# Student Learning Seminar on Galois Deformations.

**Rafah Hajjar, October 25, 2023**

These notes are based on Mazur's paper *An Introduction to the Deformation Theory of Galois Representations* and Gouvea's book *Deformations of Galois representations*.

## 1 Review of deformation conditions

Let $\overline{\rho} : \Pi \to \mathrm{GL}_n(k)$ be a residual representation that satisfies $C(\overline{\rho}) = k$. We know that in this case the deformation functor $D_{\overline{\rho}} : \mathcal{C}_\Lambda \to \mathbf{Sets}$ defined by

$$D_{\overline{\rho}}(A) = \{\text{equivalence classes of liftings } \rho : \Pi \to \mathrm{GL}_n(A) \text{ of } \overline{\rho}\}$$

is representable.

**Definition 1.1.** A *deformation condition* for $\overline{\rho}$ is a property $Q$ that satisfies

1. $\overline{\rho}$ has the property $Q$
2. If $\rho : \Pi \to \mathrm{GL}_n(A)$ has the property $Q$ and $\alpha : A \to A'$, then $\alpha_* \rho$ has the property $Q$
3. If the pushforwards of $\rho : \Pi \to \mathrm{GL}_n(A \times_k B)$ to $A$ and $B$ have property $Q$, then so does $\rho$.

A deformation condition cuts out a subfunctor $D_Q$ of $D_{\overline{\rho}}$ represented by a quotient $R_{\mathbb{Q}}$ of $R_{\overline{\rho}}$. We denote by $t_Q := D_Q(k[\varepsilon])$ the Zariski tangent space of $D_Q$. As a vector space, $t_Q$ is isomorphic to a subspace of $t_{\overline{\rho}} \cong H^1(\Pi, \mathrm{Ad}(\overline{\rho}))$, and we often write

$$t_Q =: H^1_Q(\Pi, \mathrm{Ad}(\overline{\rho})) \subseteq H^1(\Pi, \mathrm{Ad}(\overline{\rho})).$$

Recall the deformation conditions that we discussed in the previous talk.

1. **Deformations with fixed determinant:** We say a deformation $\rho$ of $\overline{\rho}$ to $R$ has determinant $\delta : \Pi \to \Lambda^\times$ if $\det \rho = \iota_R \circ \delta$, where $\iota_R$ is the structure morphism $\iota_R : \Lambda \to R$. This is a deformation condition and we have
$$H^1_{\det=\delta}(\Pi, \mathrm{Ad}(\overline{\rho})) = H^1(\Pi, \mathrm{Ad}^0(\overline{\rho}))$$
2. **Categorical deformations:** Fix $\mathcal{P}$ a full subcategory of $\Lambda[[\Pi]]$-modules of finite length closed under passage to subobjects, quotients and finite direct sums. Being of type $\mathcal{P}$ is a deformation condition.
3. **Ordinary deformations:** Let $\rho : \Pi \to \mathrm{GL}_2(R)$ be a deformation given by the $\Pi$-module $V \cong R \times R$, and let $I \subset \Pi$ be a closed subgroup. We say $\rho$ (or $V$) is $I$-ordinary if $V^I \subset V$ is a free $R$-module of rank 1 and a direct summand of $V$. Being $I$-ordinary is a deformation condition, and
$$H^1_{I-\mathrm{ord}}(\Pi, \mathrm{Ad}(\overline{\rho})) = H^1(\Pi, \mathrm{Ad}_I(\overline{\rho})),$$
where $\mathrm{Ad}_I(\overline{\rho})$ is the subspace of $\mathrm{Ad}(\overline{\rho})$ of matrices which correspond to homomorphisms that factor through $V/V^I$.

## 2 Deformation conditions of global Galois representations

### 2.1 Motivation

Consider the following standard problem. We have a representation $\rho_{E,p} : G_{\mathbb{Q}}, S \to \mathrm{GL}_2(\mathbb{Z}_p)$ arising from the $p$-torsion points of an elliptic curve $E$, and we want to get some information about it (the most frequent being its modularity).

To do so, we aim to describe $\rho_{E,p}$ as a deformation of the residual representation $\overline{\rho}_{E,p}$, since we have better knowledge of these representations (for instance Serre's conjecture asserts that every representation of $G_{\mathbb{Q}}$ over a finite field is modular), and we can try to lift its properties to $\rho_{E,p}$ using the theory of Galois deformations.

For this reason, we want to make the deformation problem as tight as possible, meaning that we only consider deformations satisfying certain local deformation conditions, so that $\rho_{E,p}$ is still a deformation captured by our problem but at the same time so that every representation captured by our problem is modular.

This also motivates the choice of deformation conditions that we consider, since these are properties that the $\rho_{E,p}$ often satisfy. For instance, we know that the determinant of $\rho_{E,p}$ is the $p$-adic cyclotomic character, and $\rho_{E,p}$ is unramified outside of $pN_E$. If $E_{/\mathbb{Q}}$ is semistable, then $\rho_{E,p}$ is semistable in the sense that it is either flat or ordinary at $p$ and has the form $\rho\mid_{I_\ell} = (1 {*} 01)$ for $\ell \neq p$.

### 2.2 Global deformation problems

Let $\Lambda$ be a coefficient ring with residue field characteristic $p$. Let $\Pi = G_{K,S}$ be the Galois group of the maximal extension of $K$ unramified outside a finite set of places $S$ containing all primes above $p$.

**Definition 2.1.** By a *global Galois deformation problem* we mean a specification of a local deformation condition for each prime $\lambda \in S$. Concretely, given $\overline{\rho} : G_{K,S} \to \mathrm{GL}_n(k)$, we specify a local deformation condition for each restriction $\overline{\rho}\mid_{G_{K_\lambda}}$

**Lemma.** A global Galois deformation problem is a deformation condition for $\overline{\rho}$.

**Theorem 2.1.** The diagram

$$\begin{array}{ccc} H^1_Q(G_{K,S}, \mathrm{Ad}(\overline{\rho})) & \longrightarrow & H^1(G_{K,S}, \mathrm{Ad}(\overline{\rho})) \\ \downarrow & & \downarrow \\ \bigoplus_{\lambda \in S} H^1_{Q_\lambda}(G_{K_\lambda}, \mathrm{Ad}(\overline{\rho})) & \longrightarrow & \bigoplus_{\lambda \in S} H^1(G_{K_\lambda}, \mathrm{Ad}(\overline{\rho})) \end{array}$$

is Cartesioan, so it identifies $H^1_Q(G_{K,S}, \mathrm{Ad}(\overline{\rho}))$ with the set of cocycles of $H^1_Q(G_{K,S}, \mathrm{Ad}(\overline{\rho}))$ which, for each $\lambda \in S$, map under restriction to the image of $H^1_{Q_\lambda}(G_{K_\lambda}, \mathrm{Ad}(\overline{\rho}))$.

This makes $H^1_Q(G_{K,S}, \mathrm{Ad}(\overline{\rho}))$ into a Selmer group, which is simply a part of the global cohomology group defined by local conditions for each $\lambda \in S$.

## 2.3 Representations that are ordinary at $p$

Let $\overline{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(k)$, and assume $\overline{\rho}$ is ordinary at $p$, meaning that its restriction to $G_{\mathbb{Q}_p}$ is $I_p$-ordinary in the sense defined in the previous section. Let $D^{\mathrm{ord}}$ be the functor

$$D^{\mathrm{ord}}(R) = \{\text{deformations of } \overline{\rho} \text{ to } R \text{ which are ordinary at } p\}$$

This is representable, and we write $R^0(\rho)$ for the universal ordinary deformation ring. This case is particularly interesting because of several things, the first being the following

**Theorem 2.2** (Wiles)**.** If $\overline{\rho}$ is modular, then any ordinary deformation of $\overline{\rho}$ is also modular.

This is proven by identifying the ring $R^0(\overline{\rho})$ with a certain $p$-adic Hecke algebra. Also, the homomorphism $R(\overline{\rho}) \to R^0(\overline{\rho})$ is well understood in many cases. One has

**Theorem 2.3** (Mazur, Martin)**.** Let $S = \{p, \infty\}$, $\overline{\rho} : G_{\mathbb{Q},S}$ ordinary at $p$. Suppose either that

- $\det \overline{\rho} \neq 1, \omega, \omega^{-1}, \omega^{\frac{p-1}{2}}$, or
- $\overline{\rho}$ is tamely ramified,

then the kernel of $R(\overline{\rho}) \to R^0(\overline{\rho})$ is generated by two elements.

Under a few more reasonable hypotheses, one can show that $R(\overline{\rho})$ is a power series ring in two variables over $R^0(\overline{\rho})$. In practical terms, studying ordinary deformations can lead to results about all deformations.

## 2.4 Application: Fermat's last theorem

**Theorem 2.4.** If $p \geq 5$ is prime and $a, b, c \in \mathbb{Z}$, then $a^p + b^p + c^p = 0 \implies abc = 0$

*Proof.* Assume there is a solution to $a^p + b^p + c^p$ for $p \geq 5$ with $abc \neq 0$, and assume WLOG $a \equiv -1 \pmod 4$ and $2 \mid b$. Let $E$ be the elliptic curve

$$E : y^2 = x(y - a^p)(x + b^p)$$

and let $\rho_{E,p}$ be the associated $p$-adic Galois representation. One can show that

- $\overline{\rho}_{E,p}$ is absolutely irreducible
- $\overline{\rho}_{E,p}$ is odd
- $\overline{\rho}_{E,p}$ is unramified outside $2p$, flat at $p$ and semistable at 2

By a theorem of Ribet, we know that no such Galois representation can be modular. Hence if we prove that $\rho_{E,p}$ is modular we get the contradiction. To do so, consider the global deformation problem for $\overline{\rho}$ given by the following deformation conditions: we say $\rho$ is of type $Q$ for some set of primes $\Sigma_Q$ disjoint of $S$ if

- $\rho$ has determinant $\chi_p$.

- $\rho$ is unramified outside $S \cup \{p\} \cup \Sigma_Q$.
- $\rho$ is semistable outside $\Sigma_Q$.
- If $p \notin \Sigma_Q$ and $\overline{\rho}$ is flat at $p$, then $\rho$ is also flat at $p$.

Under a few more hypotheses on the residual representation $\overline{\rho}$ that are satisfied by $\overline{\rho}_{E,p}$, Wiles shows that every deformation of type $Q$ is modular, by explicitly constructing a Hecke algebra $\mathcal{T}_Q$ and an isomorphism $R_Q \xrightarrow{\sim} \mathcal{T}_Q$. The proof concludes by showing that $\rho_{E,p}$ is of type $Q$ $\square$