{D}$, X, D and $\mathcal{I}$ be as in Setup 1. Let (E,θ) be a (logarithmic) Higgs bundle over $(X,D)/S$. The pair $\mathscr{E}nd(E,\theta)$ is naturally a (logarithmic) Higgs bundle. Thus one has the following Higgs complex

$$\mathrm{DR}(\mathscr{E}nd(E,\theta)): \qquad 0\to \mathscr{E}nd(E)\stackrel{\theta^{\mathrm{End}}}{\longrightarrow}\mathscr{E}nd(E)\otimes\Omega^1_{X/S}(\log D)\stackrel{\theta^{\mathrm{End}}}{\longrightarrow}\mathscr{E}nd(E)\otimes\Omega^2_{X/S}(\log D)\stackrel{\theta^{\mathrm{End}}}{\longrightarrow}\cdots \tag{4.1}$$

Tensoring with the ideal sheaf $\mathcal{I}$, one gets complex

$$\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}: \qquad 0\to \mathscr{E}nd(E)_{\mathcal{I}}\stackrel{\theta^{\mathrm{End}}}{\longrightarrow}\mathscr{E}nd(E)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\stackrel{\theta^{\mathrm{End}}}{\longrightarrow}\mathscr{E}nd(E)_{\mathcal{I}}\otimes\Omega^2_{X/S}(\log D)\stackrel{\theta^{\mathrm{End}}}{\longrightarrow}\cdots \tag{4.2}$$

Suppose (E,θ,Gr) is a graded Higgs bundle. Then Gr induces a grading structure on the complex $\mathrm{DR}(\mathscr{E}nd(E,\theta))\otimes\mathcal{I}$ with

$$\mathrm{Gr}^\ell\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}: \qquad 0\to \mathrm{Gr}^\ell\mathscr{E}nd(E)_{\mathcal{I}}\stackrel{\theta^{\mathrm{End}}}{\longrightarrow}\mathrm{Gr}^{\ell-1}\mathscr{E}nd(E)_{\mathcal{I}}\otimes\Omega^1_{X/S}(\log D)\stackrel{\theta^{\mathrm{End}}}{\longrightarrow}\cdots \tag{4.3}$$

Theorem 4.1. *Let (E,θ,Gr) be a graded Higgs bundle over $(X,D)/S$. Then*

1). *the obstruction to lifting (E,θ,Gr) to a graded Higgs bundle over $(\widetilde{X},\widetilde{D})/\widetilde{S}$ lies in $\mathbb{H}^2\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$;*

2). *if (E,θ,Gr) has a graded lifting $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$, then the lifting set is an $\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$-torsor;*

3). *the infinitesimal automorphism group of $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$ over (E,θ,Gr) is $\mathbb{H}^0\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$.*

Proof of Theorem 4.1. The proof is almost the same as that of Theorem 3.1. One only needs to replace connections with Higgs fields and filtrations with graded structures. We give an outline; the details are left to readers.

First step. As in Lemma 3.3, one may find an open covering $\{U_i\}_{i\in I}$, local liftings $(\widetilde{E}_i,\widetilde{\mathrm{Gr}}_i)$ of the graded vector bundle $(E,\mathrm{Gr})\mid_{U_i}$, local liftings $\widetilde{\theta}_i$ (as an $\mathcal{O}_{\widetilde{X}}$-linear map) of $\theta\mid_{U_i}$ with $\widetilde{\theta}_i(\widetilde{\mathrm{Gr}}^\ell\widetilde{E})\subset\widetilde{\mathrm{Gr}}^{\ell-1}\widetilde{E}\otimes\Omega^1_{\widetilde{X}/\widetilde{S}}(\log\widetilde{D})$ and local isomorphisms $f_{ij}\colon(\widetilde{E}_i,\widetilde{\theta}_i,\widetilde{\mathrm{Gr}}_i)\mid_{\widetilde{U}_{ij}}\to(\widetilde{E}_i,\widetilde{\theta}_i,\widetilde{\mathrm{Gr}}_i)\mid_{\widetilde{U}_{ij}}$ of graded vector bundles with graded linear maps. Note that here $\widetilde{\theta}_i$ is only $\mathcal{O}_{\widetilde{X}}$-linear and NOT integrable. This is the *system of local data*.

Second step. As in Lemma 3.4, one shows that

$$\widetilde{c}(E,\theta,\mathrm{Gr}):=\Big(-(f_{jk}\circ f_{ij}-f_{ik}),(\widetilde{\theta}_j\circ f_{ij}-f_{ij}\circ\widetilde{\theta}_i),(\widetilde{\theta}_i\circ\widetilde{\theta}_i)\Big) \tag{4.4}$$

defines an element $c(E,\theta,\mathrm{Gr})$ in $\mathbb{H}^2\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$, which does not depend on the choice of the system of local data $(\{U_i\}_{i\in I},\{(\widetilde{E}_i,\widetilde{\theta}_i,\widetilde{\mathrm{Gr}}_i)\}_i,\{f_{ij}\}_{ij}\})$.

Third step. Let $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$ and $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})'$ be two liftings of the graded Higgs bundle (E,θ,Gr). As in Lemma 3.5, one find open covering $\{U_i\}_{i\in I}$ and local isomorphisms $f_i\colon(\widetilde{E},\widetilde{\mathrm{Gr}})\mid_{\widetilde{U}_i}\to(\widetilde{E},\widetilde{\mathrm{Gr}})\mid_{\widetilde{U}_i}$ for each $i\in I$.

Forth step. As in Lemma 3.6, one shows that

$$\widetilde{b}((\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}}),(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})'):=((f_j-f_i),(\widetilde{\theta}'\circ f_i-f_i\circ\widetilde{\theta})) \tag{4.5}$$

defines an element $b((\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}}),(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})')$ in $\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$, which does not depend on the choice of the system of local data $(\{U_i\}_{i\in I},\{f_i\}_i\})$.

Fifth step. Let $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$ be a lifting of the graded Higgs bundle (E,θ,Gr). As in Lemma 3.7, one shows that $\mathrm{id}+\epsilon$ is an automorphism of $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$ for any $\epsilon\in\mathbb{H}^0\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$ and any automorphism of $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$ which lifts the identity map is of this form.

Sixth step. As in the proof of Theorem 3.1, one shows that

(a) if the graded Higgs bundle (E,θ,Gr) has a lifting $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$, then $[c(E,\theta,\mathrm{Gr})]=0$;

(b) if $[c(E,\theta,\mathrm{Gr})]=0$, then the graded Higgs bundle (E,θ,Gr) is liftable;
(c) Let $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})$ be a lifting of the graded Higgs bundle (E,θ,Gr). For any element ϵ in $\mathbb{H}^1(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E,\theta))\otimes I)$, there exists another lifting $(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})'$ such that $\epsilon = b((\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}}),(\widetilde{E},\widetilde{\theta},\widetilde{\mathrm{Gr}})')$.

□

Ignore all graded structure and follow the similar method, one shows the following classical result.

Proposition 4.2. *Let (E,θ) be a Higgs bundle over $(X,D)/S$. Then*

1). *the obstruction to lifting (E,θ) to a Higgs bundle over $(\widetilde{X},\widetilde{D})/\widetilde{S}$ lies in $\mathbb{H}^2\big(\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$;*
2). *if (E,θ) has lifting $(\widetilde{E},\widetilde{\theta})$, then the lifting set is an $\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$-torsor;*
3). *the infinitesimal automorphism group of $(\widetilde{E},\widetilde{\theta})$ over (E,θ) is $\mathbb{H}^0\big(\mathrm{DR}(\mathscr{E}nd(E,\theta))_{\mathcal{I}}\big)$.*

5. Uniqueness of Hodge filtration

Let S, $\widetilde{S}$, $\mathfrak{a}$, $\widetilde{X}$, $\widetilde{D}$, X, D, $\mathcal{I}$ be as in Setup 1. In this section, the main result is Theorem 5.2.
Let (V,∇,Fil) be a logarithmic filtered de Rham bundle on $(X,D)/S$. Let (E,θ,Gr) be the associated graded, which is a logarithmic graded Higgs bundle on $(X,D)/S$ as in Example 2.7. Recall the hypercohomology spectral sequence attached to a filtered complex in [?, 1.4.5]; one therefore has the following *Hodge-de Rham spectral sequence*:

$$\mathbb{H}^{p+q}(\mathrm{Gr}^p\mathrm{DR}(\mathscr{E}nd(E,\theta)_{\mathcal{I}})) \Rightarrow \mathbb{H}^{p+q}(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})). \tag{5.1}$$

Recall the complex $\mathscr{C}$ defined in (2.13). One has a short exact sequence of complexes

$$0\to \mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}) \xrightarrow{\iota} \mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}) \xrightarrow{\pi} \mathscr{C}\to 0,$$

which induces a long exact sequence

$$\begin{aligned} 0\to \mathbb{H}^0\Big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) &\xrightarrow{\iota} \mathbb{H}^0\Big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) \xrightarrow{\pi} \mathbb{H}^0\Big(\mathscr{C}\Big)\\ \xrightarrow{\delta} \mathbb{H}^1\Big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) &\xrightarrow{\iota} \mathbb{H}^1\Big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) \xrightarrow{\pi} \mathbb{H}^1\Big(\mathscr{C}\Big)\\ \xrightarrow{\delta} \mathbb{H}^2\Big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) &\xrightarrow{\iota} \mathbb{H}^2\Big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) \xrightarrow{\pi} \cdots \end{aligned} \tag{5.2}$$

Denote by $\mathrm{Def}_{(V,\nabla)}(\widetilde{X},\widetilde{D})$ the set of deformations the of de Rham bundle to $(\widetilde{X},\widetilde{D})$:

$$\mathrm{Def}_{(V,\nabla)}(\widetilde{X},\widetilde{D}) := \Big\{ \text{ de Rham bundle } (\widetilde{V},\widetilde{\nabla}) \text{ over } (\widetilde{X},\widetilde{D})/\widetilde{S} \text{ with } (\widetilde{V},\widetilde{\nabla})\mid_X = (V,\nabla)\Big\}/\sim,$$

which is an $\mathbb{H}^1(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}))$-torsor space if it is not empty. Thus for any element $(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1})\in \mathrm{Def}_{(V_n,\nabla_n)}(X_{n+1},D_{n+1})$ and any element $\varepsilon\in\mathbb{H}^1(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}))$, there exists a unique element $(\widetilde{V}'_{n+1},\widetilde{\nabla}'_{n+1})\in \mathrm{Def}_{(V_n,\nabla_n)}(X_{n+1},D_{n+1})$ such that

$$b((\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1}),(\widetilde{V}'_{n+1},\widetilde{\nabla}'_{n+1}))=\varepsilon$$

where b is defined in Lemma 3.6. We will simply denote

$$(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1})+\varepsilon := (\widetilde{V}'_{n+1},\widetilde{\nabla}'_{n+1}). \tag{5.3}$$

Similarly denote by $\mathrm{Def}_{(V,\nabla,\mathrm{Fil})}(\widetilde{X},\widetilde{D})$ the set of deformations of the filtered de Rham bundle

$$\mathrm{Def}_{(V,\nabla,\mathrm{Fil})}(\widetilde{X},\widetilde{D}) := \Big\{ \text{ filtered de Rham bundle } (\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})\text{over } (\widetilde{X},\widetilde{D})/\widetilde{S} \text{ with } (\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})\mid_X = (V,\nabla,\mathrm{Fil})\Big\}/\sim.$$

One also has the notation $(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\widetilde{\mathrm{Fil}}_{n+1})+\varepsilon$ for any $(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\mathrm{Fil})\in \mathrm{Def}_{(V,\nabla,\mathrm{Fil})}(\widetilde{X},\widetilde{D})$ and any $\varepsilon\in\mathbb{H}^1(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}}))$.

Lemma 5.1. *Suppose that the Hodge-de Rham spectral sequence (5.1) degenerates at E_1. Then the long exact sequence 5.2 splits into short exact sequences*

$$0 \to \mathbb{H}^k\Big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) \longrightarrow \mathbb{H}^k\Big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) \longrightarrow \mathbb{H}^k\Big(\mathscr{C}\Big) \to 0.$$

Proof. By the definition of E_1 degenaration, $\mathbb{H}^k\Big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) \to \mathbb{H}^k\Big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big)$ is an injective map. Thus the lemma follows. □

Theorem 5.2. *Suppose that the spectral sequence (5.1) degenerates at E_1. Then*

(1). *If (V,∇) is liftable, then (V,∇,Fil) is also liftable.*

(2). *the map $\mathrm{Def}_{(V,\nabla,\mathrm{Fil})}(\widetilde{X},\widetilde{D}) \to \mathrm{Def}_{(V,\nabla)}(\widetilde{X},\widetilde{D})$ is injective. In other words, if $(\widetilde{V},\widetilde{\nabla})$ is a lifting of the underlying de Rham bundle of (V,∇,Fil), then for two liftings $\widetilde{\mathrm{Fil}}$ and $\widetilde{\mathrm{Fil}}'$ of the Hodge filtration* Fil *there exists an isomorphism*

$$(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}) \xrightarrow{\simeq} (\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}')$$

lifting the identity over S.

Remark. (1) in Theorem 5.2 means that if (V,∇) has a lifting $(\widetilde{V},\widetilde{\nabla})$ then (V,∇,Fil) has some lifting $(\widetilde{V}',\widetilde{\nabla}',\widetilde{\mathrm{Fil}}')$. In general, the underlying de Rham bundle of $(\widetilde{V}',\widetilde{\nabla}',\widetilde{\mathrm{Fil}}')$ is not equal $(\widetilde{V},\widetilde{\nabla})$ and there exists obstruction to lift the Hodge filtration to $(\widetilde{V},\widetilde{\nabla})$.

Lemma 5.3. *Let (V,∇,Fil) be a filtered de Rham bundle. Then*

(1). *$\iota(c(V,\nabla,\mathrm{Fil})) = c(V,\nabla)$;*

(3). *Suppose $c(V,\nabla,\mathrm{Fil}) = 0$. Let $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})$ and $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})'$ be two liftings of the filtered de Rham bundle (V,∇,Fil). Then*

$$\iota(b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})')) = b((\widetilde{V},\widetilde{\nabla}),(\widetilde{V},\widetilde{\nabla})').$$

In other words, for any given lifting $(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}) \in \mathrm{Def}_{(V,\nabla,\mathrm{Fil})}(\widetilde{X},\widetilde{D})$ one has the following commutative diagram

(5.4)
$$\begin{array}{ccc}
\mathrm{Def}_{(V,\nabla,\mathrm{Fil})}(\widetilde{X},\widetilde{D}) & \longrightarrow & \mathrm{Def}_{(V,\nabla)}(\widetilde{X},\widetilde{D}) \\
{\scriptstyle (\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})'\mapsto b((\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}}),(\widetilde{V},\widetilde{\nabla},\widetilde{\mathrm{Fil}})')}\downarrow\ \uparrow{\scriptstyle \epsilon\mapsto(\widetilde{V},\widetilde{\nabla})+\epsilon} & & {\scriptstyle \epsilon\mapsto(\widetilde{V},\widetilde{\nabla})+\epsilon}\uparrow\ \downarrow{\scriptstyle (\widetilde{V},\widetilde{\nabla})'\mapsto b((\widetilde{V},\widetilde{\nabla}),(\widetilde{V},\widetilde{\nabla})')} \\
\mathbb{H}^1\Big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big) & \xrightarrow{\iota} & \mathbb{H}^1\Big(\mathrm{DR}(\mathscr{E}nd(V,\nabla)_{\mathcal{I}})\Big)
\end{array}$$

Proof. This follows the definition of c and b. □

Proof of Theorem 5.2. By Lemma 5.1, the ι's are injective. Thus lemma follows Lemma 5.3. □

6. Lifting a Higgs-de Rham flow.

In this section we will study the deformation theory of a 1-periodic Higgs-de Rham flow. The main result is Theorem 6.4.

Let k be a perfect field of odd characteristic p, and set $S_m = \mathrm{Spec}(W_m(k))$ for all $m \in \mathbb{N}$. In this section we will fix an natural number n. Then $S_n \hookrightarrow S_{n+1}$ is square-zero thickening with ideal of definition $\mathfrak{a}_{n+1} = p^n W_{n+1}(k)$. Let X_{n+1} be a proper smooth S_{n+1}-scheme with relative flat normal crossing divisor D_{n+1}. Denote $X_n = X_{n+1} \times_{S_{n+1}} S_n$, $D_n = D_{n+1} \times_{S_{n+1}} S_n$ and the square-zero ideal sheaf $\mathcal{I}_{n+1} := \mathcal{O}_{X_{n+1}} \cdot \mathfrak{a}_{n+1}$. Thus

$$(\widetilde{S},S,\widetilde{X},X,\widetilde{D},D,\mathfrak{a},\mathcal{I}) := (S_{n+1},S_n,X_{n+1},X_n,D_{n+1},D_n,\mathfrak{a}_{n+1},\mathcal{I}_{n+1}).$$

is compatible with Setup 1. Denote $X_1 = X_{n+1} \times_{S_{n+1}} S_1$ and $D_1 = D_{n+1} \times_{S_{n+1}} S_1$. Since $p \cdot \mathfrak{a} = 0$, $\mathcal{I}$ is an $\mathcal{O}_{X_1}$-module. More precisely, one has an isomorphism $\mathcal{I} \simeq \mathcal{O}_{X_1}$. Thus for any vector bundle $\mathcal{F}$ over X, one has

$$\mathcal{F}_{\mathcal{I}} \simeq \mathcal{F}\mid_{X_1}. \tag{6.1}$$

In this section, we set

$$\begin{array}{ccccc} & & (V_n, \nabla_n, \mathrm{Fil}_n) & & \\ & \overset{C^{-1}}{\nearrow} & & \overset{\mathrm{Gr}}{\searrow} & \\ (E_n, \theta_n) & & \underset{\varphi}{\longleftarrow} & & \mathrm{Gr}(V_n, \nabla_n, \mathrm{Fil}_n) \end{array} \tag{6.2}$$

to be a 1-periodic Higgs-de Rham flow over $(X_n, D_n)/S_n$ where the inverse Cartier C^{-1} is defined with respect to the lift

$$(X_n, D_n)/S_n \subset (X_{n+1}, D_{n+1})/S_{n+1}.$$

Denote

$$(V_1, \nabla_1, \mathrm{Fil}_1, E_1, \theta_1, \varphi_1) = (V, \nabla, \mathrm{Fil}, E, \theta, \varphi)\mid_{X_1}$$

By (6.1), the complex 2.11 may be rewritten as a complex of sheaves over X_1

$$0 \to \mathscr{E}nd(V_1) \xrightarrow{\nabla_1^{\mathrm{End}}} \mathscr{E}nd(V_1) \otimes \Omega^1_{X_1/S_1}(\log D_1) \xrightarrow{\nabla_1^{\mathrm{End}}} \mathscr{E}nd(V_1) \otimes \Omega^2_{X_1/S_1}(\log D_1) \xrightarrow{\nabla_1^{\mathrm{End}}} \cdots \tag{6.3}$$

which is just the de Rham complex of the de Rham bundle $\mathscr{E}nd(V_1, \nabla_1)$ over X_1.
Similar the complexes 2.12 and 2.13 may be rewritten as

$$\mathrm{Fil}^\ell \mathrm{DR}(\mathscr{E}nd(V_1, \nabla_1)): \qquad 0 \to \mathrm{Fil}^\ell \mathscr{E}nd(V_1) \xrightarrow{\nabla_1^{\mathrm{End}}} \mathrm{Fil}^{\ell-1}\mathscr{E}nd(V_1) \otimes \Omega^1_{X_1/S_1}(\log D_1) \xrightarrow{\nabla_1^{\mathrm{End}}} \cdots \tag{6.4}$$

and

$$\mathscr{C} = \frac{\mathrm{DR}(\mathscr{E}nd(V_1, \nabla_1))}{\mathrm{Fil}^0 \mathrm{DR}(\mathscr{E}nd(V_1, \nabla_1))}: \qquad 0 \to \frac{\mathscr{E}nd(V_1)}{\mathrm{Fil}^0 \mathscr{E}nd(V_1)} \xrightarrow{\nabla_1^{\mathrm{End}}} \frac{\mathscr{E}nd(V_1)}{\mathrm{Fil}^{-1}\mathscr{E}nd(V_1)} \otimes \Omega^1_{X/S}(\log D_1) \xrightarrow{\nabla_1^{\mathrm{End}}} \cdots \tag{6.5}$$

Lemma 6.1. *Setup as in the beginning of the subsection and suppose (E_n, θ_n) is a logarithmic Higgs bundle on $(X_n, D_n)/S_n$ that initiates a 1-periodic Higgs-de Rham flow as in Equation 6.2. Then the Hodge-de Rham spectral sequence associated to the reduction modulo p:*

$$E_1^{p,q} := \mathbb{H}^{p+q}\big(\mathrm{Gr}^p \mathrm{DR}(\mathscr{E}nd(E_1, \theta_1))\big) \Rightarrow \mathbb{H}^{p+q}\big(\mathrm{DR}(\mathscr{E}nd(V_1, \nabla_1))\big)$$

degenerates at E_1.

Proof. First of all, the terms of E_∞ are subquotients of that of E_1; therefore one has

$$\mathrm{rank}_k\Big(\mathbb{H}^{p+q}(\mathrm{DR}(\mathscr{E}nd(E_1, \theta_1)))\Big) \geq \mathrm{rank}_k\Big(\mathbb{H}^{p+q}(\mathrm{DR}(\mathscr{E}nd(V_1, \nabla_1)))\Big),$$

and moreover the spectral sequence is degenerate at E_1 if and only if equality holds. On the other hand, by Ogus-Vologodsky, $(V, \nabla) = C^{-1}(E, \theta)$ induces an isomorphism

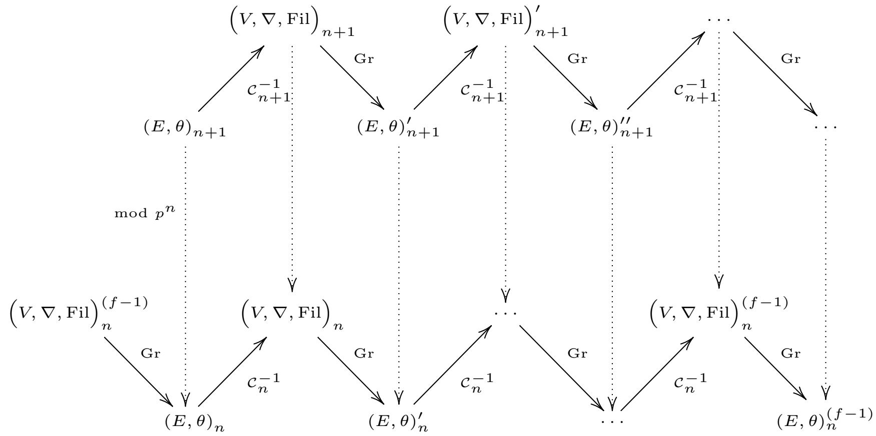
$$\sigma^* \mathbb{H}^{p+q}(\mathrm{DR}(\mathscr{E}nd(E_1, \theta_1))) \simeq \mathbb{H}^{p+q}(\mathrm{DR}(\mathscr{E}nd(V_1, \nabla_1))).$$

In particular, one has $\mathrm{rank}_k\Big(\mathbb{H}^{p+q}\big(\mathrm{DR}(\mathscr{E}nd(E_1, \theta_1))\big)\Big) = \mathrm{rank}_k\Big(\mathbb{H}^{p+q}(\mathrm{DR}(\mathscr{E}nd(V_1, \nabla_1)))\Big)$ and E_1 degeneration.

□

Theorem 6.2. *Let $(X, D)/W$ be a smooth pair with X/W projective. Let HDF_{X_n} be a periodic Higgs-de Rham flow over (X_n, D_n). For a given lifting $(E, \theta)_{X_{n+1}}$ of the initial graded Higgs bundle over (X_{n+1}, D_{n+1}), there is at most one Higgs-de Rham flow with initial term $(E, \theta)_{X_{n+1}}$ that lifts HDF_{X_n} up to isomorphism.*

Proof. Suppose there exists a Higgs-de Rham flow lifting HDF_{X_n} with initial Higgs term $(E,\theta)_{n+1}$.



As we have fixed a lifting of the initial Higgs bundle, the freedom of choice of a lifting Higgs-de Rham flow are the choices of the Hodge filtrations $\mathrm{Fil}_{n+1}, \mathrm{Fil}'_{n+1}, \mathrm{Fil}''_{n+1}, \cdots$. Applying Theorem 5.2(2) together with Lemma 6.1, the isomorphism class of this filtered de Rham bundle $(V_{n+1}, \nabla_{n+1}, \mathrm{Fil}_{n+1})$ *does not depend* on the choice of Fil_{n+1}. We therefore see that the associated graded Higgs bundle

$$(E_{n+1}, \theta_{n+1})' := \mathrm{Gr}_{\mathrm{Fil}_{n+1}}(V_{n+1}, \nabla_{n+1})$$

also and does not depend on the choice of the Hodge filtration Fil_{n+1}. Inductively, it follows that each term in the lifted Higgs-de Rham flow is uniquely determined by the initial Higgs-bundle. Thus the theorem follows. ☐

Remark. Theorem 6.2 proves that there is at most one Higgs-de Rham flow lifting HDF_{X_n} after fixing the initial graded Higgs term. However, we do not know that the lifted Higgs-de Rham flow is periodic; i.e., we don't know if the periodicity map is liftable. Even when the periodicity map is liftable, there may exist more than one lift. Therefore Theorem 6.2 does not imply that there is at most one lift as a *periodic* Higgs-de Rham flow. However, if the initial Higgs term is stable modulo p, then it easily follows that there is at most one lift, up to isomorphism, of HDF_{X_n} as a *periodic Higgs-de Rham flow.*

6.1. The torsor map induced by the inverse Cartier transform

Recall that

$$\mathrm{Def}_{(V_n,\nabla_n)}(X_{n+1}, D_{n+1}) = \left\{ \begin{array}{l} \text{de Rham bundle } (\widetilde{V}_{n+1}, \widetilde{\nabla}_{n+1}) \text{ over } (X_{n+1}, D_{n+1})/S_{n+1} \\ \text{with } (\widetilde{V}_{n+1}, \widetilde{\nabla}_{n+1}) \mid_{X_n} = (V_n, \nabla_n) \end{array} \right\} / \sim$$

and

$$\mathrm{Def}_{(V_n,\nabla_n,\mathrm{Fil}_n)}(X_{n+1}, D_{n+1}) = \left\{ \begin{array}{l} \text{filtered de Rham bundle } (\widetilde{V}_{n+1}, \widetilde{\nabla}_{n+1}, \widetilde{\mathrm{Fil}}_{n+1}) \text{ over } (X_{n+1}, D_{n+1})/S_{n_1} \\ \text{with } (\widetilde{V}_{n+1}, \widetilde{\nabla}_{n+1}, \widetilde{\mathrm{Fil}}_{n+1}) \mid_{X_n} = (V_n, \nabla_n, \mathrm{Fil}_n) \end{array} \right\} / \sim .$$

Similarly, denote

$$\mathrm{Def}_{(E_n,\theta_n)}(X_{n+1}, D_{n+1}) = \left\{ \begin{array}{l} \text{graded Higgs bundle } (\widetilde{E}_{n+1}, \widetilde{\theta}_{n+1}) \text{ over } (X_{n+1}, D_{n+1})/\widetilde{S}_{n+1} \\ \text{with } (\widetilde{E}_{n+1}, \widetilde{\theta}_{n+1}) \mid_{X_n} = (E_n, \theta_n) \end{array} \right\} / \sim$$

and

$$\mathrm{Def}_{(X_{n+1},D_{n+1})}(S_{n+2}) = \left\{ \begin{array}{l} \text{smooth log pair } (X_{n+2}, D_{n+2}) \text{ over } S_{n+2} \\ \text{with } (X_{n+2}, D_{n+2}) \times_{S_{n+2}} S_{n+1} = (X_{n+1}, D_{n+1}) \end{array} \right\} / \sim .$$

By Theorem 4.1, $\mathrm{Def}_{(E_n,\theta_n)}(X_{n+1},D_{n+1})$ is an $\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)$-torsor if $\mathrm{Def}_{(E_n,\theta_n)}(X_{n+1},D_{n+1})$ is not empty. By Theorem 3.2, $\mathrm{Def}_{(V_n,\nabla_n)}(X_{n+1},D_{n+1})$ is an $\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big)$-torsor if $\mathrm{Def}_{(V,\nabla)}(\widetilde{X},\widetilde{S})$ is not empty. It is well-known that $\mathrm{Def}_{(X_{n+1},D_{n+1})}(S_{n+2})$ is an $H^1(X_1,\mathcal{T}_{(X_1,D_1)/S_1})$-torsor if it is non-empty, see[**?**, Prop. 3.14 (iii)]. Here, $T_{(X_1,D_1)/S_1}$ refers to the *logarithmic Tangent bundle.*

The semilinear map α***.*** Denote by $F_{S_1}\colon S_1\to S_1$ the absolute Frobenius on S_1. Denote (E_1',θ_1') be the pull back Higgs bundle on X_1' via the Frobenius map $F_{S_1}\colon X_1'\to X_1$. Then (E_1',θ_1') is the image of (V_1,∇_1) under the Cartier functor constructed in Ogus-Vodogodsky, since $(V_1,\nabla_1)=C^{-1}_{X_1\subset X_2}(E_1,\theta_1)$. Thus by [**?**, Corollary 2.27], one has an isomorphism

$$\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(E_1',\theta_1'))\big)\to\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big).$$

Composing with the Frobenius map $F_{S_1}^*\colon\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)\to\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(E_1',\theta_1')\big)$, one gets a σ-semilinear map

$$\alpha\colon\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)\to\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big).$$

Since (E,θ) is a graded Higgs bundle, the $\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1)\big)$ is a direct summand of $\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1)\big)$. By abuse of notation, we still denote by α the restriction of α on this direct summand.

The semilinear map β***.*** Let's recall the a map constructed in [**?**, Section 5]. Write the Higgs field θ_1 of E_1 in the following way

$$\theta_1\colon(\mathcal{T}_{X_1/S_1},0)\to\mathscr{E}nd(E_1,\theta_1).$$

Applying the inverse Cartier transform to θ_1, one gets the corresponding morphism

$$C^{-1}_{X_1\subset X_2}(\theta_1)\colon(F^*\mathcal{T}_{X_1/S_1},\nabla_{\mathrm{can}})\to\mathscr{E}nd(V_1,\nabla_1).$$

Composing with the Frobenius map $F^*\colon\mathcal{T}_{X_1/S_1}\to F^*\mathcal{T}_{X_1/S_1}$ and then taking cohomology groups, one gets a map

$$\beta\colon H^1(X_1,\mathcal{T}_{X_1/S_1})\longrightarrow\mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big)$$

The construction of truncated inverse Cartier transform. Let $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})$ be a graded Higgs bundle over X_{n+1} and $(V_n,\nabla_n,\mathrm{Fil}_n)$ be a filtered de Rham bundle together with an isomorphism

$$\psi\colon\mathrm{Gr}_{\mathrm{Fil}_n}(V_n,\nabla_n)\to(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})\mid_{X_n}$$

Then the *inverse Cartier transform* yields filtered de Rham bundle

$$C^{-1}_{(X_{n+1},D_{n+1})\subseteq(X_{n+2},D_{n+2})}\big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(V_n,\nabla_n,\mathrm{Fil}_n),\psi\big)$$

over X_{n+1} which is constructed as following. (See also [**?**, Theorem 4.1] and its proof.)

In small affine cases. Assume X_{n+2} is small affine with coordinates $t_1,\cdots,t_d$ and a Frobenius lifting $\Phi_{n+2}\colon X_{n+2}\to X_{n+2}$. This means the following: Φ_{n+2} preserves the divisor D_{n+2} and lifts the the absolute Frobenius on X_1. We also assume all vector bundles appeared are free. By the freeness, there exists a filtered vector bundle with connection (not necessarily integrable) $(V_{n+1},\nabla_{n+1},\mathrm{Fil}_{n+1})$ over X_{n+1} which lifts $(V_n,\nabla_n,\mathrm{Fil}_n)$ and there exists an isomorphism which lifts ψ

$$\mathrm{Gr}_{\mathrm{Fil}_{n+1}}(V_{n+1},\nabla_{n+1})\xrightarrow{\sim}(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}).$$

Applying Faltings' tilde functor on $(V_{n+1},\nabla_{n+1},\mathrm{Fil}_{n+1})$ [**?**, Ch. 2] (see also the functor G_n on [**?**, p. 25-26]) one gets a vector bundle $\widetilde{H}$ over X_{n+1} with a p-connection $\widetilde{\nabla}$. Let's note that this vector bundle with p-connection

$$(\widetilde{H},\widetilde{\nabla})\tag{6.6}$$

does not depend on the choice of the lifting $(V_{n+1},\nabla_{n+1},\mathrm{Fil}_{n+1})$. Taking the Φ_{n+2}-pullback, one constructs a de Rham bundle over X_{n+1}

$$C^{-1}_{(X_{n+1},D_{n+1})\subseteq(X_{n+2},D_{n+2})}\big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(V_n,\nabla_n,\mathrm{Fil}_n),\psi\big) := \left(\Phi^*_{n+1}(\widetilde{H}),\frac{\Phi^*_{n+2}}{p}(\widetilde{\nabla})\right).$$

This de Rham bundle does not depend on the choice of the Frobenius lifting upto an canonical isomorphism. i.e. For any other Frobenius lifting Ψ_{n+2} on X_{n+2}, there is an canonical isomorphism

$$G_{\Phi_{n+2},\Psi_{n+2}}\colon \left(\Phi^*_{n+1}(\widetilde{H}),\frac{\Phi^*_{n+2}}{p}(\widetilde{\nabla})\right)\to\left(\Psi^*_{n+1}(\widetilde{H}),\frac{\Psi^*_{n+2}}{p}(\widetilde{\nabla})\right)$$

which is defined by a Taylor formula

$$m\otimes_{\Phi^*_{n+1}}1\mapsto\sum_I\widetilde{\nabla}_{n+1}(\partial)^I(m)\otimes_{\Psi^*_{n+1}}\frac{\big(\Phi^*_{n+2}(t)-\Psi^*_{n+2}(t)\big)^I}{I!\cdot p^{|I|}}$$

For more details about this type of Taylor formula, see the formula in proof of Theorem 2.3 in [?].
In general. There exists covering of small affine open subsets $\{X_{n+2,i}\}_i$ of X_{n+2} together with Frobenius liftings $\Phi_{n+2,i}$ on $X_{n+2,i}$ for each i. By taking a fine enough covering, one may assume all vector bundles appearing are free. According the construction in small affine cases, one gets a family local de Rham bundles

$$(H_i,\nabla_i)=C^{-1}_{(X_{n+1,i},D_{n+1,i})\subseteq(X_{n+2,i},D_{n+2,i})}\Big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})\mid_{X_{n+1,i}},(V_n,\nabla_n,\mathrm{Fil}_n)\mid_{X_{n,i}}\Big).$$

One obtains the de Rham bundle $C^{-1}_{(X_{n+1},D_{n+1})\subseteq(X_{n+2},D_{n+2})}\big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(V_n,\nabla_n,\mathrm{Fil}_n),\psi\big)$ by gluing (H_i,∇_i) via the canonical isomorphisms $G_{ij}:=G_{\Phi_{n+2,i},\Phi_{n+2,j}}$.

Remark. The local vector bundles with p-connections over $X_{n+1,i}$ as in (6.6) can be glued into a global vector bundles with p-connection; Lan-Sheng-Zuo [?] denote it as

$$\mathcal{T}_n(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1},V_n,\nabla_n,\mathrm{Fil}_n,\psi). \tag{6.7}$$

Once one obtains this p-connection, one can use the method in Faltings [?] to construct a de Rham bundle $C^{-1}_{(X_{n+1},D_{n+1})\subseteq(X_{n+2},D_{n+2})}\big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(V_n,\nabla_n,\mathrm{Fil}_n),\psi\big)$. Locally pull back the p-connection with local Frobenius maps and then glue via Taylor formula.

Following the explicit construction of the inverse Cartier functor over the truncated level, one has following result.

Lemma 6.3. *Fix an element* $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})\in\mathrm{Def}_{(E_n,\theta_n)}(X_{n+1},D_{n+1})$ *and* $(X_{n+2},D_{n+2})\in\mathrm{Def}_{(X_{n+1},D_{n+1})}(S_{n+2})$. *For any* $\varepsilon\in\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)$ *and any* $\eta\in H^1(X_1,\mathcal{T}_{X_1/S_1})$, *one has an isomorphism of logarithmic de Rham bundles on* $(X_n,D_n)/S_n$:

$$C^{-1}_{X_{n+1}\subset(X_{n+2}+\eta)}\big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})+\varepsilon\big)=C^{-1}_{X_{n+1}\subset X_{n+2}}(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})+\alpha(\varepsilon)+\beta(\eta).$$

Here, $X_{n+2}+\eta$ *is a lift of* (X_{n+1},D_{n+1}) *over* S_{n+2}, $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})+\varepsilon$ *is a lift of* (E_n,θ_n) *over* (X_{n+1},D_{n+1}), *and* $C^{-1}_{X_{n+1}\subset X_{n+2}}(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})+\alpha(\varepsilon)+\beta(\eta)$ *is a lift of* (V_n,∇_n) *over* (X_{n+1},D_{n+1}).
In other words, the following diagram commutes

$$\begin{array}{ccc}\mathrm{Def}_{(E_n,\theta_n)}(X_{n+1},D_{n+1})\times\mathrm{Def}_{(X_{n+1},D_{n+1})}(S_{n+2}) & \xrightarrow{\mathrm{IC}} & \mathrm{Def}_{(V_n,\nabla_n)}(X_{n+1},D_{n+1})\\ \big\downarrow{\scriptstyle 1:1} & & {\scriptstyle 1:1}\big\downarrow\\ \mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)\oplus H^1(X_1,T_{X_1}) & \xrightarrow{(\alpha,\beta)} & \mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big)\end{array} \tag{6.8}$$

where the top horizontal arrow is the (parametrized) inverse Cartier transform IC, the left vertical arrow is given by the formula:

$$((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})',(X_{n+2},D_{n+2})')\mapsto(b((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})'),b((X_{n+2},D_{n+2}),(X_{n+2},D_{n+2})')),$$

the right vertical arrow is given by the formula

$$(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1})'\mapsto b((\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1}),(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1})'),$$

where

$$(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1})=C^{-1}_{(X_{n+1},D_{n+1})\subset(X_{n+2},D_{n+2})}(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),$$

and

$$(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1})'=C^{-1}_{(X_{n+1},D_{n+1})\subset(X_{n+2},D_{n+2})'}((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})')$$

.

Proof. Set the following notation:

$$\begin{aligned}(H,\nabla)&:=C^{-1}_{X_{n+1}\subset X_{n+2}}(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1});\\(H,\nabla)'&:=C^{-1}_{X_{n+1}\subset(X_{n+2}+\eta)}\big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})\big);\\(H,\nabla)''&:=C^{-1}_{X_{n+1}\subset(X_{n+2})}\big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})+\varepsilon\big).\end{aligned}$$

Then we need to show

$$(H,\nabla)'=(H,\nabla)+\beta(\eta),\tag{6.9}$$

and

$$(H,\nabla)''=(H,\nabla)+\alpha(\varepsilon).\tag{6.10}$$

By definition, (H,∇) is the de Rham bundle glued from local de Rham bundles

$$(H_i,\nabla_i)=\left(\Phi^*_{n+1,i}(\widetilde{H}),\frac{\Phi^*_{n+2,i}}{p}(\widetilde{\nabla})\right)$$

via local isomorphisms $G_{ij}\colon(H_i,\nabla_i)\to(H_j,\nabla_j)$ over $X_{n+1,i}\cap X_{n+1,j}$ with

$$G_{ij}(m\otimes_{\Phi^*_{n+1,i}}1)=\sum_I\widetilde{\nabla}_{n+1}(\partial)^I(m)\otimes_{\Phi^*_{n+1,j}}\frac{\Big(\Phi^*_{n+2,i}(t)-\Phi^*_{n+2,j}(t)\Big)^I}{I!\cdot p^{|I|}}\tag{6.11}$$

(6.9). . Let $(\eta_{ij})_{ij}$ be a 1-cocycle representing the class η. Then $X'_{n+2}:=X_{n+2}+\eta$ is the scheme constructed by gluing the local schemes $X_{n+2,i}$ via local isomorphisms $g_{ij}\colon X_{n+2,i}\cap X_{n+2,j}\to X_{n+2,i}\cap X_{n+2,j}$, which are the unique isomorphism determined by

$$g^*_{ij}(t_k)=t_k+p^{n+1}\cdot\eta_{ij}(\mathrm{d}t_k)$$

where $t_1,\cdots,t_d$ is a system local parameters on $X_{n+2,i}\cap X_{n+2,j}$. Then $(H,\nabla)'$ is the de Rham bundle glued from (H_i,∇_i) via local isomorphisms $G'_{ij}\colon(H_i,\nabla_i)\to(H_j,\nabla_j)$ with

$$G'_{ij}(m\otimes_{\Phi^*_{n+1,i}}1)=\sum_I\widetilde{\nabla}_{n+1}(\partial)^I(m)\otimes_{\Phi^*_{n+1,j}}\frac{\Big[\Phi^*_{n+2,i}(t)-\Psi^*_{n+2,j}\big((g^{-1}_{ij})^*(t)\big)\Big]^I}{I!\cdot p^{|I|}}\tag{6.12}$$

By definition, $b((V,\nabla),(V,\nabla)')=((G'_{ij}-G_{ij})_{ij},(0)_i)$. It is a representative of the class $\beta(\eta)$ by explicit computation. Thus (6.9) holds.

(6.10). According the construction of $\mathcal{T}$ in 6.7, one has

$$(\widetilde{H},\widetilde{\nabla})'' = (\widetilde{H},\widetilde{\nabla}) + \varepsilon.$$

where

$$(\widetilde{H},\widetilde{\nabla}) = \mathcal{T}_{n+1}\Big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(V_n,\nabla_n,\mathrm{Fil}_n)\Big) \text{ and } (\widetilde{H},\widetilde{\nabla})'' = \mathcal{T}_{n+1}\Big((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})+\varepsilon,(V_n,\nabla_n,\mathrm{Fil}_n)\Big).$$

In particular, for any fix local isomorphisms $f_i\colon \widetilde{H}_i \to \widetilde{H}''_i$; then the class ε is represented by

$$\Big((f_j - f_i)_{ij}, (\widetilde{\nabla}''_i \circ f_i - f_i \circ \widetilde{\nabla}_i)_i\Big).$$

Taking the Frobenius pullback of f_i, one gets local isomorphisms $\Phi^*_{n+1,i}f_i\colon \Phi^*_{n+1,i}\widetilde{H}_i \to \Phi^*_{n+1,i}\widetilde{H}''_i$. Thus by definition $b((V,\nabla),(V,\nabla)'')$ is represented by

$$\Big(\big(G''_{ij}\circ\Phi^*_{n+1,i}(f_i) - \Phi^*_{n+1,j}(f_j)\circ G_{ij}\big), \big(\frac{\Phi^*_{n+2,i}}{p}(\widetilde{\nabla}''_i)\circ\Phi^*_{n+1,i}(f_i) - \Phi^*_{n+1,j}(f_j)\circ\frac{\Phi^*_{n+2,i}}{p}(\widetilde{\nabla}_i)\big)\Big)$$

Then by explicit computation, one shows that it is a represetation of the class $\alpha(\varepsilon)$. Thus

$$(V,\nabla)'' = (V,\nabla) + \alpha(\varepsilon).$$

□

6.2. The condition of being *ordinary*

By Theorem 5.2 and Lemma 6.1, one has an injective map

$$\mathrm{Def}_{(V_n,\nabla_n,\mathrm{Fil}_n)}(X_{n+1},D_{n+1}) \hookrightarrow \mathrm{Def}_{(V_n,\nabla_n)}(X_{n+1},D_{n+1}).$$

The image is the kernel of ob_{Fil_n}, by the definition of ob_{Fil_n}. Set

$$\mathbb{K} = \mathbb{K}_k = \mathrm{IC}^{-1}(\mathrm{Def}_{(V_n,\nabla_n,\mathrm{Fil}_n)}(X_{n+1},D_{n+1})) = \ker(ob_{\mathrm{Fil}_n}\circ\mathrm{IC}).$$

In other words, $\mathbb{K}$ consists of pairs of a logarithmic Higgs bundle $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})$ on $(X_{n+1},D_{n+1})/_{S_{n+1}}$ lifting (E_n,θ_n) and a lift $(\hat{X}_{n+2},\hat{D}_{n+2})/S_{n+2}$ of the smooth pair $(X_{n+1},D_{n+1}/S_{n+1}$ such that the inverse Cartier of $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})$ with respect to the thickening $(X_{n+1},D_{n+1})\subset(X_{n+2},D_{n+2})$ is isomorphic to a de Rham bundle (V_{n+1},∇_{n+1}) on which Fil_n lifts. Also set

$$H = (\alpha,\beta)^{-1}(\mathbb{H}^1\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big)).$$

To aid the reader, we briefly recall what this is. Lemma 6.1, one has an short exact sequence

$$0 \to \mathbb{H}^1\Big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\Big) \xrightarrow{\iota} \mathbb{H}^1\Big(\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\Big) \xrightarrow{\pi} \mathbb{H}^1\Big(\mathscr{C}\Big) \to 0.$$

Thus H is just the kernel of the σ-semilinear map $\pi\circ(\alpha,\beta)$

$$H = \ker(\pi\circ(\alpha,\beta)).$$

In partcular, H is a k vector subspace of $\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)\oplus H^1(X_1,\mathcal{T}_{X_1})$.
By lemma 6.3, if $\mathbb{K}$ is not empty then it is an H-torsor.
(6.13)

$$\begin{array}{ccccc}
\mathbb{K} & \xrightarrow{IC} & \mathrm{Def}_{(V_n,\nabla_n,\mathrm{Fil}_n)}(X_{n+1},D_{n+1}) & \xrightarrow{\mathrm{Gr}} & \mathrm{Def}_{(E_n,\theta_n)}(X_{n+1},D_{n+1}) \\
\downarrow & & \downarrow & & \\
\mathrm{Def}_{(E_n,\theta_n)}(X_{n+1},D_{n+1})\times\mathrm{Def}_{(X_{n+1},D_{n+1})}(S_{n+2}) & \xrightarrow{IC} & \mathrm{Def}_{(V_n,\nabla_n)}(X_{n+1},D_{n+1}) & & \\
 & & \downarrow{\scriptstyle ob_{\mathrm{Fil}_n}} & & \\
 & & \mathbb{H}^1(\mathscr{C}) & &
\end{array}$$

(6.14)

$$\begin{array}{ccccc} H & \xrightarrow{(\alpha,\beta)} & \mathbb{H}^1\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big) & \xrightarrow{\mathrm{Gr}} & \mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big) \\ \big\downarrow & & \big\downarrow \iota & & \\ \mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big) \oplus H^1(X_1,\mathcal{T}_{X_1}) & \xrightarrow{(\alpha,\beta)} & \mathbb{H}^1\big(\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big) & & \\ & & \big\downarrow \pi & & \\ & & \mathbb{H}^1(\mathscr{C}) & & \end{array}$$

Theorem 6.4. *Suppose $\mathbb{K}$ is not empty and the projection $H \to \mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)$ is surjective. Then there exists some finite extension k'/k and $((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(X_{n+2},D_{n+2}))\in\mathbb{K}_{k'}$ such that*

$$\mathrm{Gr}\circ\mathrm{IC}((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(X_{n+2},D_{n+2})) = (\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}).$$

In other words, $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})$ is a 1-periodic Higgs bundle under the lifting $(X_{n+1},D_{n+1})\subset(X_{n+2},D_{n+2})$.

Proof. Since the projection $H \to \mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)$ is surjective, we may choose a linear section

$$(\mathrm{id},\tau)\colon \mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big) \to H \subset \mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)\oplus H^1(X_1,T_{X_1}).$$

Choose any $((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})',(X_{n+2},D_{n+2})')\in\mathbb{K}$. Denote $(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\widetilde{\mathrm{Fil}}_{n+1})' = \mathrm{IC}((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})',(X_{n+2},D_{n+2})')$. We identify (E_m,θ_n) with $\mathrm{Gr}(V_n,\nabla_n,\mathrm{Fil}_n)$ via the periodicity map φ. Then $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})'$ and $\mathrm{Gr}((\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\widetilde{\mathrm{Fil}}_{n+1}))'$ are both liftings of (E_n,θ_n). Let Δ be the element in $\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1)\big)$ such that

$$\mathrm{Gr}(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\widetilde{\mathrm{Fil}}_{n+1})' = (\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})'+\Delta.$$

For $\epsilon\in\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1)\big)$ denote $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}) = (\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})'+\epsilon$ and $(X_{n+2},D_{n+2}) = (X_{n+2},D_{n+2})'+\tau(\epsilon)$. By the choice of τ, $((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(X_{n+2},D_{n+2}))\in\mathbb{K}$. Denote $(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\widetilde{\mathrm{Fil}}_{n+1}) = \mathrm{IC}((\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}),(X_{n+2},D_{n+2}))$. Then

$$\begin{aligned}\mathrm{Gr}\circ C^{-1}_{(X_{n+1},D_{n+1})\subset(X_{n+2},D_{n+2})}(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1}) &= \mathrm{Gr}((\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\widetilde{\mathrm{Fil}}_{n+1}))\\ &= \mathrm{Gr}\Big((\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\widetilde{\mathrm{Fil}}_{n+1})'+\alpha(\epsilon)+\beta(\tau(\epsilon))\Big)\\ &= \mathrm{Gr}\big(\widetilde{V}_{n+1},\widetilde{\nabla}_{n+1},\widetilde{\mathrm{Fil}}_{n+1}\big)'+\mathrm{Gr}\circ\alpha(\epsilon)+\mathrm{Gr}\circ\beta\circ\tau(\epsilon)\\ &= (\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})+\Delta-\epsilon+\mathrm{Gr}\circ\alpha(\epsilon)+\mathrm{Gr}\circ\beta\circ\tau(\epsilon)\end{aligned}\tag{6.15}$$

Thus $(\widetilde{E}_{n+1},\widetilde{\theta}_{n+1})$ is periodic if and only if the following equation holds.

$$\Delta-\epsilon+\mathrm{Gr}\circ\alpha(\epsilon)+\mathrm{Gr}\circ\beta\circ\tau(\epsilon)=0.\tag{6.16}$$

Since Gr and τ are linear and α, β are σ-semilinear the equation 6.16 is of *Artin-Schreier* type. Equation 6.16 is solvable after extending the base field k, and hence the theorem follows. ☐

Lemma 6.5. *If $\pi\circ\beta\colon H^1(X_1,\mathcal{T}_{X_1/S_1})\to\mathbb{H}^1(\mathscr{C})$ is surjective, then so is the projection $H\to\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)$.*

Proof. For any $\varepsilon\in\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)$, $-\pi\circ\alpha(\varepsilon)\in\mathbb{H}^1(\mathscr{C})$. Since $\pi\circ\beta$ is surjective, there exists $\eta\in H^1(X_1,\mathcal{T}_{X_1/S_1})$ such that

$$-\pi\circ\alpha(\varepsilon)=\pi\circ\beta(\eta).$$

Since $H=(\alpha,\beta)^{-1}(\mathbb{H}^1\big(\mathrm{Fil}^0\mathrm{DR}(\mathscr{E}nd(V_1,\nabla_1))\big))=\ker(\pi\circ(\alpha,\beta))$, one has $(\varepsilon,\eta)\in H$. By the arbitrary choice of ε, the projection $H\to\mathbb{H}^1\big(\mathrm{Gr}^0\mathrm{DR}(\mathscr{E}nd(E_1,\theta_1))\big)$ is surjective. ☐

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