KUDLA-RAPOPORT CONJECTURE FOR KRÄMER MODELS

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Abstract. In this paper, we propose a modified Kudla-Rapoport conjecture for the Krämer model of unitary Rapoport-Zink space at a ramified prime, which is a precise identity relating intersection numbers of special cycles to derivatives of Hermitian local density polynomials. We also introduce the notion of special difference cycles, which has surprisingly simple description. Combining this with induction formulas of Hermitian local density polynomials, we prove the modified Kudla-Rapoport conjecture when $n = 3$. Our conjecture, combining with known results at inert and infinite primes, implies arithmetic Siegel-Weil formula for all non-singular coefficients when the level structure of the corresponding unitary Shimura variety is defined by a self-dual lattice.

Contents

1. Introduction

In their seminal work [KR11] and [KR14b], Kudla and Rapoport made a conjectural local arithmetic Siegel-Weil formula (the Kudla-Rapoport conjecture) relating the intersection numbers of special divisors on unitary Rapoport-Zink spaces to the central derivative of certain local density

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polynomials. A unitary Rapoport-Zink (RZ) space is a local version of a unitary Shimura variety associated to a general unitary group $\mathrm{GU}(1, n-1)$. The Kudla-Rapoport conjecture plays a central role in the arithmetic Siegel-Weil formula for unitary Shimura varieties, which was first proposed by Kudla in [Kud97] for orthogonal Shimura varieties. When $n = 1$ or 2, the Kudla-Rapoport conjecture was proved in [KR11]. The case when $n = 3$ was proved in [Ter10]. The general case was proved recently in [LZ22a] by an ingenious induction. The Archimedean analogue of the Kudla-Rapoport conjecture was proved in [Liu11] and [GS19]. The analogue of the Kudla-Rapoport conjecture for GSpin Rapoport-Zink space is formulated and proved in [LZ22b].

Originally the Kudla-Rapoport conjecture was proposed only for good primes, namely inert primes over which the Rapoport-Zink space has hyperspecial level structure. A modified Kudla-Rapoport conjecture for Rapoport-Zink space with minuscule parahoric level structure over inert primes has been proposed in [Cho22]. For ramified primes, there are two kinds of well-understood arithmetic models of RZ spaces. One is the exotic smooth model which has good reduction, the other is the Krämer model proposed in [Krä03] which only has semi-stable reduction. The analogue of Kudla-Rapoport conjecture for the even dimensional exotic smooth model was studied in [LL22], in which case the conjecture can be proved by the same strategy as [LZ22a]. For the Krämer model, however, it was expected that serious modification of the original Kudla-Rapoport conjecture is needed. A precise formulation has not previously been known. One of the main goals of this paper is to formulate a precise conjecture (Conjecture 1.1) based on earlier work of [Shi22] and [HSY23] for the case $n = 2$. We then prove Conjecture 1.1 for $n = 3$.

In a very recent joint work with Chao Li ([HLSY22]), we proved the conjecture completely. One of the major innovations of [HLSY22] is a decomposition formula of primitive local density polynomials, which is inspired by the results in the appendix of this work. The geometric side of the 'horizontal' part in [HLSY22] essentially follows from the current work. To deal with the vertical part in general, [HLSY22] uses partial Fourier transform inspired by [LZ22b]. The current work uses explicit computation instead. Finally, it was discovered in [HLSY22] that the 'central' derivative of the primitive local density polynomials has a surprisingly simple formula.

1.1. **The naive conjecture.** Let p be an odd prime and F be a ramified quadratic field extension of a p-adic number field F_0 with residue field $\mathbb{F}_q$. Fix an algebraic closure k of $\mathbb{F}_q$. Fix a uniformizer π of F such that $\pi_0 = \pi^2$ is a uniformizer of F_0 and let v_π be the valuation on F such that $\mathrm{v}_\pi(\pi) = 1$. Let $\breve{F}_0$ be the completion of a maximal unramified extension of F_0 and $\breve{F} \coloneqq F \otimes_{F_0} \breve{F}_0$. Let $\mathcal{O}_{\breve{F}}$ and $\mathcal{O}_{\breve{F}_0}$ be the rings of integers of $\breve{F}$ and $\breve{F}_0$ respectively. For a Hermitian lattice or space M of rank n, we define its sign as

$$\chi(M) = \chi((-1)^{\frac{n(n-1)}{2}} \det(M)) = \pm 1 \tag{1.1}$$

where χ is the quadratic character of $F_0^\times$ associated to F/F_0. We call M split or non-split depending on whether $\chi(M) = 1$ or -1. For a Hermitian matrix T, define $\chi(T)$ to be the sign of its associated Hermitian lattice.

Let $\mathbb{Y}$ and $\mathbb{X}$ be pre-fixed framing Hermitian formal $\mathcal{O}_F$-modules of signature $(0,1)$ and $(1, n-1)$ respectively over $\operatorname{Spec} k$. Recall that Hermitian formal $\mathcal{O}_F$-modules are a particular kind of formal p-divisible groups with $\mathcal{O}_F$-action, see Section 2.1. The space of special quasi-homomorphisms

$$\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.2}$$

is equipped with a Hermitian form $h(,)$, see (2.2). Let $\epsilon = \chi(\mathbb{V})$. The Rapoport-Zink space $\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}$ parameterizes certain classes of supersingular Hermitian formal $\mathcal{O}_F$-modules of signature $(1, n-1)$ over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$, see Section 2.1. It is a formal scheme over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$ with semi-stable reduction and can be viewed as a regular model of the formal completion of the corresponding global unitary Shimura variety along its basic locus over p. When n is odd, $\mathcal{N}^{\mathrm{Kra}}_{n,1}$ is isomorphic to $\mathcal{N}^{\mathrm{Kra}}_{n,-1}$. We often write $\mathcal{N}^{\mathrm{Kra}}$ instead of $\mathcal{N}^{\mathrm{Kra}}_{n,\epsilon}$ for simplicity.

For each subset $L \subset \mathbb{V}$, define $\mathcal{Z}^{\mathrm{Kra}}(L)$ to be the formal subscheme of $\mathcal{N}^{\mathrm{Kra}}$ where $\mathbf{x}$ deforms to a homomorphism for any $\mathbf{x} \in L$. Let $L \subset \mathbb{V}$ be an $\mathcal{O}_F$-lattice of rank r. We say L is integral if $h(,)|_L$ is non-degenerate and takes values in $\mathcal{O}_F$. Let $\mathbf{x}_1, \dots, \mathbf{x}_r$ be a basis of L. We define

$$ {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_r)}] \in K_0(\mathcal{N}^{\mathrm{Kra}}) \tag{1.3}$$

where $\otimes^{\mathbb{L}}$ is the derived tensor product of complex of coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and $K_0(\mathcal{N}^{\mathrm{Kra}})$ is the Grothendieck groups of finite complexes of coherent locally free sheaves on $\mathcal{N}^{\mathrm{Kra}}$. By [How19, Corollary C], ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)$ is independent of the choice of basis of L. When L has rank n, we define the intersection number

$$ \mathrm{Int}(L) = \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)) \tag{1.4}$$

where χ is the Euler characteristic. One can show that $\mathrm{Int}(L)$ is finite, see Lemma 2.14.

Let L and M are Hermitian lattices of rank n and m respectively. Moreover, we assume $\mathrm{v}(M) \coloneqq \min\{\mathrm{v}_\pi(h(v,v')) \mid v, v' \in M\} \geq -1$. We use $\mathrm{Herm}_{L,M}$ to denote the scheme of Hermitian $\mathcal{O}_F$-module homomorphisms from L to M, which is a scheme of finite type over $\mathcal{O}_{F_0}$. More specifically, for an $\mathcal{O}_{F_0}$-algebra R, we define

$$ L_R \coloneqq L \otimes_{\mathcal{O}_{F_0}} R, \quad (x \otimes a, y \otimes b)_R \coloneqq \pi(x,y) \otimes_{\mathcal{O}_{F_0}} ab \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R \text{ where } x, y \in L, a, b \in R.$$

Then

$$ \mathrm{Herm}_{L,M}(R) = \{\phi \in \mathrm{Hom}_{\mathcal{O}_F}(L_R, M_R) \mid (\phi(x), \phi(y))_R \equiv (x,y)_R \text{ for all } x, y \in L_R\}.$$

To simplify the notation, let $I(M, L, d)$ denote $\mathrm{Herm}_{L,M}(\mathcal{O}_{F_0}/(\pi_0^d))$. Then direct calculation shows that

$$ \alpha(M, L) = q^{-dn(2m-n)} |I(M, L, d)| \tag{1.5}$$

become constant for sufficiently large integers $d > 0$. We call it the local density (of M representing L).

Let $\mathcal{H}$ be the (Hermitian) hyperbolic plane with Gram matrix $\mathcal{H} = \begin{pmatrix} 0 & \pi^{-1} \\ -\pi^{-1} & 0 \end{pmatrix}$. One can show that there is a (local density) polynomial $\alpha(M, L, X) \in \mathbb{Q}[X]$ such that

$$ \alpha(M \oplus \mathcal{H}^k, L) = \alpha(M, L, q^{-2k}).$$

Define its derivative by

$$ \alpha'(M, L) \coloneqq -\frac{\partial}{\partial X} \alpha(M, L, X)|_{X=1}.$$

Since $\alpha(M, L, X)$ (resp. $\alpha'(M, L)$) only depends on their gram matrices S and T, we will also denote it by $\alpha(S, T, X)$ (resp. $\alpha'(S, T)$). Let M be the unique unimodular Hermitian $\mathcal{O}_F$-lattice of rank n with $\chi(M) = -\chi(L)$. The naive analogue of the local Kudla-Rapoport conjecture is

$$ \mathrm{Int}(L) = 2\frac{\alpha'(M, L)}{\alpha(M, M)}. \tag{1.6}$$

But this conjectural formula is not even true for $n = 2$ according to the main theorem of [HSY23]. The analytic side of the conjecture needs to be modified.

1.2. **The precise conjecture.** By [Shi18, Theorem 1.2], $\mathcal{Z}^{\mathrm{Kra}}(L)$ is empty when L is not integral, so we have

$$\mathrm{Int}(L)=0.$$

On the analytic side, the right hand side of (1.6) is automatically zero only when $\mathrm{v}(L)\leq -2$, and is sometimes non-zero when $\mathrm{v}(L)=-1$. So there should be correction terms involving Hermitian lattices M with $\mathrm{v}(M)=-1$. By [Jac62], there are $n-1$ equivalent classes of Hermitian lattices which are direct sum of copies of $\mathcal{H}$ and unimodular lattices:

$$\mathcal{H}_{n,i}^{\epsilon}\coloneqq \mathcal{H}^i\obot I_{n-2i}^{\epsilon}\quad \text{for } 1\leq i\leq \frac{n}{2},\quad \epsilon=\pm 1 \tag{1.7}$$

where we use I_{n-2i}^{ϵ} to denote the unimodular Hermitian lattice of rank $n-2i$ with $\chi(I_{n-2i}^{\epsilon})=\chi(\mathcal{H}_{n,i}^{\epsilon})=\epsilon$. When $n=2r$ is even, we take $I_{0,\epsilon}=0$ and $\mathcal{H}_{n,r}^{1}=\mathcal{H}^r$. Then the local arithmetic Siegel-Weil formula, a.k.a. the KR-conjecture at a ramified prime should be of the following form:

$$\mathrm{Int}(L)=2\frac{\alpha'(I_n^{-\epsilon},L)}{\alpha(I_n^{-\epsilon},I_n^{-\epsilon})}+\sum_i c_{n,i}^{\epsilon}\frac{\alpha(\mathcal{H}_{n,i}^{\epsilon},L)}{\alpha(I_n^{-\epsilon},I_n^{-\epsilon})}, \tag{1.8}$$

where $\epsilon=\chi(L)$. Since $\mathrm{Int}(\mathcal{H}_{n,j}^{\epsilon})=0$, we should have

$$2\frac{\alpha'(I_n^{-\epsilon},\mathcal{H}_{n,j}^{\epsilon})}{\alpha(I_n^{-\epsilon},I_n^{-\epsilon})}+\sum_i c_{n,i}^{\epsilon}\frac{\alpha(\mathcal{H}_{n,i}^{\epsilon},\mathcal{H}_{n,j}^{\epsilon})}{\alpha(I_n^{-\epsilon},I_n^{-\epsilon})}=0. \tag{1.9}$$

This system of equations turns out to determine the coefficients $c_{n,i}^{\epsilon}$ uniquely by Theorem 6.1. We propose the following Kudla-Rapoport conjecture at a ramified prime.

Conjecture 1.1. *The identity* (1.8) *always holds with the coefficients* $c_{n,i}^{\epsilon}$ *uniquely determined by (1.9).*

For convenience, we set

$$\partial\mathrm{Den}(L)=2\frac{\alpha'(I_n^{-\epsilon},L)}{\alpha(I_n^{-\epsilon},I_n^{-\epsilon})}+\sum_i c_{n,i}^{\epsilon}\frac{\alpha(\mathcal{H}_{n,i}^{\epsilon},L)}{\alpha(I_n^{-\epsilon},I_n^{-\epsilon})}. \tag{1.10}$$

We remark that since $\mathrm{Int}(L)$ is always an integer, the conjecture 1.1 suggests that $\partial\mathrm{Den}(L)$ should be an integer, which is already not obvious.

The conjecture holds for $\mathcal{N}_{2,\pm 1}^{\mathrm{Kra}}$ by results in [Shi22] and [HSY23]. In this paper, we will prove the conjecture for $n=3$ and provide some partial results in general case.

Theorem 1.2. *Conjecture 1.1 is true when* $n=3$.

1.3. **Special difference cycles.** One of the novelty of the paper is the concept of special difference cycles. Let L_1 be an $\mathcal{O}_F$-lattice of $\mathbb{V}$ of rank $n_1\leq n$. Define the special difference cycle $\mathcal{D}(L_1)\in K_0(\mathcal{N}^{\mathrm{Kra}})$ by

$$\mathcal{D}(L_1)={}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L_1)+\sum_{i=1}^{n_1}(-1)^i q^{i(i-1)/2}\sum_{\substack{L_1\subset L'\subset\frac{1}{\pi}L_1\\ \dim_{\mathbb{F}_q}(L'/L_1)=i}}{}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L')\in K_0(\mathcal{N}^{\mathrm{Kra}}). \tag{1.11}$$

Here $\mathcal{D}(L_1)$ can be seen as a higher codimensional analogue of the difference divisor first introduced in [Ter10, Definition 2.10]. By the definition and a q-adic linear-algebraic inclusion-exclusion principle, we have (see Lemma 2.16)

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L_1)=\sum_{\substack{L'\text{integral}\\ L_1\subset L'\subset L_{1,F}}}\mathcal{D}(L'). \tag{1.12}$$

Here $L_F = L \otimes_{\mathcal{O}_F} F$ for an $\mathcal{O}_F$-lattice L. The above summation is in fact finite. Assume that we have a decomposition $L = L_1 \oplus L_2$ of $\mathcal{O}_F$-lattices such that L_i has rank n_i and $n_1 + n_2 = n$. Define

$$\mathrm{Int}(L)^{(n_1)} = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(L_2)) \tag{1.13}$$

where $\cdot$ is the product on $K_0(\mathcal{N}^{\mathrm{Kra}})$ induced by tensor product of complexes. Notice that this definition in fact depends on the decomposition of L.

On analytic side, we define

$$\partial\mathrm{Den}(L)^{(n_1)} := \partial\mathrm{Den}(L) - \sum_{i=1}^{n_1} (-1)^{i-1} q^{i(i-1)/2} \sum_{\substack{L_1 \subset L_1' \subset L_{1,F} \\ \dim L_1'/L_1 = i}} \partial\mathrm{Den}(L_1' \oplus L_2). \tag{1.14}$$

This definition again depends on the decomposition of L. What motivates the definition of $\partial\mathrm{Den}(L)^{(n_1)}$ and $\mathcal{D}(L_1)$ is the fact that $\partial\mathrm{Den}(L)^{(n_1)}$ is equal to the derivative of certain primitive local density polynomials, see [Kat99, Proposition 2.1] or Theorem 5.2 below. The analogue of (1.12) holds for $\partial\mathrm{Den}(L)^{(n_1)}$. As a consequence we have the following theorem (see Theorem 5.6 for a refinement).

Theorem 1.3. *Conjecture 1.1 is true if and only if for every lattice $L = L_1 \oplus L_2$ such that L_i has rank n_i, we have*

$$\mathrm{Int}(L)^{(n_1)} = \partial\mathrm{Den}(L)^{(n_1)}. \tag{1.15}$$

We speculate that $\mathcal{D}(L_1)$ is of a simple form when $n_1 = n-1$. One strong evidence for this is that the 'horizontal' part of $\mathcal{D}(L_1)$ is either empty or isomorphic to one or two copies of $\mathrm{Spf}\, W_s$ where W_s is the integer ring of an extension of $\breve{F}$ of degree q^s, see Proposition 4.6. Another evidence is that the intersection of $\mathcal{D}(L_1)$ with an exceptional divisor in $\mathcal{N}^{\mathrm{Kra}}$ is ± 1 or 0, see Lemma 3.9. When $n = 3$, we show that $\mathcal{D}(L_1)$ has a simple decomposition, see Theorem 1.4 below.

1.4. **The case $n = 3$.** The proof of Theorem 1.2 is divided into three cases, see Section 11. For $\mathrm{v}(L) < 0$, we show directly $\partial\mathrm{Den}(L) = \mathrm{Int}(L) = 0$. The case $\mathrm{v}(L) = 0$ is reduced to the case $n = 2$, which was proved in [Shi22] and [HSY23]. For $\mathrm{v}(L) > 0$, we prove that $\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}$ for a decomposition $L = L^\flat \oplus \mathrm{Span}\{\mathbf{x}\}$, and then apply Theorem 1.3 (more precisely Theorem 5.6).

In order to prove $\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}$, we need to understand the decomposition of $\mathcal{D}(L^\flat)$. We say a lattice $\Lambda \subset \mathbb{V}$ is a vertex lattice if $\pi\Lambda \subseteq \Lambda^\sharp \subseteq \Lambda$ where $\Lambda^\sharp$ is dual lattice of Λ with respect to $h(,)$ and we call $t = \dim_{\mathbb{F}_q}(\Lambda/\Lambda^\sharp)$ the type of Λ. This has to be an even integer between 0 and n. We denote the set of vertex lattices of type t by $\mathcal{V}^t$. When $n = 3$, a type 2 lattice Λ_2 corresponds to a line $\tilde{\mathcal{N}}_{\Lambda_2} \cong \mathbb{P}^1_k$ in $\mathcal{N}_3^{\mathrm{Kra}}$ and a type 0 lattice Λ_0 corresponds to a divisor $\mathrm{Exc}_{\Lambda_0} \cong \mathbb{P}^2_k$. Let H_{Λ_0} be the hyperplane class of Exc_{Λ_0}. We have the following theorem.

Theorem 1.4. *If $\mathrm{v}(L^\flat) > 0$, we have the following decomposition of cycles in $\mathrm{Gr}^2 K_0(\mathcal{N}_3^{\mathrm{Kra}})$*

$$\mathcal{D}(L^\flat) = \sum_{\substack{\Lambda_2 \in \mathcal{V}^2 \\ L^\flat \subset \Lambda_2^\sharp}} (2[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\substack{\Lambda_0 \in \mathcal{V}^0 \\ \Lambda_0 \subset \Lambda_2}} H_{\Lambda_0})$$

where $\mathrm{Gr}^\bullet K_0(\mathcal{N}_3^{\mathrm{Kra}})$ is the associated graded ring of $K_0(\mathcal{N}_3^{\mathrm{Kra}})$ with respect to the codimension filtration.

Theorem 1.4 is proved by intersecting $\mathcal{D}(L^\flat)$ with special divisors that are isomorphic to $\mathcal{N}_{2,-1}^{\mathrm{Kra}}$ and computing the intersection numbers in two different ways. One way relates the intersection numbers to the main result of [Shi22]. The other way uses the decomposition in Theorem 1.4 and detects the multiplicity of each component that shows up on the right hand side.

1.5. **Global application.** In the last part of the paper, we apply the local results above to the global intersection problem proposed by [KR14b]. For brevity and clarity of exposition we restrict our attention to the case when F is an imaginary quadratic field. We remark here that our results can be applied to the case when F is a general CM field given correct local assumptions. Let $\mathcal{M}^{\mathrm{Kra}}_{(1,n-1)}$ be the moduli functor over $\operatorname{Spec}\mathcal{O}_F$ which parametrizes principally polarized abelian varieties A with an action $\iota : \mathcal{O}_F \to \operatorname{End}(A)$, a compatible polarization $\lambda : A \to A^\vee$ and a filtration $\mathcal{F}_A \subset \operatorname{Lie} A$ which satisfies the signature $(1, n-1)$ condition (see §12.1). Let V be a Hermitian vector space over F of signature $(n-1, 1)$ containing a self-dual lattice L. The lattice L determines an open and closed substack

$$\mathcal{M} \subset \mathcal{M}_{(0,1)} \times_{\operatorname{Spec}\mathcal{O}_F} \mathcal{M}^{\mathrm{Kra}}_{(1,n-1)}$$

which is an integral model of a unitary Shimura variety. For a point in $\mathcal{M}(S)$ (S an $\mathcal{O}_F$-scheme), i.e., a pair $(E, \iota_0, \lambda_0) \in \mathcal{M}_{(0,1)}(S)$, $(A, \iota, \lambda, \mathcal{F}_A) \in \mathcal{M}^{\mathrm{Kra}}_{(1,n-1)}(S)$, define the locally free $\mathcal{O}_F$-module

$$V'(E, A) = \operatorname{Hom}_{\mathcal{O}_F}(E, A).$$

It is equipped with the Hermitian form $h'(x, y) = \iota_0^{-1}(\lambda_0^{-1} \circ y^\vee \circ \lambda \circ x)$. For a $m \times m$ nonsingular Hermitian matrix T with values in $\mathcal{O}_F$, let $\mathcal{Z}(T)$ be the stack of collections $(E, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A, \mathbf{x})$ such that $(E, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A) \in \mathcal{M}(S)$, $\mathbf{x} \in V'(E, A)^m$ with $h'(\mathbf{x}, \mathbf{x}) = T$. Then $\mathcal{Z}(T)$ is representable by a Deligne-Mumford stack which is finite and unramified over $\mathcal{M}$ ([KR14b, Proposition 2.9]). When $t \in \mathbb{Z}_{>0}$, each component of $\mathcal{Z}(t)$ can be viewed as a divisor by [How15, Proposition 3.2.3]. In general, $\mathcal{Z}(T)$ does not necessarily have the expected codimension which is the rank of T.

Let $\mathcal{C} = \{\mathcal{C}_p\}$ be an incoherent collection of local Hermitian spaces of rank n associated to V such that $\mathcal{C}_\ell \cong V_\ell$ for all finite ℓ and $\mathcal{C}_\infty$ is positive definite. It is "incoherent" in the sense that it does not come from a global Hermitian space. For a nonsingular Hermitian matrix T of rank n with values in $\mathcal{O}_F$, Let V_T be the Hermitian space with gram matrix T. Define

$$\operatorname{Diff}(T, \mathcal{C}) \coloneqq \{p \text{ a place of } \mathbb{Q} \mid \mathcal{C}_p \text{ is not isomorphic to } (V_T)_p\}. \tag{1.16}$$

Then $\mathcal{Z}(T)$ is empty if $|\operatorname{Diff}(T, \mathcal{C})| > 1$. If $\operatorname{Diff}(T, \mathcal{C}) = \{p\}$ for a finite prime p inert or ramified in F, then the support of $\mathcal{Z}(T)$ is on the supersingular locus of $\mathcal{M}$ over $\operatorname{Spec}\mathbb{F}_p$. Let e be the ramification index of $F_p/\mathbb{Q}_p$. Define the arithmetic degree

$$\widehat{\deg}_T = \chi(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(t_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_n)}) \cdot \log p^{2/e}, \tag{1.17}$$

where $\otimes^{\mathbb{L}}$ stands for derived tensor product on the category of coherent sheaves on $\mathcal{M}$, χ is Euler-characteristic and t_i ($1 \le i \le n$) are the diagonal entries of T. When $\operatorname{Diff}(T, \mathcal{C}) = \{\infty\}$, then in fact $\mathcal{Z}(T)$ is empty and one can use integration of a green current to define the arithmetic degree $\widehat{\deg}_T(v)$ with the parameter v being a positive definite hermitian matrix v of order n (which will be imaginary part of τ), see for example [LZ22a, §15.3].

On the analytic side, we consider an incoherent Eisenstein series $E(\tau, s, \Phi)$ for a non-standard section Φ in a degenerate principal series representation of $\mathrm{U}(n, n)(\mathbb{A})$, see Section 12.2. Here τ is in the Hermitian Siegel upper half space

$$\mathbb{H}_n = \{\tau = u + iv \mid u \in \operatorname{Herm}_n, v \in \operatorname{Herm}_{n,>0}\}, \tag{1.18}$$

where Herm_n (resp. $\operatorname{Herm}_{n,>0}$) is the set of $n \times n$ (positive definite) Hermitian matrices with values in $\mathbb{C}$ and $s \in \mathbb{C}$. Our local conjecture and result imply the following theorem, which extends [LZ22a, Theorem 1.3.1] to include ramified primes.

Theorem 1.5. *(Arithmetic Siegel-Weil formula for non-singular coefficients) Assume that the fundamental discriminant of F is $d_F \equiv 1 \pmod 8$ and that Conjecture 1.1 holds for every F_p with*

$p|d_F$. For any nonsingular Hermitian matrix T with values in $\mathcal{O}_F$ of size n, we have

$$E'_T(\tau,0,\Phi) = C\cdot \widehat{\deg}_T(v)\cdot q^T,\quad q^T=\exp(2\pi i \mathrm{tr}(T\tau)),$$

where $E'_T(\tau,0,\Phi)$ is the T-th Fourier coefficient of $E'(\tau,0,\Phi)$ and C is a constant that only depends on F and L. In particular, the arithmetic Siegel-Weil formula holds for $n=2,3$ for non-singular T.

In a very recent joint work with Chao Li ([HLSY22]), we proved Conjecture 1.1, and so this theorem is now unconditional.

1.6. **Notations.** For $\mathcal{O}_F$-lattices (resp. $\mathcal{O}_{\breve{F}}$-lattices) L and L', we write $L \overset{t}{\subset} L'$ if $L\subset L'\subset \frac{1}{\pi}L$ and $\dim_{\mathbb{F}_q}(L'/L)=t$ (resp. $\dim_k(L'/L)=t$). We say a vector $v\in L$ is primitive if $\frac{1}{\pi}v\notin L$.

Through out the paper, we always assume a Hermitian lattice is non-degenerate. For Hermitian lattices L and L', we use $L\obot L'$ to denote orthogonal direct sum, and $L\oplus L'$ as direct sum of lattices. Given a Hermitian lattice L with Hermitian form $(\ ,\ )$, we consider two different dual lattices of L. We use $L^\sharp$ (resp. $L^\vee$) to denote the dual lattice of L with respect to $(,)$ (resp. $\mathrm{tr}_{F/F_0}(,)$). Recall that $\mathrm{v}(L)$ is defined to be $\min\{\mathrm{v}_\pi(h(v,v'))\mid v,v'\in L\}$. For each Hermitian lattice L, there exists a Jordan decomposition $L=\obot_{i\ge s}L_i$ such that $L_i^\sharp=\pi^{-i}L_i$. We call L integral if $s\ge 0$. For an integral lattice L, we define

$$t(L) \coloneqq \sum_{i\ge 1}\mathrm{rank}_{\mathcal{O}_F}(L_i).$$

Following [LL22, Definition 2.11], for a lattice L with hermitian form $(\ ,\ )$, we may find a basis of L whose Gram matrix is

$$\left(\beta_1\pi^{2b_1}\right)\oplus\cdots\oplus\left(\beta_s\pi^{2b_s}\right)\oplus\begin{pmatrix}0 & \pi^{2c_1+1}\\ -\pi^{2c_1+1} & 0\end{pmatrix}\oplus\cdots\oplus\begin{pmatrix}0 & \pi^{2c_t+1}\\ -\pi^{2c_t+1} & 0\end{pmatrix}$$

for some $\beta_1,\dots,\beta_s\in\mathcal{O}_{F_0}^\times$ and $b_1,\dots,b_s,c_1,\dots,c_t\in\mathbb{Z}$. Moreover, we define its (unitary) fundamental invariants $(a_1,\cdots,a_n)$ to be the unique nondecreasing rearrangement of $(2b_1,\cdots,2b_s,2c_1+1,\cdots,2c_t+1)$. The partial order of $\mathbb{Z}^n$ induces a partial order on the set of fundamental invariants.

Let $\mathcal{H}_i=\begin{pmatrix}0 & \pi^i\\ (-\pi)^i & 0\end{pmatrix}$ and $\mathcal{H}=\mathcal{H}_{-1}$. We also use it to denote a Hermitian lattice with Gram matrix $\mathcal{H}_i$. Given a Hermitian lattice M, we use $M^{[k]}$ to denote $M\obot\mathcal{H}^k$. We use I_m^ϵ to denote a unimodular Hermitian lattice of rank m and $\chi(I_m^\epsilon)=\epsilon$. For a Hermitian matrix T, we define $\mathrm{v}(T)=\mathrm{v}(L)$ where L is a lattice whose Gram matrix is T. We use $\mathrm{Herm}_n(F)$ to denote the set of Hermitian matrices over F of size n. When there is no confusion, we also simply denote it as Herm_n. For $T,T'\in\mathrm{Herm}_n(F)$, we say T is equivalent to T' if there is a $U\in\mathrm{GL}_n(\mathcal{O}_F)$ such that $U^*TU=T'$, where $U^*={}^t\bar{U}$. In this case, we denote it as $T\approx T'$.

For $t\in\mathcal{O}_{F_0}$, let $\mathrm{v}(t)\coloneqq\mathrm{val}_{\pi_0}(t)$ and write $t=t_0(-\pi_0)^{\mathrm{v}(t)}$. For $x\in\mathbb{V}$, we set $q(x)=(x,x)$ and $\mathrm{v}(x)=\mathrm{v}(q(x))$. We use $\langle t\rangle$ to denote a lattice $\mathcal{O}_Fx$ of rank one with $q(x)=t$.

The notations in each section that are not mentioned here will be explained at the very beginning of the section.

1.7. **The structure of the paper.** The paper is divided into three parts. In Part 1, we prove some facts about special cycles for arbitrary n. More specifically, in Section 2 we recall some basic facts about $\mathcal{N}^{\mathrm{Kra}}$ and define special cycles and special difference cycles on it. In Section 3, we compute the intersection number between special cycles and the exceptional divisors. In Section 4, we prove a decomposition theorem for the horizontal component of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ when $L^\flat$ has rank $n-1$.

Part 2 is about Hermitian local densities. In Section 5, we study induction formulas of local density polynomials and relate the local density polynomials with primitive local densities. In

Section 6, we show that the coefficients $c^{\epsilon}_{n,i}$ in (1.9) are uniquely determined and give an algorithm to compute them. In Sections 7 and 8, we compute the local density polynomials when $n \le 3$.

In Part 3 we prove Theorem 1.2, i.e. Conjecture 1.1 for $n = 3$. In Section 9, we study the reduced locus of the special cycles for $n = 3$. In Section 10, we decompose ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ for $L^\flat$ of rank 2 and $\mathrm{v}(L^\flat) = 0$, and compute the intersection number of $\tilde{\mathcal{N}}_{\Lambda_2}$ with $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$. Finally, we prove Theorem 1.4 and finish the proof of Theorem 1.2 in Section 11 and explain its global applications in §12.

In Appendix A, we compute the primitive local densities that are used in Part 2 of the paper.

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Part 1. The geometric side

2. Rapoport-Zink space and special cycle

We denote $\bar{a}$ the Galois conjugate of $a \in F$ over F_0. Let $\mathrm{Nilp}\,\mathcal{O}_{\breve{F}}$ be the category of $\mathcal{O}_{\breve{F}}$-schemes S such that π is locally nilpotent on S. For such an S, denote its special fiber $S \times_{\mathrm{Spf}\,\mathcal{O}_{\breve{F}}} \mathrm{Spec}\,k$ by $\bar{S}$. Let σ be the Frobenius element of $\breve{F}_0/F_0$.

2.1. **RZ spaces.** Let $S \in \mathrm{Nilp}\,\mathcal{O}_{\breve{F}}$. A p-divisible strict $\mathcal{O}_{F_0}$-module over S is a p-divisible group over S with an $\mathcal{O}_{F_0}$ action whose induced action on its Lie algebra is via the structural morphism $\mathcal{O}_{F_0} \to \mathcal{O}_S$.

Definition 2.1. A formal Hermitian $\mathcal{O}_F$-module of dimension n over S is a triple (X, ι, λ) where X is a supersingular p-divisible strict $\mathcal{O}_{F_0}$-module over S of dimension n and F_0-height $2n$ (supersingular means the relative Dieudonné module of X at each geometric point of S has slope $\frac{1}{2}$), $\iota : \mathcal{O}_F \to \mathrm{End}(X)$ is an $\mathcal{O}_F$-action and $\lambda : X \to X^\vee$ is a principal polarization in the category of strict $\mathcal{O}_{F_0}$-modules such that the Rosati involution induced by λ is the Galois conjugation of F/F_0 when restricted on $\mathcal{O}_F$. We say (X, ι, λ) satisfies the signature condition $(1, n-1)$ if for all $a \in \mathcal{O}_F$ we have

(i) $\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = (T - s(a)) \cdot (T - s(\bar{a}))^{n-1}$ where $s : \mathcal{O}_F \to \mathcal{O}_S$ is the structure morphism;
(ii) The wedge condition proposed in [Pap00]:

$$\wedge^n(\iota(a) - s(a) \mid \mathrm{Lie}\, X) = 0,\ \wedge^2(\iota(a) - s(\bar{a}) \mid \mathrm{Lie}\, X) = 0.$$

Let $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be a formal Hermitian $\mathcal{O}_F$-module of dimension n over k, and N be its rational relative Dieudonne module. Then N is an $2n$-dimensional $\breve{F}_0$-vector space equipped with a σ-linear operator $\mathbf{F}$ and a σ^{-1}-linear operator $\mathbf{V}$. The $\mathcal{O}_F$-action $\iota_{\mathbb{X}} : \mathcal{O}_F \to \mathrm{End}(\mathbb{X})$ induces on N an $\mathcal{O}_F$-action commuting with $\mathbf{F}$ and $\mathbf{V}$. We still denote this induced action by $\iota_{\mathbb{X}}$ and denote $\iota_{\mathbb{X}}(\pi)$ by π. Let $\tau := \pi \mathbf{V}^{-1}$ and $C := N^\tau$. Then C is an n-dimensional F-vector space equipped with a Hermitian form $(,)_{\mathbb{X}}$ defined using the polarization $\lambda_{\mathbb{X}}$, see [Shi18, Equation (2.7)]. When n is odd, there is a unique choice of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ up to quasi-isogenies that preserves the polarization by a factor in $\mathcal{O}_{F_0}^\times$. When n is even, there are two such choices according to the sign $\epsilon = \chi(C)$ (see (1.1)) of C. See [Shi18, Remark 2.16] and [RTW14, Remark 4.2]. Fix a formal Hermitian $\mathcal{O}_F$-module $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ of signature $(0, 1)$ over $\mathrm{Spec}\, k$. It is unique up to $\mathcal{O}_F$-linear isomorphisms. Define

$$\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X}) \otimes \mathbb{Q}, \tag{2.1}$$

which is equipped with a Hermitian form

$$h(x, y) = \lambda_{\mathbb{Y}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}^0_F(\mathbb{Y}) \xrightarrow{\sim} F \tag{2.2}$$

where $y^\vee$ is the dual quasi-homomorphism of y and $\mathrm{End}_F^0(\mathbb{Y})$ is the ring of F-linear quasi-endomorphisms of $\mathbb{Y}$. The Hermitian spaces $(\mathbb{V}, h(,))$ and $(C, (,)_{\mathbb{X}})$ are related by the F-linear isomorphism

$$b : \mathbb{V} \to C, \quad \mathbf{x} \mapsto \mathbf{x}(e) \tag{2.3}$$

where e is a generator of the relative covariant Dieudonné module $M(\mathbb{Y})$ of $\mathbb{Y}$. Let $(,)_{\mathbb{Y}}$ be the analogue of $(,)_{\mathbb{X}}$ for $\mathbb{Y}$, namely the Hermition form on the rational relative Dieudonné module of $\mathbb{Y}$ defined by $\lambda_{\mathbb{Y}}$. By [Shi18, Lemma 3.6], we have

$$h(\mathbf{x}, \mathbf{x})(e, e)_{\mathbb{Y}} = (b(\mathbf{x}), b(\mathbf{x}))_{\mathbb{X}}. \tag{2.4}$$

By scaling the Hermitian form $(,)_{\mathbb{Y}}$ we can assume that

$$(e, e)_{\mathbb{Y}} = 1,$$

so $\mathbb{V}$ and C are isomorphic as Hermitian spaces. We will sometimes identify $\mathbb{V}$ and C.

Definition 2.2. Fix a formal Hermitian $\mathcal{O}_F$-module $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of dimension n over k with the sign $\epsilon = \chi(C)$. The moduli space $\mathcal{N}_{n,\epsilon}^{\mathrm{Pap}}$ is the functor sending each $S \in \mathrm{Nilp}\, \mathcal{O}_{\breve{F}}$ to the groupoid of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ where (X, ι, λ) is a formal Hermitian $\mathcal{O}_F$-module over S of signature $(1, n-1)$ and $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, k} \bar{S}$ is a quasi-morphism of formal $\mathcal{O}_F$-modules of height 0. An isomorphism between two such quardruples $(X, \iota, \lambda, \rho)$ and $(X', \iota', \lambda', \rho')$ is given by an $\mathcal{O}_F$-linear isomorphism $\alpha : X \to X'$ such that $\rho' \circ (\alpha \times_S \bar{S}) = \rho$ and $\alpha^*(\lambda')$ is a $\mathcal{O}_{F_0}^\times$ multiple of λ. We drop the subscript ϵ in $\mathcal{N}_{n,\epsilon}^{\mathrm{Pap}}$ when we do not emphasize on the sign.

By the discussion before (2.1), when n is odd, two different choices of ϵ give us isomorphic moduli spaces. When n is even, two different choices of ϵ give us two sets of non-isomorphic moduli spaces. By [RTW14], $\mathcal{N}_n^{\mathrm{Pap}}$ is representable by a formal scheme flat and of relative dimension $n-1$ over $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$. We remark here that although [RTW14] works on the category of p-divisible groups (namely when $F_0 = \mathbb{Q}_p$), their arguments and results easily extend to the category of strict formal $\mathcal{O}_{F_0}$-modules using relative Dieudonné theory or more generally the relative display theory developed in [ACZ16]. When $n = 1$, we have $\mathcal{N}_1^{\mathrm{Pap}} \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$. The universal Hermitian $\mathcal{O}_F$-module over $\mathcal{N}_1^{\mathrm{Pap}}$ is the canonical lifting $(\mathcal{G}, \iota_{\mathcal{G}}, \lambda_{\mathcal{G}}, \rho_{\mathcal{G}})$ of $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ to $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ in the sense of [Gro86]. When $n > 1$, $\mathcal{N}_n^{\mathrm{Pap}}$ is regular outside the set of superspecial points over $\mathrm{Spec}\, k$, which are the points characterized by the condition $\iota(\pi)|_{\mathrm{Lie}\, X} = 0$. The set of superspecial points is in fact the set of type 0 lattices (see Section 2.3), hence is isolated and we denote it by Sing.

Definition 2.3. Fix $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be as in Definition 2.2. The moduli space $\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}$ is the functor sending each $S \in \mathrm{Nilp}\, \mathcal{O}_{\breve{F}}$ to the groupoid of isomorphism classes of quintuples $(X, \iota, \lambda, \rho, \mathcal{F})$ where $(X, \iota, \lambda, \rho) \in \mathcal{N}_{n,\epsilon}^{\mathrm{Pap}}(S)$ and $\mathcal{F}$ is a locally free direct summand of $\mathrm{Lie}\, X$ of rank $n-1$ as an $\mathcal{O}_S$-module such that $\mathcal{O}_F$ acts on $\mathrm{Lie}\, X/\mathcal{F}$ by the structural morphism and acts on $\mathcal{F}$ by the Galois conjugate of the structural morphism. An isomorphism between two such quintuples $(X, \iota, \lambda, \rho, \mathcal{F})$ and $(X', \iota', \lambda', \rho', \mathcal{F}')$ is an isomorphism $\alpha : (X, \iota, \lambda, \rho) \to (X', \iota', \lambda', \rho')$ in $\mathcal{N}_{n,\epsilon}^{\mathrm{Pap}}(S)$ such that $\alpha^*(\mathcal{F}') = \mathcal{F}$. Again we drop the subscript ϵ in $\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}$ when we do not emphasize on the sign.

By [Krä03] (see also [Shi22, Proposition 2.7]), the natural forgetful functor $\Phi : \mathcal{N}_n^{\mathrm{Kra}} \to \mathcal{N}_n^{\mathrm{Pap}}$ forgetting $\mathcal{F}$ is the blow up of $\mathcal{N}_n^{\mathrm{Pap}}$ along its singular locus Sing. For each point $\Lambda \in \mathrm{Sing}$, its inverse image $\Phi^{-1}(\Lambda)$ is an exceptional divisor Exc_Λ isomorphic to $\mathbb{P}_k^{n-1}$.

2.2. **Special cycles.**

Definition 2.4. For an $\mathcal{O}_F$-lattice L of $\mathbb{V}$, define $\mathcal{Z}^{\mathrm{Pap}}(L)$ to be the subfunctor of $\mathcal{N}_n^{\mathrm{Pap}}$ sending each $S \in \mathrm{Nilp}\,\mathcal{O}_{\breve{F}}$ to the isomorphism classes of tuples $(X, \iota, \lambda, \rho) \in \mathcal{N}_n^{\mathrm{Pap}}(S)$ such that for any $x \in L$ the quasi-homomorphism

$$\rho^{-1} \circ x \circ \rho_{\mathcal{G}} : \mathcal{G} \times_S \bar{S} \to X \times_S \bar{S}$$

extends to a homomorphism $\mathcal{G}_S \to X$. For $\mathbf{x} \in \mathbb{V}^m$, we let $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) := \mathcal{Z}^{\mathrm{Pap}}(L)$ where $L = \mathrm{Span}\{\mathbf{x}\}$. Let

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \mathcal{Z}^{\mathrm{Kra}}(L) := \mathcal{Z}^{\mathrm{Pap}}(L) \times_{\mathcal{N}_n^{\mathrm{Pap}}} \mathcal{N}_n^{\mathrm{Kra}}.$$

By Grothendieck-Messing theory $\mathcal{Z}^{\mathrm{Pap}}(L)$ (hence $\mathcal{Z}^{\mathrm{Kra}}(L)$) is a closed formal subscheme of $\mathcal{N}_n^{\mathrm{Pap}}$. We sometimes add the subscript ${}_{n,\epsilon}$ to $\mathcal{Z}^{\mathrm{Pap}}(L)$, $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$, $\mathcal{Z}^{\mathrm{Kra}}(L)$ and $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ to indicate their ambient moduli spaces.

Definition 2.5. For an $\mathcal{O}_F$-lattice $L \subset \mathbb{V}$, define $\tilde{\mathcal{Z}}(L)$ to be the strict transform (see the definition after [Har13, Chapter II, Corollary 7.15]) of $\mathcal{Z}^{\mathrm{Pap}}(L)$ under the blow up $\mathcal{N}_n^{\mathrm{Kra}} \to \mathcal{N}_n^{\mathrm{Pap}}$.

Proposition 2.6. *Suppose $\chi(\mathbb{V}) = \epsilon$. Let L be a self-dual lattice of rank m in $\mathbb{V}$ with $\eta = \chi(L)$. We have*

$$\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(L) \cong \mathcal{N}_{n-m,\epsilon\eta}^{\mathrm{Pap}}, \text{ and } \tilde{\mathcal{Z}}_{n,\epsilon}(L) \cong \mathcal{N}_{n-m,\epsilon\eta}^{\mathrm{Kra}}.$$

Proof. Let us start with the case $L = \mathrm{Span}\{\mathbf{x}_0\}$ where $\mathbf{x}_0 \in \mathbb{V}$. Assume that $u = h(\mathbf{x}_0, \mathbf{x}_0)$. Multiplying the Hermitian form $(,)_{\mathbb{X}}$ on C by u^{-1} does not affect the various moduli spaces involved. So we can perform this and assume that $h(\mathbf{x}_0, \mathbf{x}_0) = 1$. Moreover, the sign of its orthogonal complement in $\mathbb{V}$ becomes

$$\epsilon_1 = \epsilon \cdot \chi(u^{-1}) \cdot \chi(u^{-(n-1)}) \cdot \chi(-1)^{n-1} = \epsilon\chi(u)^n\chi(-1)^{n-1}.$$

Then for $(X, \iota, \lambda, \rho) \in \mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(\mathbf{x}_0)(S)$, we define

$$\mathbf{x}_0^* := \lambda_{\mathcal{G}}^{-1} \circ \mathbf{x}_0^\vee \circ \lambda,\ e := \mathbf{x}_0 \circ \mathbf{x}_0^* \in \mathrm{End}(X).$$

By the fact that $h(\mathbf{x}_0, \mathbf{x}_0) = 1$ we know that e is an idempotent. It is routine to check that

$$((1-e)X, (1-e)\iota, (1-e^\vee)\lambda(1-e), \rho(1-e))$$

is an object in $\mathcal{N}_{n-1,\epsilon_1}^{\mathrm{Pap}}(S)$. Conversely given $(Y, \iota_Y, \lambda_Y, \rho_Y) \in \mathcal{N}_{n-1,\epsilon_1}^{\mathrm{Pap}}(S)$, the object

$$(Y \times \mathcal{G}_S, \iota_Y \times \iota_{\mathcal{G}_S}, \lambda_Y \times \lambda_{\mathcal{G}_S}, g \circ (\rho_Y \times \rho_{\mathcal{G}_S}))$$

is in $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(\mathbf{x}_0)(S)$ where $g \in \mathrm{U}(\mathbb{V})$ such that $g^{-1}\mathbf{x}_0$ is the inclusion $0 \times \mathrm{id} : \mathbb{Y} \to \mathbb{X}_{n-1} \times \mathbb{Y}$ where $\mathbb{X}_{n-1}$ is the framing object of $\mathcal{N}_{n-1,\epsilon_1}^{\mathrm{Pap}}$. The above two functors are inverse to each other. This shows that $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(\mathbf{x}_0) \cong \mathcal{N}_{n-1,\epsilon_1}^{\mathrm{Pap}}$. For general L of rank m and determinant u, find a basis with Gram matrix $\{1, \ldots, 1, u\}$ and apply the above result repeatedly. So we have $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(L) \cong \mathcal{N}_{n-m,\epsilon_m}^{\mathrm{Pap}}$ where

$$\epsilon_m = \epsilon\chi(u)^{n-m+1}\chi(-1)^{(n-m)m+\frac{m(m-1)}{2}}.$$

Notice that by scaling the Hermitian form by $(-1)^m u$ again we have $\mathcal{N}_{n-m,\epsilon_m}^{\mathrm{Pap}} = \mathcal{N}_{n-m,\epsilon\eta}^{\mathrm{Pap}}$. It then follows from [Har13, Chapter II, Corollary 7.15] that $\tilde{\mathcal{Z}}_{n,\epsilon}(L)$ is the blow up of $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(L)$ along its superspecial points, which is $\mathcal{N}_{n-m,\epsilon\eta}^{\mathrm{Kra}}$. □

Corollary 2.7. *Let L be as in Proposition 2.6 and $\mathbf{y} \in \mathbb{V}$ such that $\mathbf{y} \perp L$. Then*

$$\mathcal{Z}_{n,\epsilon}^{\mathrm{Kra}}(\mathbf{y}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(L) \cong \mathcal{Z}_{n-m,\epsilon\eta}^{\mathrm{Kra}}(\mathbf{y}).$$

Remark 2.8. *It follows directly from the definition that $\tilde{\mathcal{Z}}(L)$ is a closed sub formal scheme of $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \cdots \cap \tilde{\mathcal{Z}}(\mathbf{x}_r)$ if $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is a basis of L. However in general these two can not be identified.*

2.3. **Bruhat-Tits stratification.** For an $\mathcal{O}_{\breve{F}}$-lattice M of N, define $M^\sharp$ to be the dual lattice of M with respect to the form $(,)_{\mathbb{X}}$. Recall the following results.

Proposition 2.9. *(*[RTW14, Proposition 2.2 and 2.4]*) Let $\mathcal{N}(k)$ be the set of $\mathcal{O}_{\breve{F}}$-lattices*

$$\mathcal{N}(k) = \{M \subset C \otimes_F \breve{F} \mid M^\sharp = M,\, \pi\tau(M) \subset M \subset \pi^{-1}\tau(M),\, \dim_k(M + \tau(M))/M \leq 1\}.$$

Then the map

$$\mathcal{N}^{\mathrm{Pap}}(k) \to \mathcal{N}(k), \quad x = (X, \iota, \lambda, \rho) \mapsto M(x) = \rho(M(X)) \subset N$$

is a bijection.

We say a lattice $\Lambda \subset C$ is a vertex lattice if $\pi\Lambda \subseteq \Lambda^\sharp \subseteq \Lambda$ where $\Lambda^\sharp$ is dual lattice of Λ with respect to $(,)_{\mathbb{X}}$, and we call $t = \dim_{\mathbb{F}_q}(\Lambda/\Lambda^\sharp)$ the type of Λ. We denote the set of vertex lattices (resp. of type t) by $\mathcal{V}$ (resp. $\mathcal{V}^t$). We say two vertex lattice Λ_1 and Λ_2 are neighbours if $\Lambda_1 \subset \Lambda_2$ or $\Lambda_2 \subset \Lambda_1$. Then we can define a simplicial complex $\mathcal{L}$ as follows. When n is odd or when n is even and C is non-split, then an r-simplex is formed by $\Lambda_0, \ldots, \Lambda_r$ if any two members of this set are neighbours. When n is even and C is split, we refer to discussion before [RTW14, 3.4] for the definition of $\mathcal{L}$. We also use $\mathcal{L}_{n,\epsilon}$ to denote $\mathcal{L}$ if C has dimension n and $\chi(C) = \epsilon$. Again when n is odd, $\mathcal{L}_{n,1} = \mathcal{L}_{n,-1}$, hence we use $\mathcal{L}_n$ to denote it.

By results in Sections 4 and 6 of loc. cit., to each $\Lambda \in \mathcal{V}^t$ we can associate a Deligne-Lusztig varieties $\mathcal{N}_\Lambda$ and $\mathcal{N}_\Lambda^\circ$ of dimension $t/2$, such that

$$\mathcal{N}_\Lambda(k) = \{M \in \mathcal{N}(k) \mid M \subset \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}\},$$

and

$$\mathcal{N}_\Lambda^\circ(k) = \{M \in \mathcal{N}(k) \mid \Lambda(M) = \Lambda\}.$$

Here $\Lambda(M)$ is the minimal vertex lattice such that $\Lambda(M) \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$ contains M which always exists by [RTW14, Proposition 4.1]. By Theorem 1.1 of loc.cit., we know that

$$\mathcal{N}_\Lambda := \bigsqcup_{\Lambda' \in \mathcal{L}, \Lambda' \subseteq \Lambda} \mathcal{N}_{\Lambda'}^\circ,$$

and

$$\mathcal{N}_{\mathrm{red}}^{\mathrm{Pap}} = \bigsqcup_{\Lambda \in \mathcal{L}} \mathcal{N}_\Lambda^\circ$$

where each $\mathcal{N}_\Lambda$ is a closed subvariety of $\mathcal{N}_{\mathrm{red}}^{\mathrm{Pap}}$. By loc. cit., we also know that

$$\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda'} = \begin{cases} \mathcal{N}_{\Lambda \cap \Lambda'} & \text{if } \Lambda \cap \Lambda' \in \mathcal{V}, \\ \emptyset & \text{otherwise.} \end{cases}$$

For a lattice $L \subset \mathbb{V}$, define

$$\mathcal{V}(L) := \{\Lambda \in \mathcal{V} \mid L \subseteq \Lambda^\sharp\}, \text{ and } \mathcal{V}^t(L) := \{\Lambda \in \mathcal{V}^t \mid L \subseteq \Lambda^\sharp\}. \tag{2.5}$$

When $L = \mathrm{Span}\{\mathbf{x}\}$ we also denote $\mathcal{V}(L)$ (resp. $\mathcal{V}^t(L)$) by $\mathcal{V}(\mathbf{x})$ (resp. $\mathcal{V}^t(\mathbf{x})$). For any subset S of $\mathcal{V}$, we define $\mathcal{L}(S)$ to be the subcomplex of $\mathcal{L}$ such that a simplex is in $\mathcal{L}(S)$ if and only if every vertex in it is in S. For a lattice L of $\mathcal{V}$ and $\mathbf{x} \in C$, define

$$\mathcal{L}(L) = \mathcal{L}(\mathcal{V}(L)). \tag{2.6}$$

When $L = \mathrm{Span}\{\mathbf{x}\}$ we also denote $\mathcal{L}(L)$ by $\mathcal{L}(\mathbf{x})$.

2.4. **Horizontal and vertical part.** A formal scheme X over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ is called horizontal (resp. vertical) if it is flat over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ (resp. π is locally nilpotent on $\mathcal{O}_X$). For a formal scheme X over $\mathrm{Spf}\,O_{\breve{F}}$, its horizontal part X_h is canonically defined by the ideal sheaf $\mathcal{O}_{X,\mathrm{tor}}$ of torsion sections on $\mathcal{O}_X$. If X is noetherian, there exists a $m \in \mathbb{Z}_{>0}$ such that $\pi^m \mathcal{O}_{X,\mathrm{tor}} = 0$. We define the vertical part $X_v \subset X$ to be the closed formal subscheme defined by the ideal sheaf $\pi^m \mathcal{O}_X$. Since $\mathcal{O}_{X,\mathrm{tor}} \cap \pi^m \mathcal{O}_X = \{0\}$, we have the following decomposition by primary decomposition

$$X = X_h \cup X_v \tag{2.7}$$

as a union of horizontal and vertical formal subschemes. Notice that the horizontal part X_h is canonically defined while the vertical part X_v depends on the choice of m.

Lemma 2.10. *For a lattice $L^\flat \subset \mathbb{V}$ of rank greater or equal to $n-1$ with nondegenerate Hermitian form, $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ is noetherian.*

Proof. The lemma can be proved as in [LZ22a, Lemma 2.9.2]. □

Lemma 2.11. *For a rank $n-1$ lattice $L^\flat \subset \mathbb{V}$ with nondegenerate Hermitian form, $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v$ is supported on the reduced locus $\mathcal{N}^{\mathrm{Kra}}_{\mathrm{red}}$ of $\mathcal{N}^{\mathrm{Kra}}$, i.e., $\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v}$ is annihilated by a power of the ideal sheaf of $\mathcal{N}^{\mathrm{Kra}}_{\mathrm{red}}$.*

Proof. The proof is the same as that of [LZ22a, Lemma 5.1.1]. □

2.5. **Derived special cycles.** For a locally noetherian formal scheme X together with a formal subscheme Y, denote by $K_0^Y(X)$ the Grothendieck group of finite complexes of coherent locally free $\mathcal{O}_X$-modules acyclic outside Y. For such a complex $A^\bullet$, denote by $[A^\bullet]$ the element in $K_0^Y(X)$ represented by it. We use $K_0(X)$ to denote $K_0^X(X)$. Denote by $\mathrm{F}^i K_0^Y(X)$ the codimension i filtration on $K_0^Y(X)$ and $\mathrm{Gr}^i K_0^Y(X)$ its i-th graded piece. We have a cup product $\cdot$ on $K_0^Y(X)$ defined by tensor product of complexes:

$$[A_1^\bullet] \cdot [A_2^\bullet] = [A_1^\bullet \otimes A_2^\bullet].$$

When X is a scheme, the cup product satisfies ([SABK94, Section I.3, Theorem 1.3])

$$\mathrm{F}^i K_0^Y(X)_{\mathbb{Q}} \cdot \mathrm{F}^j K_0^Y(X)_{\mathbb{Q}} \subset \mathrm{F}^{i+j} K_0^Y(X)_{\mathbb{Q}}. \tag{2.8}$$

It is expected that (2.8) is also true when X is a formal scheme. We will only need special cases of this fact which can be checked directly, see for example Lemma 3.4 and 10.1.

Let $K_0'(Y)$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_Y$-modules on Y. When X is regular we have the following isomorphism

$$K_0^Y(X) \cong K_0'(Y). \tag{2.9}$$

In particular, $K_0(X) \cong K_0'(X)$.

When X is a regular scheme of dimension d, there is an isomorphism of graded rings defined by the Chern character:

$$\mathrm{ch} : K_0(X)_{\mathbb{Q}} \cong \bigoplus_{i=1}^{d} \mathrm{CH}^i(X)_{\mathbb{Q}}.$$

In particular we have

$$\mathrm{Gr}^i K_0(X)_{\mathbb{Q}} \cong \mathrm{CH}^i(X)_{\mathbb{Q}}.$$

Recall that for $\mathbf{x} \in \mathbb{V}$, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ is a divisor, see [How19, Proposition 4.3].

Definition 2.12. For $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_r) \in \mathbb{V}^r$, define ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ to be

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_r)}] \in K_0^{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}(\mathcal{N}^{\mathrm{Kra}}) \tag{2.10}$$

where $\otimes^{\mathbb{L}}$ is the derived tensor product of complexes of coherent locally free sheaves on $\mathcal{N}^{\mathrm{Kra}}$. By [How19, Theorem B], ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ only depends on $L \coloneqq \mathrm{Span}\{\mathbf{x}\}$, hence can be denoted as ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)$.

Definition 2.13. When L has rank n, we define the intersection number

$$\mathrm{Int}(L) = \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)) \tag{2.11}$$

where χ is the Euler characteristic.

Lemma 2.14. *$\mathcal{Z}^{\mathrm{Kra}}(L)$ is properly supported on $\mathcal{N}^{\mathrm{Kra}}_{\mathrm{red}}$. In particular,* $\mathrm{Int}(L)$ *is finite.*

Proof. This can be proved exactly the same way as [LZ22a, Lemma 2.10.1]. □

2.6. **Special Difference cycles.** Conjecture 1.1 and Theorem 5.2 motivate us to make the following definition.

Definition 2.15. For $L \subset \mathbb{V}$ a rank ℓ lattice, define the special difference cycle $\mathcal{D}(L) \in K_0^{\mathcal{Z}^{\mathrm{Kra}}(L)}(\mathcal{N}^{\mathrm{Kra}})$ by

$$\mathcal{D}(L) = {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) + \sum_{i=1}^{\ell} (-1)^i q^{i(i-1)/2} \sum_{\substack{L \subset L' \subset \frac{1}{\pi} L \\ \dim_{\mathbb{F}_q}(L'/L) = i}} {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L'). \tag{2.12}$$

One interesting observation is the following decomposition of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)$.

Lemma 2.16. *For $L \subset \mathbb{V}$ a lattice of rank ℓ, we have the following identity in $K_0^{\mathcal{Z}^{\mathrm{Kra}}(L)}(\mathcal{N}^{\mathrm{Kra}})$ where the summation is finite.*

$$ {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = \sum_{\substack{L' \text{integral} \\ L \subset L' \subset L_F}} \mathcal{D}(L'). $$

Proof. First of all, if L is not integral, neither is L' if $L \subset L'$. In this case ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = 0$ and the summation index on the right hand side of the identity in the lemma is empty. This proves the lemma when $\mathrm{v}(L) < 0$. We can now prove the identity by induction on the fundamental invariant of L. Assume that the lemma is proved for all $L' \subset L_F$ with $L \subsetneq L'$.

For L' with $L \subsetneq L' \subset \frac{1}{\pi} L$, we have

$$ {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L') = \sum_{L' \subset L'' \subset L'_F} \mathcal{D}(L'') $$

by the induction hypothesis. Combining this with (2.12), we can write

$$ {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = \sum_{L \subset L'' \subset L_F} m(L'') \mathcal{D}(L'') $$

where $m(L'') \in \mathbb{Z}$. Now it suffices to show $m(L'') = 1$ for any L'' such that $L \subset L'' \subset L_F$.

First, notice that $m(L) = 1$. For any L'' such that $L \subset L'' \subsetneq L_F$, let $M' = \frac{1}{\pi} L \cap L''$ and $m = \dim_{\mathbb{F}_q}(M'/L)$. We have

$$ m(L'') = -\sum_{i=1}^{m} (-1)^i q^{i(i-1)/2} \sum_{\substack{L \subset L' \subset M' \\ \dim_{\mathbb{F}_q}(L'/L) = i}} 1 = 1 \tag{2.13}$$

by evaluating the identity in the corollary to [Tam63, Lemma 12] at $t = 1$. □

Remark 2.17. *When $\ell = 1$ and $L = \mathrm{Span}\{\mathbf{x}\}$, the Cartier divisor*

$$\mathcal{D}(L) = \mathcal{Z}(\mathbf{x}) - \mathcal{Z}(\frac{1}{\pi}\mathbf{x})$$

is the difference divisor $\mathcal{D}(\mathbf{x})$ defined in [Ter10, Definition 2.10].

Definition 2.18. Assume $L = L_1 \oplus L_2$, where L_i is of rank n_i and $n_1 + n_2 = n$. We define

$$\mathrm{Int}(L)^{(n_1)} = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L_1) \cdot^{\mathbb{L}} \mathcal{Z}^{\mathrm{Kra}}(L_2)). \tag{2.14}$$

Notice that $\mathrm{Int}(L)^{(n_1)}$ depends on the decomposition $L = L_1 \oplus L_2$.

3. Special cycles and exceptional divisors

For a formal subscheme $\mathcal{Z}$ of $\mathcal{N}^{\mathrm{Kra}}$, we use the notation $\otimes_{\mathcal{Z}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{Z}}$) instead of $\otimes_{\mathcal{O}_{\mathcal{Z}}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{Z}}}$). We also simply write $\otimes$ (resp. $\otimes^{\mathbb{L}}$) instead of $\otimes_{\mathcal{N}^{\mathrm{Kra}}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{N}^{\mathrm{Kra}}}$). Let us first recall the following distribution law of derived tensor product. In this section, we identify $\mathbb{V}$ with C by the isomorphism b defined in (2.3).

Lemma 3.1. *Assume that $\mathcal{A}_i$ ($1 \leq i \leq k$) is in the derived category of bounded coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and $i : \mathcal{Z} \to \mathcal{N}^{\mathrm{Kra}}$ is a closed embedding of formal subscheme. Then the following identity holds in the derived category of bounded coherent sheaves on $\mathcal{Z}$.*

$$i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) = i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) \otimes^{\mathbb{L}}_{\mathcal{Z}} \ldots \otimes^{\mathbb{L}}_{\mathcal{Z}} i^*(\mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}).$$

Proof. We can take locally free representatives of $A_i^\bullet$ of $\mathcal{A}_i$. Then $A_1^\bullet \otimes \ldots \otimes A_k^\bullet$ is again a complex of locally free sheaves on $\mathcal{N}^{\mathrm{Kra}}$, hence a locally free representatives of $\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k$. Hence $i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}})$ can be represented by $A_1^\bullet \otimes \ldots \otimes A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}}$. Meanwhile $A_i^\bullet \otimes \mathcal{O}_{\mathcal{Z}}$ is a representative of $\mathcal{A}_i \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}$ in the derived category of bounded coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and is also a complex of locally free sheaves on $\mathcal{Z}$. Hence $i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) \otimes^{\mathbb{L}}_{\mathcal{Z}} \ldots \otimes^{\mathbb{L}}_{\mathcal{Z}} i^*(\mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}})$ can be represented by $(A_1^\bullet \otimes \mathcal{O}_{\mathcal{Z}}) \otimes_{\mathcal{Z}} \ldots \otimes_{\mathcal{Z}} (A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}})$. Now by the distribution law of tensor products we have

$$A_1^\bullet \otimes \ldots \otimes A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}} = (A_1^\bullet \otimes \mathcal{O}_{\mathcal{Z}}) \otimes_{\mathcal{Z}} \ldots \otimes_{\mathcal{Z}} (A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}}).$$

This finishes the proof of the lemma. □

Proposition 3.2. *Assume that the dimension of $\mathbb{V}$ is $n \geq 2$. Then for each $\mathbf{x} \in \mathbb{V}$, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ is a divisor. Moreover, we have the following decomposition of Cartier divisors*

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \tilde{\mathcal{Z}}(\mathbf{x}) + \sum_{\Lambda \in \mathcal{V}^0, \mathbf{x} \in \Lambda} (m_\Lambda(\mathbf{x}) + 1)\mathrm{Exc}_\Lambda \tag{3.1}$$

where $m_\Lambda(\mathbf{x})$ is the largest integer m such that $\pi^{-m} \cdot \mathbf{x} \in \Lambda$.

Proof. The fact that $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ is a divisor is due to [How19, Proposition 4.3]. By [Shi18, Proposition 3.7], the superspecial point corresponding to a type zero lattice Λ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$ if and only if $\mathbf{x} \in \Lambda$. Hence $\mathrm{Exc}_\Lambda \subset \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ if and only if $\mathbf{x} \in \Lambda$. Since $\mathcal{N}^{\mathrm{Kra}}_{n,\epsilon}$ is regular, we must have a decomposition as in (3.1) and the only job left is to determine the multiplicity of each Exc_Λ.

Fix a type zero lattice Λ and let $m \coloneqq m_\Lambda(\mathbf{x})$. Then $\pi^{-m} \cdot \mathbf{x}$ is a primitive vector in Λ. By Lemma A.3, there exists a decomposition

$$\Lambda = \Lambda_2 \ominus \Lambda'$$

where Λ_2 and Λ' are unimodular lattices of rank 2 and $n-2$ respectively and $\pi^{-m} \cdot \mathbf{x} \in \Lambda_2$. Let $\eta = \chi(\Lambda')$. By applying Proposition 2.6, we see that $\tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') \cong \mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. Moreover we have the following proper intersections

$$\mathcal{Z}^{\mathrm{Kra}}_{n,\epsilon}(\mathbf{x}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \mathcal{Z}^{\mathrm{Kra}}_{2,\epsilon\eta}(\mathbf{x}),\ \ \tilde{\mathcal{Z}}_{n,\epsilon}(\mathbf{x}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x}),$$

and

$$\mathrm{Exc}_\Lambda \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \mathrm{Exc}_{\Lambda_2},$$

where Exc_{Λ_2} is the exceptional divisor in $\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$ corresponding to the vertex lattice Λ_2. Hence the multiplicity of Exc_Λ in $\mathcal{Z}^{\mathrm{Kra}}_{n,\epsilon}(\mathbf{x})$ is the same as the multiplicity of Exc_{Λ_2} in $\mathcal{Z}^{\mathrm{Kra}}_{2,\epsilon\eta}(\mathbf{x})$. Now the proposition follows from [Shi22, Theorem 4.6] and [HSY23, Theorem 4.1]. $\square$

The Chow ring $\mathrm{CH}^\bullet(\mathrm{Exc}_\Lambda) \cong \mathrm{Gr}^\bullet K_0(\mathrm{Exc}_\Lambda)$ is isomorphic to $\mathbb{Z}[H_\Lambda]/(H_\Lambda^{n-1}-1)$ where H_Λ is the hyperplane class of Exc_Λ represented by any $\mathbb{P}^{n-2}_k$ in Exc_Λ.

Proposition 3.3. *Assume* $\dim \mathbb{V} = n \geq 2$. *Assume* $\mathbf{x} \in \mathbb{V}$ *such that* $h(\mathbf{x},\mathbf{x}) \neq 0$ *and* Λ *is a type* 0 *vertex lattice containing* $\mathbf{x}$. *Let* $m := m_\Lambda(\mathbf{x})$ *as in Proposition 3.2. Then* $\tilde{\mathcal{Z}}(\mathbf{x})$ *and* Exc_Λ *intersect properly and*

$$[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})\cap \mathrm{Exc}_\Lambda}] = (2m+1)H_\Lambda \in \mathrm{CH}^1(\mathrm{Exc}_\Lambda).$$

Proof. First $\tilde{\mathcal{Z}}(\mathbf{x})$ and Exc_Λ are Cartier divisors with no common component, so they intersect properly. Let $m = m_\Lambda(\mathbf{x})$ and $\mathbf{x}' := \pi^{-m}\cdot \mathbf{x}$. By assumption $m \geq 0$. By Proposition 5.9, we have

$$\{v \in \Lambda \mid h(\mathbf{x}', v) = 0\} = \mathrm{Span}\{\mathbf{y}\} \oplus \Lambda'$$

where $\mathrm{v}(\mathbf{y}) = \mathrm{v}(\mathbf{x}')$ and Λ' is unimodular. Let $\eta = \chi(\Lambda')$ and

$$\Lambda_2 := \{v \in \Lambda \mid v \perp \Lambda'\}.$$

Λ_2 is rank 2 unimodular and contains $\mathbf{x}'$.

By Proposition 2.6, we have $\tilde{\mathcal{Z}}(\Lambda') \cong \mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. In particular, $\tilde{\mathcal{Z}}(\Lambda')$ is regular. By Corollary 2.7, we know that $\tilde{\mathcal{Z}}(\Lambda') \cap \tilde{\mathcal{Z}}(\mathbf{x}) = \tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x})$. In particular $\tilde{\mathcal{Z}}(\Lambda')$ and $\tilde{\mathcal{Z}}(\mathbf{x})$ intersect properly as $\tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x})$ is a divisor in $\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. On the other hand $\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda$ is the exceptional divisor Exc_{Λ_2} in $\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. Since $\mathrm{Exc}_\Lambda \cong \mathbb{P}^{n-1}_k$, it is also regular. Our strategy is to compute the intersection number

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')})$$

in two different ways. By Lemma 3.1, one way is

$$\chi(\tilde{\mathcal{Z}}(\Lambda'), \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')\cap\tilde{\mathcal{Z}}(\mathbf{x})} \otimes^{\mathbb{L}}_{\tilde{\mathcal{Z}}(\Lambda')} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')\cap \mathrm{Exc}_\Lambda}) \tag{3.2}$$

where we use the fact that the intersections $\tilde{\mathcal{Z}}(\Lambda') \cap \tilde{\mathcal{Z}}(\mathbf{x})$ and $\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda$ are proper (see for example [Zha21, Lemma B.2]). The other way is, by Lemma 3.1,

$$\chi(\mathrm{Exc}_\Lambda, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})\cap \mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')\cap \mathrm{Exc}_\Lambda}). \tag{3.3}$$

When $\epsilon\eta = -1$, by Proposition 3.11 and Theorem 4.5 of [Shi22], we know that (3.2) is equal to $2m+1$. When $\epsilon\eta = 1$, by Lemma 3.10, Theorem 4.1 and Lemma 5.2 of [HSY23], we know that (3.2) is equal to $2m+1$ as well. Since the intersection number of H_Λ with $\mathrm{Exc}_{\Lambda_2} \cong \mathbb{P}^1_k$ in Exc_Λ is 1, the proposition follows. $\square$

3.1. **Intersection numbers involving the exceptional divisors.**

Lemma 3.4. *The class of* $\underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{m}$ *in* $\mathrm{CH}^{m-1}(\mathrm{Exc}_\Lambda)$ *is* $(-2H_\Lambda)^{m-1}$.

Proof. To study this intersection, it suffices to consider the local model N^{Kra} constructed in [Krä03]. Let N^{Kra}_s be its special fiber. Recall by equation (4.11) loc. cit., we have

$$N^{\mathrm{Kra}}_s = \mathrm{Exc} + Z_2$$

as Cartier divisors where Exc is the exceptional divisor of N^{Kra} and Z_2 is a divisor in N^{Kra} which intersect properly with Exc. Their intersection is $2H$ where H is the hyperplane class of Exc. Since Exc is properly supported on N^{Kra}, we have

$$[\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}} \mathcal{O}_{N_s^{\mathrm{Kra}}}] = 0.$$

Hence

$$\begin{aligned} 0 &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{N_s^{\mathrm{Kra}}}] \\ &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{\mathrm{Exc}}] + [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{Z_2}] \\ &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{\mathrm{Exc}}] + 2H. \end{aligned}$$

This proves the lemma when $m = 2$. The general case now follows from Lemma 3.1. □

Corollary 3.5. *Let $\Lambda \in \mathcal{V}^0$ and $\mathbf{x} \in \Lambda$. Then we have the following identity in $\mathrm{CH}^1(\mathrm{Exc}_\Lambda)$:*

$$[\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}] = -H_\Lambda.$$

Proof. By Propositions 3.2, 3.3 and Lemma 3.4, we have the following identity in $\mathrm{CH}^1(\mathrm{Exc}_\Lambda)$:

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}] = [(2m_\Lambda(\mathbf{x})+1) - 2(m_\Lambda(\mathbf{x})+1)]H_\Lambda = -H_\Lambda.$$

This finishes the proof of the corollary. □

Corollary 3.6. *Assume that $n - m \geq 1$ and $\mathrm{Exc}_\Lambda \subset \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cap \ldots \cap \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)$, then*

$$\chi(\mathcal{N}_n^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \ldots \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)} \otimes^{\mathbb{L}} \underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{n-m}) = (-1)^{n-1} \cdot 2^{n-m-1}.$$

Proof. By Corollary 3.5, Lemmas 3.1 and 3.4, we have

$$\begin{aligned} &\chi(\mathcal{N}_n^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \ldots \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)} \otimes^{\mathbb{L}} \underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{n-m}) \\ &= \chi(\mathrm{Exc}_\Lambda, \underbrace{(-H_\Lambda) \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \cdots \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} (-H_\Lambda)}_{m} \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \underbrace{(-2H_\Lambda) \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \cdots \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} (-2H_\Lambda)}_{n-m-1}) \\ &= (-1)^m \cdot (-2)^{n-m-1}. \end{aligned}$$

□

For $\Lambda \in \mathcal{V}^0$, let $\mathbb{P}^1_\Lambda$ be any $\mathbb{P}^1_k$ in Exc_Λ, and

$$\mathrm{Int}_\Lambda(\mathbf{x}) = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}). \tag{3.4}$$

Corollary 3.7. *For $\Lambda \in \mathcal{V}^0$, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) = -2. \tag{3.5}$$

Proof. By Lemma 3.4, we have

$$\begin{aligned} &\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) \\ &= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} (\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda})) \\ &= \chi(\mathrm{Exc}_\Lambda, (\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) \\ &= -2\chi(\mathrm{Exc}_\Lambda, H_\Lambda \cdot [\mathcal{O}_{\mathbb{P}^1_\Lambda}]) \\ &= -2. \end{aligned}$$

□

Corollary 3.8. *For $\Lambda \in \mathcal{V}^0$, we have*

$$\mathrm{Int}_\Lambda(\mathbf{x}) = -1_\Lambda(\mathbf{x}).$$

Proof. If $\mathbf{x} \notin \Lambda$, then the intersection number is apparently 0. Otherwise we have by Corollary 3.5

$$\begin{aligned}&\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda})\\ &= \chi(\mathrm{Exc}_\Lambda, (\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda})\\ &= -\chi(\mathrm{Exc}_\Lambda, H_\Lambda \cdot [\mathcal{O}_{\mathbb{P}^1_\Lambda}])\\ &= -1.\end{aligned}$$

□

The above results suggest that the difficulty to compute $\mathrm{Int}(L)$ mainly lies in computing

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(x_n)}).$$

We end this section by studying the intersection number of difference cycle with exceptional divisors.

Lemma 3.9. *If $L^\flat$ has rank $n-1$, then for any $\Lambda \in \mathcal{V}^0(L^\flat)$, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = \begin{cases}(-1)^{n-1} & \text{if } L^\flat = \Lambda \cap L^\flat_F,\\ 0 & \text{otherwise.}\end{cases}$$

Remark 3.10. *$L^\flat = \Lambda \cap L^\flat_F$ if and only if $L^\flat$ is of type (see* (4.2) *and Lemma 4.3 below)* 1 *or* 0 *and Λ is at the boundary of the $\mathcal{L}(L^\flat)$.*

Proof. Define

$$M' \coloneqq \frac{1}{\pi} L^\flat \cap \Lambda \text{ and } m \coloneqq \dim_{\mathbb{F}_q}(M'/L^\flat).$$

Then for L' such that $L^\flat \subset L' \subset \frac{1}{\pi}L^\flat$, we know that $\mathcal{Z}^{\mathrm{Kra}}(L')$ intersects Exc_Λ if and only if $L' \subset M'$. For such L', by Corollary 3.6, we have

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L') \cdot [\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = (-1)^{n-1}. \tag{3.6}$$

Hence

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = (-1)^{n-1}[1 + \sum_{i=1}^{m} (-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat \subset L' \subset M'\\ \dim_{\mathbb{F}_q}(L'/L^\flat) = i}} 1].$$

Notice that $m = 0$ if and only if $M' = L^\flat$ which is equivalent to the condition $L^\flat = \Lambda \cap L^\flat_F$. In this case the summation in (4.8) is over an empty set hence (4.8) is equal to 1. If $m > 0$ we know (4.8) is equal to 0 by (2.13). □

4. Horizontal components of special cycles

Given an integral Hermitian lattice L we can have its Jordan decomposition:

$$L = \oplus_{t \geq 0} L_t \tag{4.1}$$

where L_t is π^t-modular, see [Jac62]. Define the type of L to be

$$t(L) = \sum_{t \geq 1} \mathrm{rank}_{\mathcal{O}_F}(L_t). \tag{4.2}$$

4.1. **Quasi-canonical lifting cycles.** Assume that $\dim(\mathbb{V})=2$. When $\chi(\mathbb{V})=-1$, for $\mathbf{y}\in\mathbb{V}$, by [Shi22, Theorem 4.5], we have the following equality of Cartier divisors on $\mathcal{N}_{2,-1}^{\mathrm{Kra}}$.

$$\tilde{\mathcal{Z}}_{2,-1}(\mathbf{y})=\mathcal{Z}_0+\sum_{s=1}^{\mathrm{v}(\mathbf{y})}(\mathcal{Z}_s^{+}+\mathcal{Z}_s^{-}).$$

Here $\mathcal{Z}_0$ (resp. $\mathcal{Z}_s^{\pm}$) is a canonical (resp. quasi-canonical) lifting cycle of level 0 (resp. s), see [Shi22, §3]. Moreover by [Shi22, Proposition 3.12], $\mathcal{Z}_s^{+}$ and $\mathcal{Z}_s^{-}$ do not intersect when $s\geq 1$. Let $\mathcal{O}_s:=\mathcal{O}_{F_0}+\mathcal{O}_F\cdot\pi_0^s$ and M_s be the finite abelian extension of $\breve{F}$ corresponding to the subgroup $\mathcal{O}_s^{\times}$ under local class field theory. Let W_s be the integral closure of $\mathcal{O}_{\breve{F}}$ in M_s. Then we have $\mathcal{Z}_0\cong\operatorname{Spf}\mathcal{O}_{\breve{F}}$ and $\mathcal{Z}_s^{\pm}\cong\operatorname{Spf}W_s$. Define the primitive part of $\tilde{\mathcal{Z}}_{2,-1}(\mathbf{y})$ to be

$$\tilde{\mathcal{Z}}_{2,-1}(\mathbf{y})^{\circ}:=\begin{cases}\mathcal{Z}^{+}_{\mathrm{v}(\mathbf{y})}+\mathcal{Z}^{-}_{\mathrm{v}(\mathbf{y})} & \text{if } \mathrm{v}(\mathbf{y})>0,\\ \mathcal{Z}_0 & \text{if } \mathrm{v}(\mathbf{y})=0.\end{cases}$$

When $\chi(\mathbb{V})=1$, for $\mathbf{y}\in\mathbb{V}$ such that $\mathrm{v}(\mathbf{y})\geq 0$, by [HSY23, Theorem 4.1], we have the following equality of Cartier divisors on $\mathcal{N}_{2,1}^{\mathrm{Kra}}$.

$$\tilde{\mathcal{Z}}_{2,1}(\mathbf{y})=\mathcal{Z}_0+\mathcal{Z}_v(\mathbf{y}),$$

where $\mathcal{Z}_0\cong\operatorname{Spf}\mathcal{O}_{\breve{F}}$ is a canonical lifting cycle and $\mathcal{Z}_v(\mathbf{y})$ is a Cartier divisor whose structure sheaf is annihilated by π^N for some $N>0$. Define the primitive horizontal part of $\tilde{\mathcal{Z}}_{2,1}(\mathbf{y})$ to be

$$\tilde{\mathcal{Z}}_{2,1}(\mathbf{y})^{\circ}:=\begin{cases}0 & \text{if } \mathrm{v}(\mathbf{y})>0,\\ \mathcal{Z}_0 & \text{if } \mathrm{v}(\mathbf{y})=0.\end{cases}$$

4.2. **Horizontal cycles.**

Definition 4.1. Let $M^{\flat}$ be a rank $n-1$ integral lattice in $\mathbb{V}$. We say that $M^{\flat}$ is horizontal if one of the following conditions is satisfied

(1) $M^{\flat}$ is unimodular.
(2) $M^{\flat}$ is of the form $M^{\flat}=M\obot\operatorname{Span}\{\mathbf{y}\}$ where M is a unimodular sublattice of rank $n-2$ such that $(M_F)^{\perp}$ (the perpendicular complement of M_F in $\mathbb{V}$) is nonsplit.

Notice that condition (2) is independent of the choice of M. We denote the set of horizontal lattices by Hor.

For a rank $n-1$ integral lattice $L^{\flat}$, define

$$\operatorname{Hor}(L^{\flat}):=\{M^{\flat}\in\operatorname{Hor}\mid L^{\flat}\subseteq M^{\flat}\}. \tag{4.3}$$

Let $M^{\flat}\subset\mathbb{V}$ be a lattice of rank $n-1$ and type 1 or 0. We can decompose $M^{\flat}$ as

$$M^{\flat}=M\obot\operatorname{Span}\{\mathbf{y}\}, \tag{4.4}$$

for some unimodular lattice M of rank $n-2$. Then Proposition 2.6 and its corollary imply that

$$\tilde{\mathcal{Z}}(M^{\flat})\cong\tilde{\mathcal{Z}}_{2,\chi((M_F)^{\perp})}(\mathbf{y}).$$

Under this isomorphism, define $\tilde{\mathcal{Z}}(M^{\flat})^{\circ}$ to be the formal subscheme of $\tilde{\mathcal{Z}}(M^{\flat})$ isomorphic to $\tilde{\mathcal{Z}}_{2,\chi((M_F)^{\perp})}(\mathbf{y})^{\circ}$. By the discussion in §4.1, $\tilde{\mathcal{Z}}(M^{\flat})^{\circ}$ is nonempty if and only if $M^{\flat}\in\operatorname{Hor}$, in which case it consists of the union of irreducible components of $\tilde{\mathcal{Z}}(M^{\flat})$ isomorphic to $\operatorname{Spf}W_s$. In particular, $\tilde{\mathcal{Z}}(M^{\flat})^{\circ}$ is independent of the choice of M.

Theorem 4.2. *Let $L^\flat$ be a rank $n-1$ integral lattice in $\mathbb{V}$, then*

$$\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h = \bigcup_{M^\flat \in \mathrm{Hor}(L^\flat)} \tilde{\mathcal{Z}}(M^\flat)^\circ. \tag{4.5}$$

In particular, $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h$ is of pure dimension 1. Moreover we have the following identity in $\mathrm{Gr}^{n-1}K_0(\mathcal{N}^{\mathrm{Kra}})$:

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h}] = \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ}].$$

Proof. The proof largely follows [LZ22a, Section 4.4]. Let K be a finite extension of $\breve{F}$. Assume that z is an irreducible component of $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)(\mathcal{O}_K) = \mathcal{Z}^{\mathrm{Pap}}(L^\flat)(\mathcal{O}_K)$, and let G be the corresponding formal $\mathcal{O}_F$-module over $\mathcal{O}_K$. Define

$$L := \mathrm{Hom}_{\mathcal{O}_F}(T_p\mathcal{G}, T_pG)$$

where $\mathcal{G}$ is the canonical lifting and T_p is the integral p-adic Tate module. L is an $\mathcal{O}_F$-module of rank n equipped with the Hermitian form

$$\{x,y\} = \lambda_{\mathcal{G}}^\vee \circ y^\vee \circ \lambda_G \circ x,$$

under which it is self-dual. We have two inclusions (preserving Hermitian forms)

$$i_K : \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}, G)_F \to L_F,$$

and

$$i_k : \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}, G)_F \to \mathbb{V}.$$

By Lemma 4.4.1 of loc.cit., we have

$$\mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}, G) = i_K^{-1}(L). \tag{4.6}$$

Let

$$M^\flat := (L^\flat_F) \cap i_k(i_K^{-1}(L)) \cong \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}, G).$$

Then $z \subset \mathcal{Z}(M^\flat)(\mathcal{O}_K)$. Lemma 4.3 below implies that $t(M^\flat) \le 1$. Hence we know that z is one of the irreducible component of $\tilde{\mathcal{Z}}(M^\flat)^\circ \cong \tilde{\mathcal{Z}}_{2,\chi((M_F)^\perp)}(\mathbf{y})$ assuming the decomposition of $M^\flat$ as in (4.4). The nonemptiness of $\tilde{\mathcal{Z}}(M^\flat)^\circ$ implies that $M^\flat \in \mathrm{Hor}$. It remains to prove that z has multiplicity 1 in $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$. Consider R-points of both sides of (4.5), where $R := \mathcal{O}_K[x]/(x^2)$. As in [Krä03] (see [RZ96, Appendix of Chapter 3]) we know

$$\mathbb{D}(\mathcal{G})(R) \cong \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R, \text{ and } \mathbb{D}(G)(R) \cong (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)^n$$

where $\mathbb{D}$ is the $\mathcal{O}_{F_0}$-relative Dieudonné crystal. Define

$$\tilde{e}_0 = 1 \otimes 1 \in \mathbb{D}(\mathcal{G})(R), \quad \tilde{f}_0 = \pi \otimes 1 \in \mathbb{D}(\mathcal{G})(R).$$

Then the Hodge submodule $\mathcal{F}_0$ of $\mathbb{D}(\mathcal{G})(R)$ is spanned by

$$(1 \otimes \pi)\tilde{e}_0 + \tilde{f}_0.$$

$\mathbb{D}(G)(R)$ is equipped with an $\mathcal{O}_F$-invariant symplectic form $\langle , \rangle$ and we can assume that $\mathbb{D}(G)(R)$ has a basis $\{\tilde{e}_1, \ldots, \tilde{e}_n, \tilde{f}_1, \ldots, \tilde{f}_n\}$ such that

$$(\pi \otimes 1)\tilde{e}_i = \tilde{f}_i, \quad \langle \tilde{e}_i, \tilde{f}_j \rangle = \delta_{ij}.$$

Since any element in $L^\flat$ is $\mathcal{O}_F$-linear, we can arrange a change of basis if necessary and assume that

$$L^\flat((1\otimes\pi)\tilde{e}_0 + \tilde{f}_0) = \mathrm{Span}_R\{(1\otimes\pi^{a_1})((1\otimes\pi)\tilde{e}_1 + \tilde{f}_1), \ldots, (1\otimes\pi^{a_{n-1}})((1\otimes\pi)\tilde{e}_{n-1} + \tilde{f}_{n-1})\}.$$

Now $\mathbb{D}(G)(\mathcal{O}_K) = \mathbb{D}(G)(R) \otimes_R \mathcal{O}_K$. Let $e_i = \tilde{e}_i \otimes 1$ and $f_i = \tilde{f}_i \otimes 1$ respectively. There is an exact sequence of free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K$-modules (the Hodge filtration)

$$0 \to \mathrm{Fil} \to \mathbb{D}(G)(\mathcal{O}_K) \to \mathrm{Lie}\, G \to 0$$

where Fil is isotropic with respect to $\langle , \rangle$. We must have $L^\flat((1 \otimes \pi)e_0 + f_0) \subset \mathrm{Fil}$. Hence we have

$$(1 \otimes \pi)e_1 + f_1, \ldots, (1 \otimes \pi)e_{n-1} + f_{n-1} \subset \mathrm{Fil}.$$

Since Fil is isotropic and by the signature condition, we have

$$\mathrm{Fil} = \mathrm{Span}_{\mathcal{O}_K}\{(1 \otimes \pi)e_1 + f_1, \ldots, (1 \otimes \pi)e_{n-1} + f_{n-1}, (1 \otimes \pi)e_n - f_n\}.$$

Since $(x) \subset R$ has a nilpotent p.d. structure, by Grothendieck-Messing theory, a lift $\tilde{z}$ of z to $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)(R)$ corresponds to a lift of Fil to an isotropic $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-module $\widetilde{\mathrm{Fil}}$ in $\mathbb{D}(G)(R)$ containing the image of $L^\flat$. By the same reasonning as above, we must have

$$\widetilde{\mathrm{Fil}} = \mathrm{Span}_R\{(1 \otimes \pi)\tilde{e}_1 + \tilde{f}_1, \ldots, (1 \otimes \pi)\tilde{e}_{n-1} + \tilde{f}_{n-1}, (1 \otimes \pi)\tilde{e}_n - \tilde{f}_n\}.$$

Hence such lift is unique. This implies that the multiplicity of z in $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ is one. □

Lemma 4.3. *Let L be a self-dual Hermitian lattice of rank n and W be a $n-1$ dimensional subspace of L_F. Then $t(M^\flat) \leq 1$ for $M^\flat = L \cap W$.*

Proof. This is exactly the same as the proof of [LZ22a, Lemma 4.5.1]. Notice that in our case we may need some blocks $\begin{pmatrix} 0 & \pi^a \\ (-\pi)^a & 0 \end{pmatrix}$ in the upper left $(n-1) \times (n-1)$ block of T as in loc.cit. Alternatively, see [LL22, Lemma 2.24(2)]. □

We end this subsection with the following lemma.

Lemma 4.4. *Assume $M^\flat \in \mathrm{Hor}$. Then $\tilde{\mathcal{Z}}(M^\flat)^\circ$ intersects the special fiber of $\mathcal{N}^{\mathrm{Kra}}_{n,\epsilon}$ at a unique Exc_Λ for some $\Lambda \in \mathcal{V}^0$. Moreover*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) = \begin{cases} 1 & \text{if } M^\flat \text{ is unimodular,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. By the definition of Hor, we can find a decomposition of $M^\flat$

$$M^\flat = M \obot \{\mathbf{x}\}$$

such that M is self dual. Let Λ be any vertex lattice containing $M^\flat$. If $M^\flat$ is unimodular, then Λ has to be of the form $M^\flat \obot L'$ where L' is the unique unimodular lattice in $(M^\flat_F)^\perp$. If $M^\flat$ is of the form $M \obot L'$ such that M is of rank $n-2$ and $(M_F)^\perp$ is nonsplit, then the proof of [Shi18, Theorem 3.10] implies that there is a unique vertex lattice Λ' in $(M_F)^\perp$ which is of unimodular (this fact is the same as the fact that the Bruhat-Tits building of $(M_F)^\perp$ has only one point). Then Λ must be of the form $M \obot \Lambda'$. In both cases, Λ is unique and is of type 0.

Assume $\chi(M) = \eta$. By Proposition 2.6, $\tilde{\mathcal{Z}}(M) \cong \mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. Moreover $\tilde{\mathcal{Z}}(M) \cap \mathrm{Exc}_\Lambda = \mathbb{P}^1_k$ is an exceptional divisor in $\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. Hence by Lemma 3.1, we have

$$\chi(\mathcal{N}^{\mathrm{Kra}}_{n,\epsilon}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) = \chi(\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ} \otimes^{\mathbb{L}}_{\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}} \mathcal{O}_{\mathbb{P}^1_k}).$$

Now the lemma follows from [HSY23, Lemma 5.2] when $\epsilon\eta = 1$, and from [Shi22, Proposition 3.11] when $\epsilon\eta = -1$. □

4.3. **The horizontal part of special difference cycles.** Definition 2.15 motivates us to make the following definition.

Definition 4.5. When $L^\flat$ is a rank $n-1$ integral lattice, define $\mathcal{D}(L^\flat)_h \in \mathrm{Gr}^{n-1}K_0(\mathcal{N}^{\mathrm{Kra}})$ by

$$\mathcal{D}(L^\flat)_h = [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h}] + \sum_{i=1}^{n-1}(-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat\subset L'\subset \frac{1}{\pi}L^\flat \\ \dim_{\mathbb{F}_q}(L'/L^\flat)=i}} [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L')_h}]. \tag{4.7}$$

Proposition 4.6. *Assume $L^\flat$ is a rank $n-1$ integral lattice, then*

$$\mathcal{D}(L^\flat)_h = \begin{cases} \tilde{\mathcal{Z}}(L^\flat)^\circ & \text{if } L^\flat \in \mathrm{Hor}, \\ 0 & \text{if } L^\flat \notin \mathrm{Hor}. \end{cases}$$

Proof. By Theorem 4.2, it suffices to compute the multiplicity of an irreducible component in $\tilde{\mathcal{Z}}(M^\flat)^\circ$ in $\mathcal{D}(L)_h$ for all $M^\flat \in \mathrm{Hor}(L^\flat)$ (see (4.3)). For such a $M^\flat$, define

$$M' \coloneqq \frac{1}{\pi}L^\flat \cap M^\flat \text{ and } m \coloneqq \dim_{\mathbb{F}_q}(M'/L^\flat).$$

Then for a lattice L' with $L^\flat \subset L' \subset \frac{1}{\pi}L^\flat$, we know that $\tilde{\mathcal{Z}}(M^\flat)^\circ$ is in $\mathcal{Z}^{\mathrm{Kra}}(L')_h$ if and only if $L' \subset M'$. Hence the multiplicity of an irreducible components in $\tilde{\mathcal{Z}}(M^\flat)^\circ$ in $\mathcal{D}(L)_h$ is

$$1 + \sum_{i=1}^{m}(-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat\subset L'\subset M' \\ \dim_{\mathbb{F}_q}(L'/L^\flat)=i}} 1. \tag{4.8}$$

Notice that $m=0$ if and only if $M' = M^\flat = L^\flat$, in this case the summation in (4.8) is over an empty set, hence (4.8) is equal to 1. If $m>0$, (4.8) is equal to 0 by (2.13). ☐

Part 2. The analytic side

5. Induction formula and primitive local density

In this section, we study various induction formulas of local density polynomials. Let M be a Hermitian $\mathcal{O}_F$-lattice of rank m with $\mathrm{v}(M) \coloneqq \min\{\mathrm{v}_\pi(h(v,v')) \mid v,v' \in M\} \geq -1$. and let $M^{[k]} = \mathcal{H}^k \oplus M$ for an integer $k \geq 0$. Let L be a Hermitian $\mathcal{O}_F$-lattice of rank n with Gram matrix T.

There is a polynomial $\alpha(M,L,X)$ of X—the local density polynomial—such that

$$\alpha(M,L,q^{-2k}) = \int_{\mathrm{Herm}_n(F)} \int_{(M^{[k]})^n} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x}\, dY, \tag{5.1}$$

where $T(\mathbf{x})$ is the moment matrix of $\mathbf{x}$, $d\mathbf{x}$ is the Haar measure on $(M^{[k]})^n$ with total volume 1, dY is the Haar measures on $\mathrm{Herm}_n(F)$ such that $\mathrm{Herm}_n(\mathcal{O}_F)$ has total volume 1, and ψ is an additive character of F_0 with conductor $\mathcal{O}_{F_0}$. Finally, we define $\langle X, Y\rangle = \mathrm{Tr}(XY)$ on Herm_n. We will also denote $\alpha(M,L) = \alpha(M,L,1)$ and

$$\alpha'(M,L) = -\frac{\partial}{\partial X}\alpha(M,L,X)|_{X=1}. \tag{5.2}$$

There is another way to define $\alpha(M,L,X)$ as follows. We use $\mathrm{Herm}_{L,M}$ to denote the scheme of Hermitian $\mathcal{O}_F$-module homomorphisms from L to M, which is a scheme of finite type over $\mathcal{O}_{F_0}$. More specifically, for an $\mathcal{O}_{F_0}$-algebra R, we define

$$L_R \coloneqq L \otimes_{\mathcal{O}_{F_0}} R, \quad (x\otimes a, y \otimes b)_R \coloneqq \pi(x,y) \otimes_{\mathcal{O}_{F_0}} ab \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R \text{ where } x,y \in L, a,b \in R.$$

Then

$$\mathrm{Herm}_{L,M}(R) = \{\phi \in \mathrm{Hom}_{\mathcal{O}_F}(L_R, M_R) \mid (\phi(x), \phi(y))_R \equiv (x, y)_R \text{ for all } x, y \in L_R\}.$$

To simplify the notation, we let

$$I(M, L, d) := \mathrm{Herm}_{L,M}(\mathcal{O}_{F_0}/(\pi_0^d)). \tag{5.3}$$

Then a direct calculation as in [Shi22, Lemma 6.1] shows that

$$\alpha(M, L) = q^{-dn(2m-n)}|I(M, L, d)| \tag{5.4}$$

for sufficiently large integers $d > 0$. Since $\alpha(M, L, X)$ only depends on the Gram matrices of M and L, we may also denote it by $\alpha(S, T, X)$ if S and T are the Gram matrices of M and L.

Now we define primitive local density polynomials. For $1 \le \ell \le n$, let

$$(M^{[k]})^{n,(\ell)} = \{\mathbf{x} = (x_1, \cdots, x_n) \in (M^{[k]})^n \mid \dim \mathrm{Span}\{x_1, \cdots, x_\ell\} = \ell \text{ in } M^{[k]}/\pi M^{[k]}\}. \tag{5.5}$$

For $L = L_1 \oplus L_2$, where $L_1 = \mathrm{Span}\{l_1, \cdots, l_\ell\}$ and $L_2 = \mathrm{Span}\{l_{\ell+1}, \cdots, l_n\}$, we define the local ℓ-primitive density to be

$$\beta(M^{[k]}, L_1 \oplus L_2)^{(\ell)} = \int_{\mathrm{Herm}_n(F)} \int_{(M^{[k]})^{n,(\ell)}} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x}\, dY. \tag{5.6}$$

When $\ell \neq n$, the above definition depends on a choice of $L = L_1 \oplus L_2$. Hence we always fix such a decomposition $L = L_1 \oplus L_2$ in this case. When $L = L_1 \obot L_2$, and L_i is represented by T_i, we also denote $\beta(M, L_1 \obot L_2)^{(\ell)}$ as $\beta(S, \mathrm{Diag}(T_1, T_2))^{(\ell)}$. When $\ell = n$, we simply denote $\beta(M, L_1 \oplus L_2)^{(\ell)}$ as $\beta(M, L)$.

Lemma 5.1. *Assume $L = L_1 \oplus L_2$ where $\mathrm{rank}(L_1) = n_1$. Then*

$$\alpha(M, L, X) = \sum_{L_1 \subset L_1' \subset L_{1,F}} (q^{n-m}X)^{\ell(L_1'/L_1)} \beta(M, L_1' \oplus L_2, X)^{(n_1)},$$

where $\ell(L_1'/L_1) = \mathrm{length}_{\mathcal{O}_F} L_1'/L_1$.

Proof. This is the analogue of [Kit83, Lemma 3]. Let $G = \mathrm{GL}_{n_1}(F) \cap \mathrm{M}_{n_1,n_1}(\mathcal{O}_F)$ and $U = \mathrm{GL}_{n_1}(\mathcal{O}_F)$. By choosing a basis $\{l_1, \cdots, l_{n_1}\}$ of L_1, we may identify $U\backslash G$ with $\{L_1' \mid L_1 \subset L_1' \subset L_{1,F}\}$ by sending g to $L_1 \cdot g^{-1}$. Then the identity we want to prove is equivalent to

$$\alpha(M, L, X) = \sum_{g \in U\backslash G} |\det g|^{2k+m-n} \beta(M, L_1 \cdot g^{-1} \obot L_2, X)^{(n_1)},$$

where $|\pi| = q^{-1}$. By a partition of M_k^n, we have

$$\begin{aligned}\alpha(M, L, X) &= \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^n} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x} \\ &= \sum_{g \in U\backslash G} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)} \cdot g_1} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x},\end{aligned}$$

where $g_1 = \mathrm{Diag}(g, I_{n-n_1})$, and the action of g_1 is simply matrix multiplication on the n components of $M^{n,(n_1)}$. Now

$$\begin{aligned}
&\int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}\cdot g_1} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x} \\
&= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}} \psi(\langle Y, T(\mathbf{x}g_1) - T\rangle) d\mathbf{x} \\
&= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}} \psi(\langle Y, (T(\mathbf{x}) - T[g_1^{-1}])[g_1]\rangle) d\mathbf{x} \\
&= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}} \psi(\langle Y[g_1^*], T(\mathbf{x}) - T[g_1^{-1}]\rangle) d\mathbf{x} \\
&= |\det g_1|^{2k+m-n} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}} \psi(\langle Y, T(\mathbf{x}) - T[g_1^{-1}]\rangle) d\mathbf{x} \\
&= |\det g_1|^{2k+m-n} \beta(M^{[k]}, L\cdot g_1^{-1})^{(n_1)}.
\end{aligned}$$

Here $T[g] \coloneqq g^*Tg$. Now the lemma is clear. ☐

Theorem 5.2. *Let L be as in Lemma 5.1. Then*

$$\alpha(M, L, X) = \sum_{i=1}^{n_1} (-1)^{i-1} q^{i(i-1)/2+i(n-m)} X^i \cdot \sum_{\substack{L_1\subset L_1'\subset \pi^{-1}L_1 \\ \dim L_1'/L_1 = i}} \alpha(M, L_1'\oplus L_2, X) + \beta(M, L, X)^{(n_1)}.$$

Proof. This is an analogue of [Kat99, Proposition 2.1]. The proof follows from a combination of the argument (in a reverse order) in 2.16 and Lemma 5.1. ☐

Motivated by Theorem 5.2, we give the following definition.

Definition 5.3. Let $L = L_1 \oplus L_2$ be as in Lemma 5.1. We define

$$\partial\mathrm{Den}(L)^{(n_1)} \coloneqq \partial\mathrm{Den}(L) - \sum_{i=1}^{n_1} (-1)^{i-1} q^{i(i-1)/2} \sum_{\substack{L_1\subset L_1'\subset L_{1,F} \\ \dim L_1'/L_1 = i}} \partial\mathrm{Den}(L_1'\oplus L_2). \tag{5.7}$$

Corollary 5.4. *Let $L = L_1 \oplus L_2$ be as in Lemma 5.1, and $\epsilon = \chi(L)$. Then*

$$\partial\mathrm{Den}(L)^{(n_1)} = \frac{1}{\alpha(I_n^{-\epsilon}, I_n^{-\epsilon})}\left(2\beta'(I_n^{-\epsilon}, L)^{(n_1)} + \sum_i c_\epsilon^{n,i} \beta(\mathcal{H}_{n,i}^{\epsilon}, L)^{(n_1)}\right).$$

As a corollary of Lemma 5.1, we have

Corollary 5.5. *Let $L = L_1 \oplus L_2$ be as in Lemma 5.1. Then we have the following identity where the summation is finite:*

$$\partial\mathrm{Den}(L) = \sum_{L_1\subset L_1'\subset L_{1,F}} \partial\mathrm{Den}(L_1'\oplus L_2)^{(n_1)}.$$

We may reduce the identity $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$ to a primitive version as the following theorem shows.

Theorem 5.6. *Let $L = L_1 \oplus L_2 \subset \mathbb{V}$ be as in Lemma 5.1.*

(1) Conjecture 1.1 is true for L if for every $L_1 \subset L_1' \subset L_{1,F}$, we have

$$\mathrm{Int}(L_1'\oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1'\oplus L_2)^{(n_1)}.$$

(2) *If Conjecture 1.1 holds for all lattices $L' = L_1' \oplus L_2$ of $\mathbb{V}$ of rank n with $L_1 \subset L_1' \subset L_{1,F}$, then*

$$\mathrm{Int}(L_1 \oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1 \oplus L_2)^{(n_1)}.$$

(3) *For $1 \le n_1 \le n$, Conjecture 1.1 is true if and only if for every lattice $L = L_1 \oplus L_2 \subset \mathbb{V}$ with $\mathrm{rank}(L_1) = n_1$, one has*

$$\mathrm{Int}(L_1 \oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1 \oplus L_2)^{(n_1)}.$$

Proof. (1) follows from Lemma 2.16 and Corollary 5.5. (2) follows from Definitions 2.15 and 5.3. (3) follows from (1) and (2). □

For the rest of this section, we assume that M is unimodular of rank m with a Gram matrix $\mathrm{Diag}(I_{m-1}, \nu)$. To go further with the calculation of $\alpha(M, L, X)$, we need an induction formula for $\beta(M, L, X)^{(\ell)}$ as follows. The proof is essentially the same as that of Corollary 9.11 of [KR11], and is left to the reader.

Proposition 5.7. *Let $L = L_1 \obot L_2$, where L_j is of rank n_j. Let $C(M^{[k]}, L_1)$ be the $\mathrm{U}(M^{[k]})$-orbits of sublattices $M(i) \subset M^{[k]}$ such that $M(i)$ is isometric to L_1, and write $C(M^{[k]}, L_1) = \sqcup_{i\in J}\{M(i)\}$. Then*

$$\beta(M, L, X)^{(n_1)} = \sum_{i\in J} |M : M(i) \obot M(i)^{\perp}|^{-n_2} |M(i)^{\vee} : M(i)|^{n_2} \beta_i(M, L_1, X)\alpha(M(i)^{\perp}, L_2),$$

where

$$\beta_i(M, L_1, X) = \lim_{d\to\infty} q^{-dn_1(2m+4k-n_1)} \#\{\phi \in I(M^{[k]}, L_1, d)^{(n_1)} \mid \exists\ \Phi \in \mathrm{U}(M) \text{ with } \phi(L_1) = \Phi(M(i))\},$$

and

$$I(M^{[k]}, L_1, d)^{(n_1)} := \{\phi \in I(M^{[k]}, L_1, d) \mid \mathrm{rank}_{\mathbb{F}_q} \phi(L_1) \otimes_{\mathcal{O}_F} \mathbb{F}_q = n_1\}.$$

Recall that $I(M^{[k]}, L_1, d)$ is defined in (5.3).

One special case is that $L = \mathcal{H}^i \obot L_2$. Since any sublattice of $M^{[k]} = M \obot \mathcal{H}^k$ isometric to $\mathcal{H}^i$ is always a direct summand of $M^{[k]}$ and $\alpha(M, \mathcal{H}^i, X) = \beta(M, \mathcal{H}^i, X)^{(2i)} = \beta(M, \mathcal{H}^i, X)$, the above proposition specializes to

Corollary 5.8. *Assume $L = \mathcal{H}^i \obot L_2$, then*

$$\alpha(M, L, X) = \beta(M, \mathcal{H}^i, X)\alpha(M, L_2, q^{2i}X) = \alpha(M, \mathcal{H}^i, X)\alpha(M, L_2, q^{2i}X). \tag{5.8}$$

We end this section with two more special cases of Proposition 5.7. Proofs are given in Appendix A.

Proposition 5.9. *Let the notation be as in Proposition 5.7. Assume $n_1 = 1$ and $L_1 = \langle t\rangle$ where $t \in \mathcal{O}_{F_0}$.*

1. *There always exists a primitive vector $M(1) \in \mathcal{H}^k$ with $q(M(1)) = t$, and*

$$M(1)^{\perp} \cong \mathcal{H}^{k-1} \obot I_m^{\chi(M)} \obot \langle -t\rangle.$$

Here $\langle t\rangle$ stands for a lattice $\mathcal{O}_F v$ of rank one with $(v, v) = t$.

2. *If $\mathrm{v}(t) = 0$, then there exist a primitive vector $M(0) \in M$ with $q(M(0)) = t$, and*

$$M(0)^{\perp} \cong \mathcal{H}^k \obot I_{m-2}^{\epsilon_{m-2}} \obot \langle \nu t\rangle.$$

Here $\epsilon_{m-2} = \chi((-1)^{\frac{(m-2)(m-3)}{2}})$.

(3) *If* $\mathrm{v}(t) > 0$*, then there exist a primitive vector* $M(0) \in M$ *with* $q(M(0)) = t$ *only when* M *is isotropic (i.e.* $\exists\, v \in M$ *with* $q(v) = 0$*). In this case,*

$$M(0)^\perp \cong \mathcal{H}^k \oplus I_{m-2}^{\chi(M)} \oplus \langle -t \rangle.$$

Assuming the existence of $M(1)$ *and* $M(0)$*, we have*

$$|M^{[k]} : M(i) \oplus M(i)^\perp|^{-1} |M(i)^\vee : M(i)| = \begin{cases} 1 & \text{if } i = 1, \\ q & \text{if } i = 0. \end{cases}$$

(4) *Under the action of* $\mathrm{U}(M^{[k]})$*,* v *is either in the same orbit of a fixed vector* $M(1) \in \mathcal{H}^k$ *or a fixed vector* $M(0) \in M$*.*

(5) *We have the following induction formula:*

$$\beta(M, L, X)^{(1)} = \beta_1(M, L_1, X)\alpha(M(1)^\perp, L_2) + q^{n-1}\beta_0(M, L_1, X)\alpha(M(0)^\perp, L_2).$$

Moreover,

(a) *For any* L_1*,*

$$\beta_1(M, L_1, X) = 1 - X.$$

(b) *Assume* $\mathrm{v}(t) = 0$*, then*

$$\beta_0(M, L_1, X) = \begin{cases} (1 + \chi(M)\chi(L)q^{-\frac{m-1}{2}})X & \text{if } m \text{ is odd}, \\ (1 - \chi(M)q^{-\frac{m}{2}})X & \text{if } m \text{ is even}. \end{cases}$$

(c) *Assume* $\mathrm{v}(t) > 0$*, then*

$$\beta_0(M, L_1, X) = \begin{cases} (1 - q^{1-m})X & \text{if } m \text{ is odd}, \\ \left(1 - q^{1-m} + \chi(M)(q-1)q^{-\frac{m}{2}}\right) X & \text{if } m \text{ is even}. \end{cases}$$

Proof. Parts (1)—(4) are proved in subsection A.1. The induction formula for $\beta(M, L, X)^{(1)}$ follows from Proposition 5.7. For the formula of $\beta_i(M, L_1, X)$, see Corollaries A.10 and A.12. □

Proposition 5.10. *Let the notation be as in Proposition 5.7. Assume* $\mathrm{v}(L_1) > 0$ *and* $n_1 = 2$*. Then we have a partition of* $C(M^{[k]}, L_1) = \bigsqcup_{i=0}^{2} C_i(M^{[k]}, L_1)$ *such that for any* $M(i) \in C_i(M^{[k]}, L_1)$*,* $M(i)^\perp$ *is isometric to*

$$(-L_1) \oplus \mathcal{H}^{k-i} \oplus M^{(i)}.$$

Here $M^{(i)}$ *is a unimodular* $\mathcal{O}_F$*-lattice of rank* $m - 2(2-i)$ *and has determinant* $(-1)^i \det L$*.*

Moreover, we have

$$\beta(M, L, X)^{(2)} = \sum_{i=0}^{2} q^{(2-i)(n-2)} \beta_i(M, L_1, X)\alpha(M(i)^\perp, L_2, X), \tag{5.9}$$

where

$$\beta_2(M, L_1, X) = (1 - X)(1 - q^2 X),$$

$$\beta_1(M, L_1, X) = q(q+1)\left((1 - q^{1-m}) + \delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}}\right) X(1 - X),$$

$$\beta_0(M, L_1, X) = \begin{cases} q(1 - q^{1-m})(1 - q^{3-m})X^2 & \text{if } m \text{ is odd}, \\ q\left((1 - q^{2-m}) + \chi(M)(q^2 - 1)q^{-\frac{m}{2}}\right)(1 - q^{2-m})X^2 & \text{if } m \text{ is even}. \end{cases}$$

Here $\delta_e(m) = 1$ *or* 0 *depending on whether* m *is even or odd.*

Proof. Equation (5.9) follows from Proposition 5.7 and Proposition A.5. For the formula of $\beta_i(S, L_1, X)$, see Corollaries A.10, A.13 and Proposition A.14. □

6. The modified Kudla-Rapoport conjecture

Recall that the Hermitian lattices used to define the correction terms are of the following forms:

$$\mathcal{H}^{\epsilon}_{n,i} := \mathcal{H}^i \oplus I^{\epsilon}_{n-2i} \quad \text{for } 1 \le i \le \frac{n}{2}, \quad \epsilon = \pm 1 \tag{6.1}$$

where I^{ϵ}_{n-2i} is the unimodular Hermitian lattice of rank $n-2i$ with $\chi(I^{\epsilon}_{n-2i}) = \chi(\mathcal{H}^{\epsilon}_{n,i}) = \epsilon$. When $n = 2r$ is even, we take $I^{\epsilon}_0 = 0$ and $\mathcal{H}^1_{n,r} = \mathcal{H}^r$.

Theorem 6.1. *Let $r_\epsilon = \frac{n-1}{2}$ when n is odd, and $r_\epsilon = \lfloor \frac{n+\epsilon}{2} \rfloor$ when n is even. In the following we just write r_ϵ as r.*

$$A^{\epsilon} = (A^{\epsilon}_{i,j}) = \begin{pmatrix} \alpha(\mathcal{H}^{\epsilon}_{n,1}, \mathcal{H}^{\epsilon}_{n,1}) & \alpha(\mathcal{H}^{\epsilon}_{n,2}, \mathcal{H}^{\epsilon}_{n,1}) & \cdots & \alpha(\mathcal{H}^{\epsilon}_{n,r}, \mathcal{H}^{\epsilon}_{n,1}) \\ 0 & \alpha(\mathcal{H}^{\epsilon}_{n,2}, \mathcal{H}^{\epsilon}_{n,2}) & \cdots & \alpha(\mathcal{H}^{\epsilon}_{n,r}, \mathcal{H}^{\epsilon}_{n,2}) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \alpha(\mathcal{H}^{\epsilon}_{n,r}, \mathcal{H}^{\epsilon}_{n,r}) \end{pmatrix},$$

$$B^{\epsilon} = {}^t(\alpha'(I^{-\epsilon}_n, \mathcal{H}^{n,1}_{\epsilon}), \cdots, \alpha'(I^{-\epsilon}_n, \mathcal{H}^{\epsilon}_{n,r})),$$

and

$$C^{\epsilon} = {}^t(c^{n,1}_{\epsilon}, \cdots, c^{n,r}_{\epsilon}),$$

where $c^{\epsilon}_{n,i}$ is as in Conjecture 1.1.

Then C^{ϵ} is the solution of the equation

$$A^{\epsilon} C^{\epsilon} = -2B^{\epsilon}. \tag{6.2}$$

Moreover,

$$A^{\epsilon}_{j,j} = 2q^{\frac{(n-2j)(n-2j-1)}{2}} \prod_{0<s\le j} (1-q^{-2s}) \prod_{1\le s\le \lfloor \frac{n-2j-1}{2} \rfloor} (1-q^{-2s}) \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1-\epsilon q^{-\frac{n-2j}{2}} & \text{if } n \text{ is even.} \end{cases} \tag{6.3}$$

Finally, for $i<j$,

$$A^{\epsilon}_{i,j} = A^{\epsilon}_{j,j} \cdot \begin{cases} I(n-2i, \frac{n-2i-1}{2}, j-i) & \text{if } n \text{ is odd,} \\ I(n-2i, \frac{n-2i-1+\epsilon}{2}, j-i) & \text{if } n \text{ is even,} \end{cases} \tag{6.4}$$

where

$$I(n,d,k) := \prod_{s=1}^{k} \frac{(q^{d-s+1}-1)(q^{n-d-s}+1)}{q^s-1}.$$

Proof. First notice that $\alpha(\mathcal{H}^{\epsilon}_{n,i}, \mathcal{H}^{\epsilon}_{n,j}) = 0$ if $i<j$. So (1.9) is indeed equivalent to (6.2), and there exists a unique solution C_ϵ.

Now we compute $A^{\epsilon}_{j,j}$ explicitly. Corollary 5.8 and Lemma A.9 imply that

$$\alpha(\mathcal{H}^{\epsilon}_{n,j}, \mathcal{H}^{\epsilon}_{n,j}) = \alpha(\mathcal{H}^j, \mathcal{H}^j)\alpha(I^{\epsilon}_{n-2j}, I^{\epsilon}_{n-2j}).$$

According to Lemma A.8,

$$\alpha(\mathcal{H}^j, \mathcal{H}^j) = \prod_{0<s\le j} (1-q^{-2s}).$$

By Lemma A.11,

$$\alpha(I^{\epsilon}_{n-2j}, I^{\epsilon}_{n-2j}) = |\mathrm{O}(\overline{I}^{\epsilon}_{n-2j})(\mathbb{F}_q)|,$$

where $\overline{I}^\epsilon_{n-2j} = I^\epsilon_{n-2j} \otimes_{\mathcal{O}_F} \mathcal{O}_F/(\pi)$ is the space over $\mathbb{F}_q$ with the naturally induced quadratic form. Now (6.3) follows from the well-known formula:

$$|\mathrm{O}(\overline{I}^\epsilon_{n-2j})(\mathbb{F}_q)| = \begin{cases} 2q^{\frac{(n-2j)(n-2j-1)}{2}} \prod_{s=1}^{\frac{n-2j-1}{2}} (1-q^{-2s}) & \text{if } n \text{ is odd,} \\ 2q^{\frac{(n-2j)(n-2j-1)}{2}} (1-\epsilon q^{-\frac{n-2j}{2}}) \prod_{s=1}^{\frac{n-2j}{2}-1} (1-q^{-2s}) & \text{if } n \text{ is even.} \end{cases}$$

To obtain (6.4), notice that (Corollary 5.8)

$$\alpha(\mathcal{H}^\epsilon_{n,j}, \mathcal{H}^\epsilon_{n,i}) = \alpha(\mathcal{H}^\epsilon_{n,j}, \mathcal{H}^i)\alpha(\mathcal{H}^\epsilon_{n-2i,j-i}, I^\epsilon_{n-2i}),$$

and

$$\alpha(\mathcal{H}^\epsilon_{n,j}, \mathcal{H}^\epsilon_{n,j}) = \alpha(\mathcal{H}^\epsilon_{n,j}, \mathcal{H}^i)\alpha(\mathcal{H}^\epsilon_{n-2i,j-i}, \mathcal{H}^\epsilon_{n-2i,j-i}).$$

Hence

$$\frac{A^\epsilon_{i,j}}{A^\epsilon_{j,j}} = \frac{\alpha(\mathcal{H}^\epsilon_{n,j}, \mathcal{H}^\epsilon_{n,i})}{\alpha(\mathcal{H}^\epsilon_{n,j}, \mathcal{H}^\epsilon_{n,j})} = \frac{\alpha(\mathcal{H}^\epsilon_{n-2i,j-i}, I^\epsilon_{n-2i})}{\alpha(\mathcal{H}^\epsilon_{n-2i,j-i}, \mathcal{H}^\epsilon_{n-2i,j-i})}.$$

Fix an $\mathcal{O}_F$-lattice L that is represented by I^ϵ_{n-2i}. According to Lemma 6.2, to compute $\frac{A^\epsilon_{i,j}}{A^\epsilon_{j,j}}$, we need to count the number of lattices L' in L_F such that contain $L \subset L'$ and $L' \cong \mathcal{H}^\epsilon_{n-2i,j-i}$, which is equivalent to the following condition:

$$\pi L \overset{j-i}{\subset} \pi L' \overset{n-2j}{\subset} (L')^\sharp \overset{j-i}{\subset} L \overset{j-i}{\subset} L'.$$

Since L' and $\pi L'$ determine each other, we just need to count $\pi L'$ satisfying the above condition. We regard $\pi L'/\pi L$ as a $(j-i)$-dimensional subspace of $L/\pi L$, where $L/\pi L$ is equipped with quadratic form $(x, y)/\pi$.

Claim: The condition

$$\pi L' \subset (L')^\sharp$$

is equivalent to the condition that $\pi L'/\pi L$ is an isotropic subspace of $L/\pi L$.

Indeed, assume $\pi L'/\pi L$ is an isotropic subspace of $L/\pi L$. Then $(\pi x, \pi y) \in \pi\mathcal{O}_F$ for any $x, y \in L'$, which is equivalent to $(x, \pi y) \in \mathcal{O}_F$ for any $x, y \in L'$. The latter condition is the same as $L' \subset (L')^\sharp$. The other direction is clear.

Therefore $\frac{A^\epsilon_{i,j}}{A^\epsilon_{j,j}}$ is the number of $(j-i)$-dimensional isotropic subspaces of $L/\pi L$. According to [LZ22b, Lemma 3.2.2], it equals to

$$\begin{cases} I(n-2i, \frac{n-2i-1}{2}, j-i) & \text{if } n \text{ is odd,} \\ I(n-2i, \frac{n-2i-1+\epsilon}{2}, j-i) & \text{if } n \text{ is even.} \end{cases}$$

□

According to Theorem 6.1, in order to solve C^ϵ, we need to know B^ϵ and A^ϵ. Here, B^ϵ can be calculated by applying Corollary 5.8 and Proposition 5.9 inductively. The following lemma can be used to compute A^ϵ.

Lemma 6.2. *Let F/F_0 be a quadratic p-adic field extension, and let L and M be two $\mathcal{O}_F$-Hermitian lattices of rank n. Then $\frac{\alpha(M,L)}{\alpha(M,M)}$ is equal to the number of lattices L' in L_F containing L and isometric to M.*

Proof. The proof is a generalization of that of Proposition 10.2 of [KR14b] and works for both inert and ramified primes.

Let us assume that there is an isometric embedding from L into M, otherwise both sides of the identity in the lemma are zero. In this case, we have a fixed $L_F \cong M_F$. Let α (resp. β) be a top degree translation invariant form on L_F^n (resp. $\mathrm{Herm}_n(F)$). Let $\nu_p = \alpha/h^*(\beta)$ where

$$h: \quad L_F^n \to \mathrm{Herm}_n(F), \quad x \mapsto (x, x).$$

Define X to be the set of F-linear isometric embeddings from L into M. By fixing a basis of L_F and regarding $\phi \in X$ as a linear isometry from L_F to itself, we identify X as a subset of L_F^n. By the argument in Section 3 of [GY00] (in particular Lemma 3.4), we know that

$$\alpha(M, L) = \mathrm{vol}(X, d\nu_p) \frac{\mathrm{vol}(\mathrm{Herm}_n(\mathcal{O}_F), d\beta)}{\mathrm{vol}((M)^n, d\alpha)}. \tag{6.5}$$

For any $\phi \in X$ regarded as a linear isometry from L_F to itself, the lattice $L_\phi := \phi^{-1}(M)$ is a lattice containing L. Conversely, for any L' containing L and isometric to M, there is a $\phi \in X$ such that $L_\phi = L'$. Hence we have a partition

$$X = \bigsqcup_{L \subset L'} X_{L'}, \quad X_{L'} := \{\phi \in X \mid L_\phi = L'\}.$$

Since each L' is isomorphic to M, all the $X_{L'}$ have the same volume as that of X_M. Specializing (6.5) to $L = M$, we see

$$\alpha(M, M) = \mathrm{vol}(X_M, d\nu_p) \frac{\mathrm{vol}(\mathrm{Herm}_n(\mathcal{O}_F), d\beta)}{\mathrm{vol}((M)^n, d\alpha)}. \tag{6.6}$$

Dividing equation (6.5) by (6.6), we prove the lemma. □

Remark 6.3. *When F/F_0 is unramified and M is unimodular, the lemma was proved by equation (3.6.1.1) of* [LZ22a].

Now we specialize Theorem 6.1 to the case $n = 3$.

Lemma 6.4. *Assume $n = 3$ and $\epsilon = \chi(L)$. Then $c^\epsilon_{3,1} = \frac{q^2}{1+q}$, hence*

$$\partial\mathrm{Den}(L) = 2\frac{\alpha'(I_3^{-\epsilon}, L)}{\alpha(I_3^{-\epsilon}, I_3^{-\epsilon})} + \frac{q^2}{1+q}\frac{\alpha(\mathcal{H}^\epsilon_{3,1}, L)}{\alpha(I_3^{-\epsilon}, I_3^{-\epsilon})}.$$

Proof. First of all, according to Theorem 6.1,

$$\alpha(\mathcal{H}^\epsilon_{3,1}, \mathcal{H}^\epsilon_{3,1}) = 2(1 - q^{-2}). \tag{6.7}$$

By Corollary 5.8, we have

$$\alpha(I_3^{-\epsilon}, \mathcal{H}^\epsilon_{3,1}, X) = \alpha(I_3^{-\epsilon}, \mathcal{H}, X)\alpha(I_3^{-\epsilon}, I_1^\epsilon, q^2X).$$

According to Lemmas A.8 and A.9,

$$\alpha(I_3^{-\epsilon}, \mathcal{H}, X) = \beta(\mathcal{H}^k, \mathcal{H}) = 1 - X.$$

Lemma 7.1 gives that

$$\alpha(I_3^{-\epsilon}, I_1^\epsilon, q^2X) = 1 - qX.$$

Hence

$$\alpha(I_3^{-\epsilon}, \mathcal{H}^\epsilon_{3,1}, X) = (1 - X)(1 - qX),$$

and

$$\alpha'(I_3^{-\epsilon}, \mathcal{H}^\epsilon_{3,1}) = 1 - q.$$

Combining this with (6.7), we solve (6.2) and obtain

$$c^{\epsilon}_{3,1} = \frac{q^2}{1+q}.$$

Now the lemma follows from (1.10). $\square$

7. Local density formula when $\mathrm{rank}(T) \leq 2$

The main purpose of this section is to give an explicit formula for $\alpha(I, T, X)$ where $I = \mathrm{Diag}(I_{m-1}, \nu)$ with $\nu \in \mathcal{O}_{F_0}^{\times}$ and $\mathrm{rank}(T) \leq 2$.

7.1. **The case $T = (t)$.** In order to apply induction formulas to calculate $\alpha(I, T, X)$ for T with $\mathrm{rank}(T) = 2$, we need to consider the case $T = (t)$ first. Write $t = t_0(-\pi_0)^{\mathrm{v}(t)}$ for $t_0 \in \mathcal{O}_{F_0}^{\times}$, and

$$I_{a,b} = \mathrm{Diag}(I, \nu_1(-\pi_0)^a, \nu_2(-\pi_0)^b) = \mathrm{Diag}(s_1, \ldots, s_{m+2}) \tag{7.1}$$

for integers $0 \leq a \leq b$.

Lemma 7.1. *Assume $0 \leq a \leq b \leq \mathrm{v}(t)$.*

(1) If m is odd, then

$$\alpha(I_{a,b}, (t), X) = 1 + \chi(I)\chi(-\nu_1)(q-1) \sum_{s=a+1}^{b} q^{-ms+a+\frac{m-1}{2}} X^s + \chi(I_{a,b})\chi(t_0) q^{-(m+1)\mathrm{v}(t)+a+b-\frac{m+1}{2}} X^{\mathrm{v}(t)+1}.$$

(2) If m is even, then

$$\alpha(I_{a,b}, (t), X) = 1 + \chi(I)(q-1) \sum_{s=1}^{a} q^{-(m-1)s+\frac{m}{2}-1} X^s + \chi(I_{a,b}) q^{a+b} \left((q-1) \sum_{s=b+1}^{\mathrm{v}(t)} q^{-(m+1)s+\frac{m}{2}} X^s - q^{-(m+1)\mathrm{v}(t)-1-\frac{m}{2}} X^{\mathrm{v}(t)+1} \right).$$

Proof. Direct calculation gives

$$\begin{aligned}
\alpha(I_{a,b}, (t), X) &= \int_{F_0} dY \int_{\mathcal{O}_F^{2k+m+2}} \psi(\langle Y, \mathrm{Diag}(\mathcal{H}^k, I_{a,b})[\mathbf{x}] - t\rangle)\, d\mathbf{x} \\
&= \int_{F_0} \psi(-tY)\, dY \int_{\mathcal{O}_F^{2k} \times \mathcal{O}_F^{m+2}} \psi(Y \sum_{i=1}^{k} \mathrm{tr}(\frac{1}{\pi} x_i \bar{y}_i) + Y \sum_{l=1}^{m+2} s_l z_l \bar{z}_l) \prod_i dx_i dy_i \prod_l dz_l \\
&= 1 + \sum_{s=1}^{\infty} \int_{\mathrm{v}(Y)=-s} I_k(Y) I_{I_{a,b}}(Y) \psi(-tY)\, dY.
\end{aligned}$$

Here, according to [Shi22, Lemma 7.6],

$$I_k(Y) = \int_{\mathcal{O}_F^{2k}} \psi(Y \sum_{i=1}^{k} \mathrm{tr}(\frac{1}{\pi} x_i \bar{y}_i)) \prod dx_i dy_i = q^{-2ks},$$

and

$$I_{I_{a,b}}(Y) = \int_{\mathcal{O}_F^{m+2}} \psi(Y \sum_{l=1}^{m+2} s_l z_l \bar{z}_l) \prod dz_l = \prod_{l=1}^{m+2} J(s_l Y),$$

where

$$J(t) = \int_{\mathcal{O}_F} \psi(tz\bar{z})dz = \begin{cases} 1 & \text{if } \mathrm{v}(t) \geq 0, \\ q^{\mathrm{v}(t)}\chi(-t_0)g(\chi, \psi_{\frac{1}{\pi_0}}) & \text{if } \mathrm{v}(t) < 0, \end{cases} \tag{7.2}$$

and

$$g(\chi, \psi_{\frac{1}{\pi_0}}) = \sum_{x \in \mathcal{O}_{F_0}/\pi_0} \chi(x)\psi(\frac{x}{\pi_0})$$

is the Gauss sum. Write $\psi' = \psi_{\frac{1}{\pi_0}}$. Then

$$\begin{aligned} \alpha(I_{a,b}, (t), X) = 1 &+ \sum_{s=1}^{a} q^s \int_{\mathcal{O}_{F_0}^\times} q^{-2ks} \cdot q^{-ms}\chi(\nu(-Y)^m)g(\chi, \psi')^m\psi(-(-\pi_0)^s Yt)dY \\ &+ \sum_{s=a+1}^{b} \int_{\mathcal{O}_{F_0}^\times} q^{-2ks} \cdot q^{-ms+a}\chi(\nu_1\nu(-Y)^{m+1})g(\chi, \psi')^{m+1}\psi(-(-\pi_0)^{-s}Yt)\, dY \\ &+ \sum_{s=b+1}^{\infty} \int_{\mathcal{O}_{F_0}^\times} q^{-2ks} \cdot q^{-(m+1)s+a+b}\chi(\nu_1\nu_2\nu(-Y)^{m+2})g(\chi, \psi')^{m+2}\psi(-(-\pi_0)^{-s}Yt)\, dY. \end{aligned}$$

Recall the well-known facts that

$$\begin{aligned} g(\chi, \psi')^2 &= \chi(-1) \cdot q, \\ \int_{\mathcal{O}_{F_0}^\times} \psi((-\pi_0)^{-s}Yt)dY &= \mathrm{Char}(\pi_0^s\mathcal{O}_{F_0})(t) - q^{-1}\,\mathrm{Char}(\pi_0^{s-1}\mathcal{O}_{F_0})(t), \\ \int_{\mathcal{O}_{F_0}^\times} \chi(Y)\psi((-\pi_0)^{-s}Yt)dY &= \chi(-t_0)q^{-1}g(\chi, \psi')\,\mathrm{Char}(\pi_0^{s-1}\mathcal{O}_{F_0}^\times)(t). \end{aligned} \tag{7.3}$$

When m is odd, we have

$$\begin{aligned} \alpha(I_{a,b}, (t), X) = 1 &+ \chi((-1)^{\frac{m+1}{2}}\nu_1\nu)(q-1)\sum_{s=a+1}^{b} q^{-ms+a+\frac{m-1}{2}}X^s \\ &+ \chi((-1)^{\frac{m+1}{2}}\nu_1\nu_2\nu t_0)q^{-(m+1)(\mathrm{v}(t)+1)+a+b+\frac{m+1}{2}}X^{\mathrm{v}(t)+1}. \end{aligned}$$

When m is even, we have

$$\begin{aligned} &\alpha(I_{a,b}, (t), X) \\ &= 1 + \chi((-1)^{\frac{m}{2}}\nu)(q-1)\sum_{s=1}^{a} q^{-(m-1)s+\frac{m}{2}-1}X^s + \chi((-1)^{\frac{m+2}{2}}\nu_1\nu_2\nu) \\ &\quad \cdot \left((q-1)\sum_{s=b+1}^{\mathrm{v}(t)} q^{-(m+1)s+a+b+\frac{m}{2}}X^s - q^{-(m+1)(\mathrm{v}(t)+1)+a+b+\frac{m}{2}}X^{\mathrm{v}(t)+1} \right). \end{aligned}$$

Finally, notice that for I of rank m we have

$$\chi(I) = \begin{cases} \chi((-1)^{\frac{m-1}{2}}\nu) & \text{if } m \text{ is odd}, \\ \chi((-1)^{\frac{m}{2}}\nu) & \text{if } m \text{ is even}. \end{cases}$$

Now the lemma is clear. □

Similarly, we have the following lemma.

Lemma 7.2. *Let I be unimodular with odd rank m. Then*

$$\alpha(I \oplus \mathcal{H}_i, (t), X) = \begin{cases} 1 + \chi(I)\chi(t_0) q^{-(\mathrm{v}(t)+1)(m+1)+\frac{m+1}{2}+i} X^{\mathrm{v}(t)+1} & \text{if } i \leq 2\mathrm{v}(t), \\ 1 + \chi(I)\chi(t_0) q^{-(\mathrm{v}(t)+1)(m-1)+\frac{m-1}{2}} X^{\mathrm{v}(t)+1} & \text{if } i > 2\mathrm{v}(t). \end{cases}$$

7.2. **The case $T = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b)$.** In this subsection, we compute $\alpha(I, T, X)$ for I unimodular of rank $m \geq 2$ and $T = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b)$ with $0 \leq a \leq b$. Notice that $\alpha(I, T, X) = 0$ when $a < 0$.

Proposition 7.3. *Assume $T = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b)$ and that I is isotropic of even rank $m \geq 2$, then*

$$\begin{aligned}
\alpha(I, T, X) =& (1 - X) \left(\sum_{i=0}^{a} (q^{2-m} X)^i + \gamma_e(I, T, X) \right) \\
&+ qX (q^{2-m} X)^a (1 - \chi(I) q^{-\frac{m}{2}})(1 + \chi(I)\chi(T) q^{\frac{m-2}{2}} (q^{2-m} X)^{b+1}) \\
&+ \left(1 - q^{-(m-1)} + (q-1)\chi(I) q^{-\frac{m}{2}}\right) X \\
&\quad \cdot \left(q \sum_{i=0}^{a-1} (q^{2-m} X)^i + \gamma_e(I, T, X) - \chi(I)\chi(T) q^{\frac{m}{2}} (q^{2-m} X)^{a+b+1} \right),
\end{aligned}$$

where

$$\gamma_e(I, T, X) = \chi(I) q^{\frac{m}{2}} \left(\sum_{d=1}^{a} (q^d - 1)(q^{2-m} X)^d + \chi(T) q^a (q^{2-m} X)^{b+1} \sum_{i=0}^{a} (q^{1-m} X)^i \right).$$

Proof. Since I is of even rank, $u_1 \cdot I^{[k]} \approx I^{[k]}$, and we may assume T is of the form $\mathrm{Diag}((-\pi_0)^a, u(-\pi_0)^b)$ without loss of generality.

According to Theorem 5.2 and Proposition 5.9, we have

$$\begin{aligned}
\alpha(I, T, X) = &\beta_1(I, (-\pi_0)^a, X) \alpha(M(1)^\perp, u(-\pi_0)^b) \\
&+ q\beta_0(I, (-\pi_0)^a, X) \alpha(M(0)^\perp, u(-\pi_0)^b) \\
&+ q^{2-m} X \alpha(I, \mathrm{Diag}((-\pi_0)^{a-1}, u(-\pi_0)^b), X)
\end{aligned}$$

where $M(1)^\perp = \mathrm{Diag}(\mathcal{H}^{k-1}, -(-\pi_0)^i, I)$ and

$$M(0)^\perp = \mathrm{Diag}(\mathcal{H}^k, \underbrace{-(-\pi_0)^i, 1, \ldots, 1, -\nu}_{m-1}).$$

Continuing this process, we obtain

$$\begin{aligned}
&\alpha(I, T, X) \\
&= \sum_{i=0}^{a} (q^{2-m} X)^{a-i} \cdot \left(\beta_1(I, (-\pi_0)^i, X) \alpha(M(1)^\perp, u(-\pi_0)^b) + q\beta_0(I, (-\pi_0)^i, X) \alpha(M(0)^\perp, u(-\pi_0)^b) \right).
\end{aligned}$$

By the formulas in Proposition 5.9 and Lemma 7.1, the above equals to

$$\begin{aligned}
&\sum_{i=0}^{a}(q^{2-m}X)^{a-i}(1-X)\\
&\cdot\left(1+(q-1)\chi(I)q^{\frac{m-2}{2}}\sum_{s=-i}^{-1}(q^{(m-1)}(q^2X)^{-1})^s+\chi(I)\chi(T)q^{i+\frac{m}{2}}(q^{2-m}X)^{b+1}\right)\\
&+q(q^{2-m}X)^aX(1-q^{-\frac{m}{2}}\chi(I))(1+\chi(I)\chi(T)q^{-(b+1)(m-2)+\frac{m-2}{2}}X^{b+1})\\
&+q\sum_{i=1}^{a}(q^{2-m}X)^{a-i}X\left((1-q^{-(m-1)})+(q-1)\chi(I)q^{-(m-1)+\frac{m-2}{2}}\right)\\
&\cdot\left(1+(q-1)\chi(I)q^{\frac{m-4}{2}}\sum_{s=-i}^{-1}(q^{(m-3)}X^{-1})^s+\chi(I)\chi(T)q^{i+\frac{m-2}{2}}(q^{2-m}X)^{b+1}\right).
\end{aligned}$$

Now the transformation

$$\sum_{i=0}^{a}\sum_{s=1}^{i}q^s(q^{2-m}X)^{a-i+s}=\sum_{d=1}^{a}\sum_{s=1}^{d}q^s(q^{2-m}X)^d$$

and some calculation give us the result we want. □

The case that I is anisotropic (i.e. when $m=2$ and $\chi(I)=-1$) can be computed similarly and is simpler. We omit the detail here. In particular, we may recover the following formula.

Proposition 7.4. [Shi22, Theorem 6.2(1)] *Assume* $I=\mathrm{Diag}(1,\nu)$*, then*

$$\begin{aligned}
&\alpha(I,T,X)\\
&=(1-X)(1+\chi(I)+q\chi(I))\sum_{e=0}^{\alpha}(qX)^e-\chi(T)q^{\alpha+1}X^{\beta+1}(1-X)\sum_{e=0}^{\alpha}(q^{-1}X)^e\\
&\quad-\chi(T)(1+q)(X^{\alpha+\beta+2}+\chi(I)\chi(T))+(1+\chi(I))q^{\alpha+1}X^{\alpha+1}(1+\chi(T)X^{\beta-\alpha}).
\end{aligned}$$

Moreover, a similar computation yields the following, and we leave the detail to reader.

Proposition 7.5. *Assume that* I *is unimodular of odd rank* $m\geq 3$*. Then*

$$\begin{aligned}
&\alpha(I,T,X)\\
&=(1-X)\left(\sum_{i=0}^{a}(q^{2-m}X)^i+\gamma_{o,1}(I,T,X)\right)+(1-q^{-(m-1)})X\left(q\sum_{i=0}^{a-1}(q^{2-m}X)^i+\gamma_{o,0}(S,T,X)\right)\\
&\quad+qX(q^{2-m}X)^a\left(1+\chi(I)\chi(u_1)q^{-\frac{m-1}{2}}\right)\left(1-\chi(I)\chi(u_1)q^{(2-m)b-\frac{m-1}{2}}X^{b+1}\right),
\end{aligned}$$

where $\gamma_{o,1}(I,T,X)$ *equals*

$$\chi(I)\chi(u_1)q^{\frac{m-1}{2}}\left(\sum_{d=a+1}^{a+b}(q^{a+b+1-d}-1)(q^{2-m}X)^d-\sum_{i=b+1}^{a+b+1}(q^{2-m}X)^i\right),$$

and $\gamma_{o,0}(I,T,X)$ *equals*

$$\chi(I)\chi(u_1)q^{\frac{m-1}{2}}\left(\sum_{d=a+1}^{a+b}(q^{a+b+1-d}-q)(q^{2-m}X)^d-\sum_{i=b+1}^{a+b}(q^{2-m}X)^i\right).$$

8. Local density formula when $\mathrm{rank}(T) = 3$

In this section, we always assume $\mathrm{rank}(T) = 3$ and $S = I_3^{-\chi(T)}$. The aim of this section is to compute $\partial\mathrm{Den}(T)$ explicitly. We treat the case $\mathrm{v}(T) \le -1$ in the first subsection. In the second subsection, we deal with the case when $T = \mathrm{Diag}(1, T_2)$ for T_2 diagonal. In the last subsection, instead of $\partial\mathrm{Den}(T)$, we compute $\partial\mathrm{Den}(T)^{(2)}$ for T of the form not covered by previous subsections.

8.1. $\partial\mathrm{Den}(T)$ for T with $\mathrm{v}(T) \le -1$.

Proposition 8.1. *If* $\mathrm{v}(T) \le -1$, *then* $\mathrm{Int}(T) = \partial\mathrm{Den}(T) = 0$.

Proof. If $\mathrm{v}(T) < -1$, then $\partial\mathrm{Den}(T) = 0$ since $\mathrm{v}(S^{[k]}) \ge -1$ where $S \oplus \mathcal{H}^k$. If $\mathrm{v}(T) = -1$, then T is of the form $\mathrm{Diag}(\mathcal{H}, (u_3(-\pi_0)^c))$ with $\chi(T) = \chi(u_3)$. In this case, according to Corollary 5.8, Lemmas A.8 and A.9, we have

$$\alpha(S, T, X) = (1 - X)\alpha(S, (u_3(-\pi_0)^c), q^2 X).$$

Similarly, we have

$$\begin{aligned}\alpha(\mathcal{H}^{3,1}_{\chi(T)}, T) &= \beta(\mathcal{H}^{3,1}_{\chi(T)}, \mathcal{H})\alpha(u_3, (u_3(-\pi_0)^c)) \\ &= (1 - q^{-2})\alpha(u_3, (u_3(-\pi_0)^c)).\end{aligned}$$

Hence, applying Lemma 7.1 to $I_{0,0} = S$ where I is of rank 1, we have

$$\begin{aligned}\partial\mathrm{Den}(T) &= 2\alpha(S, (u_3(-\pi_0)^c), q^2) + \frac{q^2}{1+q}(1 - q^{-2})\alpha((u_3), (u_3(-\pi_0)^c)) \\ &= 2(1 + \chi(S)\chi(u_3)q) + 2(q - 1) \\ &= 2(1 - q) + 2(q - 1) \\ &= 0.\end{aligned}$$

Here we are using the fact $\chi(S)\chi(T) = \chi(S)\chi(u_3) = -1$. □

8.2. $\partial\mathrm{Den}(T)$ for $T = \mathrm{Diag}(1, T_2)$ with T_2 diagonal.

In this subsection, we assume $T = \mathrm{Diag}(1, T_2)$, where $T_2 = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b)$ with $0 \le a \le b$. Let $u = u_1u_2$. Also, let $S = \mathrm{Diag}(1, 1, \nu)$ and $S_2 = \mathrm{Diag}(1, \nu)$. We compare $\partial\mathrm{Den}(T)$ and $\partial\mathrm{Den}(T_2)$ in this subsection.

Recall that

$$\partial\mathrm{Den}(T) = 2\frac{\alpha'(S, T)}{\alpha(S, S)} + \frac{q^2}{1+q}\frac{\alpha(\mathcal{H}^{3,1}_{\chi(T)}, T)}{\alpha(S, S)}.$$

Moreover, according to [Shi22, Theorem 1.3] and [HSY23, Theorem 1.1], the analytic side in the case $n = 2$ is

$$\partial\mathrm{Den}(T_2) = 2\frac{\alpha'(S_2, T_2)}{\alpha(S_2, S_2)} - \frac{2q^2}{q^2 - 1}\frac{\alpha(\mathcal{H}, T_2)}{\alpha(S_2, S_2)}.$$

Proposition 8.2.

$$\partial\mathrm{Den}(T) - \partial\mathrm{Den}(T_2) = \begin{cases} 1 + 2\sum_{i=1}^{a} q^i & \text{if } \chi(T) = 1, \\ 1 & \text{if } \chi(T) = -1. \end{cases}$$

Proof. Proposition 5.9 implies that $\alpha(S, T, X)$ equals

$$(1 - X)\alpha(\mathrm{Diag}(-1, S), T_2, q^2 X) + q^2(1 + q^{-1}\chi(S))X\alpha(S_2, T_2, X).$$

Hence

$$\alpha'(S, T) = \alpha(\mathrm{Diag}(-1, S), T_2, q^2) + q^2(1 + q^{-1}\chi(S))\alpha'(S_2, T_2). \tag{8.1}$$

According to Lemma A.11, one can check that $\alpha(S,S)=\beta(S,S)=2q(q^2-1)$, and $\alpha(S_2,S_2)=2(q-\chi(S_2))$. Then

$$\frac{\alpha'(S,T)}{\alpha(S,S)}-\frac{\alpha'(S_2,T_2)}{\alpha(S_2,S_2)}=\frac{\alpha(\mathrm{Diag}(-1,S),T_2,q^2)}{\alpha(S,S)}. \tag{8.2}$$

Hence we just need to check that

$$\begin{aligned}&2\frac{\alpha(\mathrm{Diag}(-1,S),T_2,q^2)}{\alpha(S,S)}+\frac{q^2}{1+q^2}\frac{\alpha(\mathcal{H}^{3,1}_{\chi(T)},T)}{\alpha(S,S)}+\frac{2q^2}{q^2-1}\frac{\alpha(\mathcal{H},T_2)}{\alpha(S_2,S_2)}\\ &=\begin{cases}1+2\sum_{i=1}^a q^i & \text{if } \chi(T)=1,\\ 1 & \text{if } \chi(T)=-1.\end{cases}\end{aligned}$$

By Proposition 7.3, we may check that

$$2\alpha(\mathrm{Diag}(-1,S),T_2,q^2)=\begin{cases}2(2q^{a+2}-(q+1)^2)(q-1) & \text{if } \chi(T)=1,\\ 2(q-1)(q^2-1) & \text{if } \chi(T)=-1.\end{cases} \tag{8.3}$$

To compute $\frac{q^2}{1+q}\alpha(\mathcal{H}^{3,1}_{\chi(T)},T)$, we may choose $\mathcal{H}^{3,1}_{\chi(T)}=\mathrm{Diag}(\mathcal{H},1)$ when $\chi(T)=1$. By Corollary 5.8, Proposition 7.4, and a direct calculation, we have

$$\frac{q^2}{1+q}\frac{\alpha(\mathcal{H}^{3,1}_{\chi(T)},T)}{\alpha(S,S)}=\frac{1}{2q(q^2-1)}\cdot\begin{cases}(q-1)\alpha(\mathrm{Diag}(-1,1),T_2)+\frac{2q^2}{q-1}\alpha(\mathcal{H},T_2) & \text{if } \chi(T)=1\\ (q-1)\alpha(\mathrm{Diag}(-1,-u),T_2) & \text{if } \chi(T)=-1.\end{cases}$$

Combining this with the formulas in [HSY23, Theorem 6.1], we have

$$\frac{q^2}{1+q^2}\frac{\alpha(\mathcal{H}^{3,1}_{\chi(T)},T)}{\alpha(S,S)}+\frac{2q^2}{q^2-1}\frac{\alpha(\mathcal{H},T_2)}{\alpha(S_2,S_2)}=\frac{1}{q(q^2-1)}\cdot\begin{cases}4q^{a+2}-q^2-2q-1 & \text{if } \chi(T)=1\\ (q^2-1) & \text{if } \chi(T)=-1.\end{cases} \tag{8.4}$$

Now a direct computation combined with (8.3) and (8.4) proves the proposition. ∎

Corollary 8.3. *Assume L is a Hermitian lattice with Gram matrix T, then*

$$\partial\mathrm{Den}(T)-\partial\mathrm{Den}(T_2)=|\{\mathcal{V}^0(L)\}|. \tag{8.5}$$

Proof. We can write $L=L^\flat\oplus\mathcal{O}_F\mathbf{x}$ where $q(\mathbf{x})=1$. If $L^\flat$ is non-split, then $|\{\mathcal{V}^0(L)\}|=1$.

If $L^\flat$ is split, then $|\{\mathcal{V}^0(L)\}|=1+2\sum_{i=1}^a q^i$ since $\mathcal{L}_3(L)$ can be identified with $\mathcal{L}_{2,1}(L^\flat)$, which is a ball in $\mathcal{L}_{2,1}$ centered at a vertex lattice of type 0 with radius a (see [HSY23] for more detail). Here $\mathcal{L}_{2,1}$ is the Bruhat-Tits tree associated with $\mathcal{N}^{\mathrm{Kra}}_{2,1}$ and $\mathcal{L}_{2,1}(L^\flat)$ is the subtree of $\mathcal{L}_{2,1}$ associated with $L^\flat$. ∎

8.3. **$\partial\mathrm{Den}(T)^{(2)}$.** In this subsection, we assume $T=\mathrm{Diag}(T_2,u_3(-\pi_0)^c)$ with $\mathrm{v}(T_2)>0$, and compute $\partial\mathrm{Den}(T)^{(2)}$. Recall that $\partial\mathrm{Den}(T)^{(2)}=\partial\mathrm{Den}(L^\flat\oplus\mathcal{O}_F\mathbf{x})^{(2)}$ where the Gram matrix of $L=L^\flat\oplus\mathcal{O}_F\mathbf{x}$ is T. We consider two cases separately in Propositions 8.4 and 8.5.

Proposition 8.4. *Let $T=\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b,u_3(-\pi_0)^c)$ where $0<a\le b\le c$. Then*

$$\partial\mathrm{Den}(T)^{(2)}=1+\chi(-u_2u_3)q^a(q^a-q^b)-q^{a+b}.$$

Proof. Recall that

$$\partial\mathrm{Den}(T)^{(2)}=\frac{1}{2q(q^2-1)}\left(2\beta'(S,T)^{(2)}+\frac{q^2}{1+q}\beta(\mathcal{H}^{3,1}_{\chi(T)},T)^{(2)}\right).$$

We compute $\beta'(S,T)^{(2)}$ first. According to Proposition 5.10, $\beta_0(S,T_2,X)=0$ and

$$\begin{aligned}\beta(S,T,X)^{(2)} &= \beta_2(S,T_2,X)\alpha(\mathrm{Diag}(S,-T_2),u_3(-\pi_0)^c,q^4X)\\ &\quad + q\beta_1(S,T_2,X)\alpha(\mathrm{Diag}(-\nu,-T_2),u_3(-\pi_0)^c,q^2X)\\ &= (1-X)(1-q^2X)\alpha(\mathrm{Diag}(S,-T_2),u_3(-\pi_0)^c,q^4X)\\ &\quad + (q+1)(q^2-1)X(1-X)\alpha(\mathrm{Diag}(-\nu,-T_2),u_3(-\pi_0)^c,q^2X).\end{aligned}$$

According to Lemma 7.1,

$$\alpha(\mathrm{Diag}(S,-T_2),u_3(-\pi_0)^c,q^4X) = 1+\chi(S)\chi(u_1)(q-1)\sum_{s=a+1}^{b} q^{a+1}(qX)^s + \chi(u_1u_2u_3\nu)q^{a+b+2}X^{c+1},$$

and

$$\alpha(\mathrm{Diag}(-\nu,-T_2),u_3(-\pi_0)^c,q^2X) = 1+\chi(S)\chi(u_1)(q-1)q^a\sum_{s=a+1}^{b}(qX)^s + \chi(u_1u_2u_3\nu)q^{a+b+1}X^{c+1}.$$

The relation $\chi(u_1u_2u_3\nu)=\chi(S)\chi(T)=-1$ and a direct calculation show that

$$\beta'(S,T_2)^{(2)} = 1+\chi(-u_2u_3)q^a(q^a-q^b)-q^{a+b}.$$

Finally, $\beta(\mathcal{H}^{3,1}_{\chi(T)},T)^{(2)}=0$ by Proposition 5.10. The proposition is proved. □

Proposition 8.5. *Recall that* $\mathcal{H}_a = \begin{pmatrix} 0 & \pi^a \\ (-\pi)^a & 0\end{pmatrix}$. *Let* $T=\mathrm{Diag}(\mathcal{H}_a,u_3(-\pi_0)^c)$ *where* a *is a positive odd integer and* $c\geq 0$. *Then*

$$\partial\mathrm{Den}(T)^{(2)} = \begin{cases}(1-q^a) & \text{if } a\leq 2c,\\ (1-q^{2c+1}) & \text{if } a>2c.\end{cases}$$

Proof. Recall that

$$\partial\mathrm{Den}(T)^{(2)} = \frac{1}{2q(q^2-1)}\left(2\beta'(S,T)^{(2)}+\frac{q^2}{1+q}\beta(\mathcal{H}^{3,1}_{\chi(T)},T)^{(2)}\right).$$

We need to compute $\beta'(S,T)^{(2)}$ and $\beta(\mathcal{H}^{3,1}_{\chi(T)},T)^{(2)}$.

According to Proposition 5.10, $\beta_0(S,T_2,X)=0$ and

$$\begin{aligned}\beta(S,T,X)^{(2)} &= \beta_2(S,\mathcal{H}_a,X)\alpha(\mathrm{Diag}(S,\mathcal{H}_a),u_3(-\pi_0)^c,q^4X)\\ &\quad + q\beta_1(S,\mathcal{H}_a,X)\alpha(\mathrm{Diag}(-\nu,\mathcal{H}_a),u_3(-\pi_0)^c,q^2X)\\ &= (1-X)\Big((1-q^2X)\alpha(\mathrm{Diag}(S,\mathcal{H}_a),u_3(-\pi_0)^c,q^4X)\\ &\quad + (q+1)(q^2-1)X\alpha(\mathrm{Diag}(-\nu,\mathcal{H}_a),u_3(-\pi_0)^c,q^2X)\Big).\end{aligned}$$

According to Lemma 7.2,

$$\alpha(\mathrm{Diag}(S,\mathcal{H}_a),u_3(-\pi_0)^c,q^4X) = \begin{cases}1+\chi(S)\chi(u_3)q^{2+a}X^{c+1} & \text{if } a\leq 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+3}X^{c+1} & \text{if } a>2c,\end{cases}$$

and

$$\alpha(\mathrm{Diag}(-\nu,\mathcal{H}_a),u_3(-\pi_0)^c,q^2X) = \begin{cases}1+\chi(S)\chi(u_3)q^{1+a}X^{c+1} & \text{if } a\leq 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+2}X^{c+1} & \text{if } a>2c.\end{cases}$$

A short computation shows that

$$\beta'(S,T)^{(2)} = q(q^2-1)\cdot \begin{cases} 1+\chi(S)\chi(u_3)q^a & \text{if } a\le 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+1} & \text{if } a>2c.\end{cases}$$

Notice that $\chi(S)\chi(u_3)=\chi(S)\chi(T)=-1$. Finally, $\beta(\mathcal{H}^{3,1}_{\chi(T)},T)^{(2)}=0$ by Proposition 5.10. The proposition is proved. □

Part 3. Proof of the main theorem

9. Reduced locus of special cycle

As remarked in §2, results of [RTW14] extend to the category of strict formal $\mathcal{O}_{F_0}$-modules using relative Dieudonné theory.

9.1. **The Bruhat-Tits building for $n=3$.** From now on we assume $n=3$ and $\mathcal{L}=\mathcal{L}_3$ as in Section 2.3.

Lemma 9.1.

(1) For every $\Lambda_2\in\mathcal{V}^2$, $\mathcal{N}_{\Lambda_2}$ is isomorphic to the projective line $\mathbb{P}^1$ over k. Its $q+1$ rational points correspond to all $\Lambda_0\in\mathcal{V}^0$ contained in Λ_2.

(2) Every $\Lambda_0\in\mathcal{V}^0$ is contained in $q+1$ type 2 lattices. In other words, there are $q+1$ projective lines in $(\mathcal{N}_3^{\mathrm{Pap}})_{\mathrm{red}}$ passing through the superspecial point $\mathcal{N}_{\Lambda_0}(k)$. Moreover

$$\bigcap_{\Lambda_2\in\mathcal{V}_2,\,\Lambda_0\subset\Lambda_2}\Lambda_2^\sharp=\pi\Lambda_0. \tag{9.1}$$

Proof. Suppose $z\in\mathcal{N}(k)$ and $M:=M(z)\subset N$ is defined as in Proposition 2.9. Since $n=3$, by [RTW14, Proposition 4.1] we have $\Lambda(M)\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}=M+\tau(M)$.

Proof of (1): Suppose $z\in\mathcal{N}_{\Lambda_2}(k)$, i.e. $M\subset\Lambda_2$.

If $M=\tau(M)$, then $M=\Lambda_0\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}$ for some $\Lambda_0\in\mathcal{V}^0$ contained in Λ_2.

If $M\neq\tau(M)$, then by taking the dual of $M\subset\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}$ we have the following sequence of inclusions

$$(\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F})^\sharp\overset{1}{\subset}M\overset{1}{\subset}M+\tau(M)=\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}. \tag{9.2}$$

In both cases the class of M in $\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}/(\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F})^\sharp\cong k^2$ is a line. This finishes the proof of (1).

Proof of (2): For each $\Lambda_0\in\mathcal{V}^0$ we just need to count the number of lattices $\Lambda_2\in\mathcal{V}^2$ that contains Λ_0. We have the following sequence of inclusions

$$\pi\Lambda_0\overset{2}{\subset}\Lambda_2^\sharp\overset{1}{\subset}\Lambda_0\overset{1}{\subset}\Lambda_2.$$

With respect to the quadratic form $(,)\pmod\pi$ on $\Lambda_0/\pi\Lambda_0$, the dual lattice $\Lambda_2^\sharp$ corresponds to the 2-dimensional subspaces $U:=\Lambda_2^\sharp/\pi\Lambda_0$ in $\Lambda_0/\pi\Lambda_0$ such that $U^\perp\overset{1}{\subset}U$. So we just need to count the number of isotropic lines $U^\perp$. Assume that $\{e_1,e_2,e_3\}$ is a basis of $\Lambda_0/\pi\Lambda_0$ whose Gram matrix with respect to the quadratic form $(,)_{\mathbb{X}}\pmod\pi$ is

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & \epsilon\end{pmatrix}.$$

It is easy to see that the isotropic lines are $\mathrm{Span}\{e_1\}$, $\mathrm{Span}\{e_2\}$ and $\mathrm{Span}\{e_1-\frac{\epsilon a^2}{2}e_2+ae_3\}$ ($a\in\mathbb{F}_q^\times$). Finally, equation (9.1) can be checked directly using this basis. □

It is well-known that $\mathcal{L}$ is a tree, see for example [Bro89, Section 3 of Chapter VI]. More specifically, the vertices of $\mathcal{L}$ correspond to vertex lattices of type 2 or 0. There is an edge between $\Lambda \in \mathcal{V}^2$ and $\Lambda_0 \in \mathcal{V}^0$ if $\Lambda_0 \subset \Lambda$. We give each edge length $\frac{1}{2}$. This defines a metric $d(,)$ on $\mathcal{L}$. Recall that we have defined $\mathcal{L}(L)$ in (2.6). Then the boundary of $\mathcal{L}(L)$ is the set

$$\mathcal{B}(L) = \{\Lambda \in \mathcal{V}^0(L) \mid \exists\, \Lambda_2 \in \mathcal{V}^2 \text{ such that } \Lambda \subset \Lambda_2, \Lambda_2 \notin \mathcal{L}(L)\}. \tag{9.3}$$

Recall we have the isomorphism $b : \mathbb{V} \to C$ defined in (2.3). Recall from [KR14a] or [HSY23] that the vertices of $\mathcal{L}_{2,1}$ correspond to vertex lattices of type 2, and an edge corresponds to a vertex lattice of type 0. Each vertex of $\mathcal{L}_{2,1}$ is contained in $q+1$ edges and each edge connects exactly two vertices. For $\mathbf{x} \in \mathbb{V}$ with $\mathrm{v}(\mathbf{x}) = 0$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ split, recall that $\mathcal{L}_{2,1}$ is the Bruhat-Tits tree of $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}_{2,1}^{\mathrm{Pap}}$. Then $\mathbf{x}$ determines an embedding $\mathcal{L}_{2,1} \hookrightarrow \mathcal{L}$ defined as follows. First we send each vertex of $\mathcal{L}_{2,1}$ corresponding to a vertex lattice $\Lambda \subset \mathrm{Span}_F\{b(\mathbf{x})\}^\perp$ of type 2 to the vertex of $\mathcal{L}$ corresponding to the type 2 lattice $\Lambda \oplus \mathrm{Span}\{b(\mathbf{x})\}$. An edge of $\mathcal{L}_{2,1}$ corresponding to a type zero lattice $\Lambda_0 \subset \mathrm{Span}_F\{b(\mathbf{x})\}^\perp$ is broken into two pieces evenly and sent to the union of the two edges in $\mathcal{L}$ joining the two vertices corresponding to $\Lambda \oplus \mathrm{Span}\{b(\mathbf{x})\}$ and $\Lambda' \oplus \mathrm{Span}\{b(\mathbf{x})\}$ where Λ and Λ' are the two type 2 lattices containing Λ_0.

9.2. **Rank** 1 **case.**

Lemma 9.2. *A point $z \in \mathcal{N}_3^{\mathrm{Pap}}(k)$ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $b(\mathbf{x}) \in M(z)$.*

1. *Assume $\Lambda_0 \in \mathcal{V}^0$, then the superspecial point $\mathcal{N}_{\Lambda_0}(k)$ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $b(\mathbf{x}) \in \Lambda_0$.*
2. *Assume $\Lambda_2 \in \mathcal{V}^2$, then*

$$\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k) \cap \mathcal{N}_{\Lambda_2}(k) = \begin{cases} \mathcal{N}_{\Lambda_2}(k) & \text{if } b(\mathbf{x}) \in \Lambda_2^\sharp, \\ \text{a superspecial point in } \mathcal{N}_{\Lambda_2}(k) & \text{if } b(\mathbf{x}) \in \Lambda_2 \backslash \Lambda_2^\sharp, \\ \emptyset & \text{if } b(\mathbf{x}) \notin \Lambda_2. \end{cases}$$

Proof. By Dieudonné theory, $z \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $\mathbf{x}(M(\mathbb{Y})) \subset M(z)$ if and only if $b(\mathbf{x}) \in M(z)$ since e is a generator of $M(\mathbb{Y})$. For $z = \mathcal{N}_{\Lambda_0}(k)$ where $\Lambda_0 \in \mathcal{V}^0$, we have $M(z) = \Lambda_0 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$. Hence (1) immediately follows.

Now we proceed to prove (2). If $b(\mathbf{x}) \in \Lambda^\sharp$, then (9.2) tells us that $z \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ for any $z \in \mathcal{N}_{\Lambda_2}^\circ(k)$. The fact that $\Lambda_2^\sharp \subset \Lambda_0$ for any $\Lambda_0 \in \mathcal{L}^0$ contained in Λ_2 implies that $\mathcal{N}_{\Lambda_0}(k) \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$. So $\mathcal{N}_{\Lambda_2}(k) \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$.

If $b(\mathbf{x}) \in \Lambda_2 \backslash \Lambda_2^\sharp$, then $\Lambda_0 := \Lambda^\sharp + \mathrm{Span}\{b(\mathbf{x})\}$ is a type 0 lattice contained in Λ_2 and $\mathcal{N}_{\Lambda_0}(k) \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$. On the other hand, since $\tau(\Lambda_0 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}) = \Lambda_0 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$, equation (9.2) tells us that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$ does not contain any point in $\mathcal{N}_{\Lambda_2}^\circ(k)$.

If $b(\mathbf{x}) \notin \Lambda_2$, then $b(\mathbf{x}) \notin M(z)$ for any $z \in \mathcal{N}_{\Lambda_2}(k)$, hence $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k) \cap \mathcal{N}_{\Lambda_2}(k) = \emptyset$. □

Corollary 9.3. *Let $L \subset \mathbb{V}$. Assume $z \in \mathcal{Z}^{\mathrm{Pap}}(L)(k)$ and $z \in \mathcal{N}_\Lambda(k)$ where $\Lambda \in \mathcal{V}^2$. Then $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\pi L)$.*

Corollary 9.4. *Assume $\mathbf{x} \in \mathbb{V}$ and $\mathrm{v}(\mathbf{x}) > 0$. Assume $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{\mathrm{red}}$ where $\Lambda \in \mathcal{V}^2$, then either $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\frac{1}{\pi}\mathbf{x})_{\mathrm{red}}$ or $\mathcal{N}_\Lambda \cap \mathcal{Z}^{\mathrm{Pap}}(\frac{1}{\pi}\mathbf{x})_{\mathrm{red}}$ is a unique superspecial point.*

Lemma 9.5. *For $L \subset \mathbb{V}$ a lattice of arbitrary rank, $\mathcal{Z}^{\mathrm{Pap}}(L)_{\mathrm{red}}$ is connected.*

Proof. Suppose $\mathcal{Z}^{\mathrm{Pap}}(L)_{\mathrm{red}}$ has two different connected components U_1 and U_2. Since $\mathrm{SU}(\mathbb{V})$ acts transitively on $\mathcal{L}$, we can find a $\mathbf{x} \in \mathbb{V}$ such that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}_{2,1}^{\mathrm{Pap}}$ (i.e. $\{\mathbf{x}\}^\perp$ is split) and $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{\mathrm{red}} \cap U_i \neq \emptyset$ for $i = 1, 2$. Hence the reduced locus of

$$\mathcal{Z}^{\mathrm{Pap}}(L \oplus \mathrm{Span}\{\mathbf{x}\}) \cong \mathcal{Z}_{2,1}^{\mathrm{Pap}}(L')$$

is not connected where L' is the orthogonal projection of L onto $\{\mathbf{x}\}^\perp$. This contradicts Corollaries 3.13, 3.15 and Lemma 3.16 of [HSY23]. □

Recall that for a lattice $L \subset \mathbb{V}$ (resp. $\mathbf{x} \in \mathbb{V}$), we have defined $\mathcal{V}(L)$ and $\mathcal{L}(L)$ (resp. $\mathcal{V}(\mathbf{x})$ and $\mathcal{L}(\mathbf{x})$) in Section 2.3.

Proposition 9.6. *Assume that $\mathbf{x} \in \mathbb{V}$ such that $h(\mathbf{x},\mathbf{x}) \neq 0$. Then we have*

$$\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{\mathrm{red}} = \bigcup_{\Lambda \in \mathcal{V}(\mathbf{x})} \mathcal{N}_\Lambda,$$

where $\mathcal{V}(\mathbf{x})$ is given as follows.

(1) When $\mathrm{v}(\mathbf{x}) = 0$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ is non-split, there is a unique vertex lattice $\Lambda_{\mathbf{x}} \in \mathcal{V}^0$ containing $b(\mathbf{x})$. In this case $\mathcal{V}(\mathbf{x}) = \{\Lambda_{\mathbf{x}}\}$.

(2) When $\mathrm{v}(\mathbf{x}) = d$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ is non-split, we have

$$\mathcal{V}(\mathbf{x}) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d\}$$

where $\Lambda_{\mathbf{x}/\pi^d}$ is as in (1).

(3) When $\mathrm{v}(\mathbf{x}) = 0$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ is split, $\mathcal{L}(\mathbf{x})$ is the tree $\mathcal{L}_{2,1}$.

(4) When $\mathrm{v}(\mathbf{x}) = d$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ is split, we have

$$\mathcal{V}(\mathbf{x}) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\mathbf{x}/\pi^d)) \leq d\}$$

where $\mathcal{L}(\mathbf{x}/\pi^d)$ is as in (3).

(5) When $h(\mathbf{x},\mathbf{x}) \notin \mathcal{O}_{F_0}$, $\mathcal{V}(\mathbf{x})$ is empty.

Proof. Proof of (1): This is a direct consequence of Proposition 2.6 and the fact that $\mathcal{N}^{\mathrm{Pap}}_{2,-1}$ has only one reduced point, see [Shi22, Section 2] or [RSZ18, Section 8]. Alternatively since $\mathrm{Span}_F\{b(\mathbf{x})\}^\perp$ is non-split of dimension 2, it contains a unique self dual lattice Λ', then $\Lambda_{\mathbf{x}} \coloneqq \mathrm{Span}\{b(\mathbf{x})\} \oplus \Lambda'$ is the unique type 0 lattice containing $b(\mathbf{x})$.

Proof of (3): Applying Proposition 2.6,we see that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}^{\mathrm{Pap}}_{2,1}$ is the Drinfeld p-adic half space, see [KR14a] and [HSY23]. The required properties of $\mathcal{L}(\mathbf{x})$ and $\mathcal{V}(\mathbf{x})$ follow.

Proof of (2): We prove this by induction. The case $d = 0$ is just (1). Now we assume $d > 0$ and that the statement holds for $d-1$, i.e.

$$\mathcal{V}(\mathbf{x}/\pi) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d-1\}.$$

Then applying Corollary 9.3 to the lattice $L = \mathrm{Span}\{\mathbf{x}/\pi\}$ we have

$$\bigcup_{\Lambda \in \mathcal{V}^2,\ d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d} \mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{\mathrm{red}}.$$

Corollary 9.4 and the induction hypothesis imply that every $\Lambda_2 \in \mathcal{V}^2(\mathbf{x})$ satisfies $d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d$. By Lemma 9.5 there is no isolated $\Lambda_0 \in \mathcal{V}^0(\mathbf{x})$, i.e. every $\Lambda_0 \in \mathcal{V}^0(\mathbf{x})$ is contained in some $\Lambda_2 \in \mathcal{V}^2(\mathbf{x})$ if $\mathrm{v}(\mathbf{x}) > 0$. This finishes the proof of (2).

Similarly we can prove (4) by an induction on d, the case $d = 0$ is just (3).

(5) follows directly from Lemma 9.2. □

9.3. **Rank 2 case.**

Proposition 9.7. *Assume that $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\} \subset \mathbb{V}$ is integral of rank 2. Then*

$$\mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{\mathrm{red}} = \bigcup_{\Lambda \in \mathcal{V}(L^\flat)} \mathcal{N}_\Lambda$$

is a finite union, where $\mathcal{V}(L^\flat)$ is the set of vertices of the tree $\mathcal{L}(L^\flat)$ described as follows.

(1) Assume $L^\flat \approx \mathcal{H}_{2a+1}$ *for some* $a \in \mathbb{Z}_{\geq 0}$*. Then* $\mathcal{L}(L^\flat)$ *is a ball centered at a vertex lattice of type* 2 *with radius* $\frac{2a+1}{2}$*.*

(2) Assume $L^\flat = \mathrm{Span}\{\pi^a \mathbf{x}_1, \pi^a \mathbf{x}_2\}$ *where* $\mathrm{v}(\mathbf{x}_1) = 0$, $\mathrm{v}(\mathbf{x}_2) \geq 0$ *and* $\mathrm{Span}_F\{\mathbf{x}_1\}^\perp$ *is nonsplit. Then* $\mathcal{L}(L^\flat)$ *is a ball centered at a vertex lattice of type* 0 *with radius* a*.*

(3) Assume $L^\flat = \mathrm{Span}\{\pi^a \mathbf{x}_1, \pi^{a+r} \mathbf{x}_2\}$ *where* $\mathbf{x}_1 \perp \mathbf{x}_2$, $\mathrm{v}(\mathbf{x}_1) = \mathrm{v}(\mathbf{x}_2) = 0$, $r \geq 0$ *and* $\mathrm{Span}_F\{\mathbf{x}_1\}^\perp$ *is split. Then*

$$\mathcal{L}(L^\flat) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a} L^\flat)) \leq a\},$$

where

$$\mathcal{L}(\pi^{-a} L^\flat) = \{\Lambda \in \mathcal{L}(\mathbf{x}_1) \mid d(\Lambda, \Lambda_0) \leq r\},$$

$\mathcal{L}(\mathbf{x}_1)$ *is described in (3) of Proposition 9.6 and* Λ_0 *is the unique type* 0 *vertex lattice containing* $\{\mathbf{x}_1, \mathbf{x}_2\}$*.*

Proof. As in the proof of Proposition 9.2, for a $\Lambda \in \mathcal{V}$, $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{\mathrm{red}}$ if and only if $\Lambda^\sharp$ contains $b(\mathbf{x}_1), b(\mathbf{x}_2)$.

We first prove (1) when $a = 0$. Suppose $\Lambda \in \mathcal{V}^2(L^\flat)$. Extend $\{b(\mathbf{x}_1), b(\mathbf{x}_2)\}$ to a basis $\{b(\mathbf{x}_1), b(\mathbf{x}_2), b_3\}$ of $\mathbb{V}$ with Gram matrix $\mathcal{H}_1 \oplus \{-\epsilon\}$. Choose a basis $\{v_1, v_2, v_3\}$ of $\Lambda^\sharp$ with the same Gram matrix $\mathcal{H}_1 \oplus \{-\epsilon\}$. Then $b(\mathbf{x}_i) \in \Lambda^\sharp$ $(i = 1, 2)$ by Lemma 9.2 and

$$b(\mathbf{x}_i) = a_{i1} v_1 + a_{i2} v_2 + a_{i3} v_3$$

where $a_{ij} \in \mathcal{O}_F$ $(j = 1, 2, 3)$. The fact that $(b(\mathbf{x}_i), b(\mathbf{x}_j))_{1 \leq i,j \leq 2} = T$ implies $a_{i3} \in \pi \mathcal{O}_F$ for $i = 1, 2$ and $(a_{ij})_{1 \leq i,j \leq 2}$ is in $\mathrm{GL}_2(\mathcal{O}_F)$. This guarantees that $L^\flat$ is a direct summand of $\Lambda^\sharp$ by Gram-Schmit process. Hence $\Lambda^\sharp$ is in fact the lattice $\mathrm{Span}_{\mathcal{O}_F}\{b(\mathbf{x}_1), b(\mathbf{x}_2), b_3\}$. The fact that all $\Lambda_0 \in \mathcal{V}^0(L^\flat)$ are in Λ follows from Lemma 9.5.

When $a = 0$, (2) follows from the fact that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1) = \mathcal{N}_{2,-1}^{\mathrm{Pap}}$ (by Proposition 2.6) and $\mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{\mathrm{red}} = \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1)_{\mathrm{red}}$ is a unique superspecial point. Similarly when $a = 0$, (3) follows from the fact that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1) = \mathcal{N}_{2,1}^{\mathrm{Pap}}$ and [HSY23, Corollary 3.13].

Now we prove (1), (2) and (3) for general a. First of all, $\mathcal{L}(L^\flat) = \mathcal{L}(\pi^a \mathbf{x}_1) \cap \mathcal{L}(\pi^a \mathbf{x}_2)$ by definition. By Corollary 9.3 we have

$$\{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a} L^\flat)) \leq a\} \subset \mathcal{L}(\pi^a \mathbf{x}_1) \cap \mathcal{L}(\pi^a \mathbf{x}_2).$$

Notice that for a sub-tree $\mathcal{L}'$ of a tree $\mathcal{L}$ and a vertex $x \in \mathcal{L} \setminus \mathcal{L}'$, there is a unique geodesic segment joining x with $\mathcal{L}'$. Given $\Lambda \in \mathcal{L}(L^\flat) = \mathcal{L}(\pi^a \mathbf{x}_1) \cap \mathcal{L}(\pi^a \mathbf{x}_2)$, let γ be the unique geodesic segment joining Λ with $\mathcal{L}(\pi^{-a} L^\flat)$. Assume that γ intersects $\mathcal{L}(\pi^{-a} L^\flat)$ at $\Lambda(L^\flat)$. Since $\mathcal{L}(\pi^{-a} L^\flat) = \mathcal{L}(\mathbf{x}_1) \cap \mathcal{L}(\mathbf{x}_2)$, γ necessarily intersects both $\mathcal{L}(\mathbf{x}_1)$ and $\mathcal{L}(\mathbf{x}_2)$. Without loss of generality we assume that γ intersects $\mathcal{L}(\mathbf{x}_1)$ at $\Lambda(\mathbf{x}_1)$ first. Hence the intersection of γ with $\mathcal{L}(\mathbf{x}_2)$ is $\Lambda(L^\flat)$ and

$$d(\Lambda, \Lambda(\mathbf{x}_1)) = d(\Lambda, \mathcal{L}(\mathbf{x}_1)) \leq d(\Lambda, \Lambda(L^\flat)) = d(\Lambda, \mathcal{L}(\mathbf{x}_2)).$$

Now by Proposition 9.6, we have

$$d(\Lambda, \mathcal{L}(\mathbf{x}_1)) \leq a, \;\; d(\Lambda, \mathcal{L}(\mathbf{x}_2)) \leq a.$$

Hence $d(\Lambda, \mathcal{L}(\pi^{-a} L^\flat)) \leq a$. This shows that

$$\{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a} L^\flat)) \leq a\} = \mathcal{L}(\pi^a \mathbf{x}_1) \cap \mathcal{L}(\pi^a \mathbf{x}_2).$$

The general case of (1), (2) and (3) follows from the above equation and the case $a = 0$.

Notice that (1), (2) and (3) have covered all possibilities of $L^\flat$ due to the classification of Hermitian lattices. Notice that in every case $\mathcal{V}(L^\flat)$ is finite. This finishes the proof of the proposition. □

Definition 9.8. Assume that $L^\flat$ is an integral lattice of rank 2 in $\mathbb{V}$. Define $\mathcal{S}(L^\flat)$, the skeleton of $\mathcal{L}(L^\flat)$, as follows. If the fundamental invariant of $L^\flat$ is $(2a, b)$ $(b \geq 2a)$, define $\mathcal{S}(L^\flat) \coloneqq \mathcal{L}(\pi^{-a}L^\flat)$. If the fundamental invariant of $L^\flat$ is $(2a+1, 2a+1)$, define $\mathcal{S}(L^\flat) \coloneqq \emptyset$.

Remark 9.9. *The skeleton $\mathcal{S}(L^\flat)$ is isomorphic to a ball in the Bruhat-Tits tree of $\mathcal{N}^{\mathrm{Pap}}_{2,\pm 1}$.*

Corollary 9.10. *For each $\Lambda_2 \in \mathcal{V}^2(L^\flat)$ not on the skeleton $\mathcal{S}(L^\flat)$, one can find $\Lambda_0 \in \mathcal{V}^0(L^\flat)$ such that Λ_2 has the largest distance to the boundary $\mathcal{B}(L^\flat)$ of $\mathcal{L}(L^\flat)$ among all type 2 lattices in $\mathcal{V}^2(L^\flat)$ containing Λ_0.*

Proof. Assmue the fundamental invariant of $L^\flat$ is $(2a, b)$ or $(2a+1, 2a+1)$. Define $M^\flat \coloneqq \pi^{-a}L^\flat$. Let b be the unique integer such that $\Lambda_2 \in \mathcal{L}(\pi^b M^\flat) \setminus \mathcal{L}(\pi^{b-1}M^\flat)$. Choose any $\Lambda_0 \in \mathcal{B}(\pi^b M^\flat)$ such that $\Lambda_0 \subset \Lambda$. Then by Proposition 9.7, Λ_0 satisfies the assumption of the corollary. □

9.4. **The Krämer model.** For $\Lambda \in \mathcal{V}^2$, let $\tilde{\mathcal{N}}_\Lambda$ be the strict transform of $\mathcal{N}_\Lambda$ under the blow-up $\mathcal{N}^{\mathrm{Kra}} \to \mathcal{N}^{\mathrm{Pap}}$. Since the strict transform of a regular curve along any of its closed point is an isomorphism, we know $\tilde{\mathcal{N}}_\Lambda \cong \mathbb{P}^1$.

Lemma 9.11. *For $\Lambda \neq \Lambda' \in \mathcal{V}^2$, $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect.*

Proof. If $\mathcal{N}_\Lambda$ and $\mathcal{N}_{\Lambda'}$ do not intersect in $\mathcal{N}^{\mathrm{Pap}}$, then obviously $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect. Without loss of generality we can assume $\Lambda = \mathrm{Span}\{e_1, e_2, e_3\}$ and $\Lambda' = \mathrm{Span}\{\pi^{-1}e_1, \pi e_2, e_3\}$ where the Gram matrix of $\{e_1, e_2, e_3\}$ is $\mathrm{Diag}(\mathcal{H}, \epsilon)$. Take $\mathbf{x}_0 = e_3$. Then by Proposition 9.7, both $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ are in $\tilde{\mathcal{Z}}(\mathbf{x}_0) \cong \mathcal{N}^{\mathrm{Kra}}_{2,1}$. Now by [HSY23, Lemma 5.3], $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect. □

Lemma 9.12. *Let $\Lambda \in \mathcal{V}^2$ and $\Lambda_0 \in \mathcal{V}^0$. When $\Lambda_0 \subset \Lambda$, $\tilde{\mathcal{N}}_\Lambda$ intersects properly with Exc_{Λ_0} and*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) = 1. \tag{9.4}$$

When Λ_0 is not contained in Λ, $\tilde{\mathcal{N}}_\Lambda$ does not intersect with Exc_{Λ_0}.

Proof. First assume $\Lambda_0 \subset \Lambda$. Since $\tilde{\mathcal{N}}_\Lambda$ is a strict transformation of a curve, it intersects the exceptional divisor properly. Let $\mathbf{x}_0$ be as in the proof of Lemma 9.11. Then $\tilde{\mathcal{N}}_\Lambda$ is in $\tilde{\mathcal{Z}}(\mathbf{x}_0) \cong \mathcal{N}^{\mathrm{Kra}}_{2,1}$.

$$\begin{aligned}\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) &= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) \\ &= \chi(\tilde{\mathcal{Z}}(\mathbf{x}_0), \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\mathrm{Exc}'}).\end{aligned}$$

Here $\mathrm{Exc}' \cong \mathbb{P}^1_k$ is the exceptional divisor on $\tilde{\mathcal{Z}}(\mathbf{x}_0)$ corresponding to the rank 2 self-dual lattice

$$\Lambda' = \{v \in \Lambda_0 \mid v \perp \mathbf{x}_0\}.$$

By [HSY23, Lemma 5.2(a)], we know $\chi(\tilde{\mathcal{Z}}(\mathbf{x}_0), \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\mathrm{Exc}'}) = 1$. When Λ_0 is not contained in Λ, the superspecial point $\mathcal{N}_{\Lambda_0}(k)$ is not contained in $\mathcal{N}_\Lambda$, hence $\tilde{\mathcal{N}}_\Lambda$ does not intersect with Exc_{Λ_0}. □

10. Intersection of vertical components and special divisors

In this section we study the intersection of $\tilde{\mathcal{N}}_\Lambda$ and special divisors. The main result is Theorem 10.2. To proceed we first study the decomposition of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ when $\mathrm{v}(L^\flat) = 0$. Since n is odd, we can without loss of generality assume that $\chi(\mathbb{V}) = \chi(C) = 1$. In the rest of the paper, we identify $\mathbb{V}$ with C by the isomorphism b defined in (2.3).

10.1. **Decomposition of $^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$.** Let $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$ are linearly independent and the Hermitian form restricted to L is non-degenerate.

Lemma 10.1. *$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}] \in K_0(\mathcal{N}^{\mathrm{Kra}})$ is in fact in $\mathrm{F}^2K_0(\mathcal{N}^{\mathrm{Kra}})$. Moreover we have the decomposition in $\mathrm{Gr}^2K_0^{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)}(\mathcal{N}^{\mathrm{Kra}})$*

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = \mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h + {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v. \tag{10.1}$$

where $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h$ is described in Theorem 4.2 and $^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)_v \in \mathrm{Gr}^2K_0^{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v}(\mathcal{N}^{\mathrm{Kra}})$.

Proof. By Lemma 2.10, $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ is Noetherian and has a decomposition

$$\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = \mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h \cup \mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v.$$

Expressing $\mathcal{Z}(\mathbf{x}_i)$ $(i = 1, 2)$ as in (3.1) and applying Propositions 3.2, 3.3 and Lemma 3.4, $^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ equals

$$[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_2)}] + \sum_{\Lambda_0 \in \mathcal{V}^0(L^\flat)} (2m_{\Lambda_0}(\mathbf{x}_1)m_{\Lambda_0}(\mathbf{x}_2) + m_{\Lambda_0}(\mathbf{x}_1) + m_{\Lambda_0}(\mathbf{x}_2))H_{\Lambda_0}.$$

$\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h$ is contained in $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \tilde{\mathcal{Z}}(\mathbf{x}_2)$ and has dimension 1 by Theorem 4.2. $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \tilde{\mathcal{Z}}(\mathbf{x}_2)_v$ also has dimension 1 as it is supported on the reduced locus of $\mathcal{N}^{\mathrm{Kra}}$ by Lemma 2.11 and does not contain any exceptional divisor Exc_{Λ_0}. Hence

$$[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_2)}] = [\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \tilde{\mathcal{Z}}(\mathbf{x}_2)}] \in \mathrm{F}^2K_0(\mathcal{N}^{\mathrm{Kra}}), \tag{10.2}$$

see for example [Zha21, Lemma B.2]. Hence we know that $^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) \in \mathrm{F}^2K_0(\mathcal{N}^{\mathrm{Kra}})$. The desired decomposition then follows from Theorem 4.2. $\square$

By Lemma 10.1 and (2.9) we know that $^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v \in K_0'(Y)$ where we can take Y to be the reduced locus of $\mathcal{N}^{\mathrm{Kra}}$. By the Bruhat-Tits stratification of $\mathcal{N}^{\mathrm{Kra}}$ and the fact that $\mathrm{Gr}^1K_0^{\mathrm{Exc}_{\Lambda_0}}(\mathcal{N}^{\mathrm{Kra}}) \cong \mathrm{CH}^1(\mathrm{Exc}_{\Lambda_0})$ is generated by H_{Λ_0}, we have the following decomposition in $\mathrm{Gr}^2K_0(\mathcal{N}^{\mathrm{Kra}})$:

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v = \sum_{\Lambda_2 \in \mathcal{V}^2(L^\flat)} m(\Lambda_2, L^\flat)[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\Lambda_0 \in \mathcal{V}^0(L^\flat)} m(\Lambda_0, L^\flat)H_{\Lambda_0}. \tag{10.3}$$

We will determine the multiplicities $m(\Lambda_2, L^\flat)$ and $m(\Lambda_0, L^\flat)$ when $\mathrm{v}(L^\flat) = 0$ in this section and deal with the general case in Section 11.

Now assume $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ with Gram matrix $\mathrm{Diag}(u_1, u_2(-\pi_0)^n)$ with $u_1, u_2 \in \mathcal{O}_{F_0}^\times$. Applying Proposition 3.2 to $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)$, we find

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}] + \sum_{\Lambda_0 \in \mathcal{V}^0(\mathbf{x}_1)} [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}].$$

By Proposition 2.6, we know the intersection $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)$ is proper and is isomorphic to $\mathcal{Z}^{\mathrm{Kra}}_{2,\chi(u_1)}(\mathbf{x}_2)$. Combining this with Corollary 3.5 we obtain

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = i_*({}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}_{2,\chi(u_1)}(\mathbf{x}_2)) - \sum_{\Lambda_0 \in \mathcal{V}^0(L^\flat)} H_{\Lambda_0}. \tag{10.4}$$

where i_* is the map $\mathrm{Gr}^1K_0(\mathcal{N}^{\mathrm{Kra}}_{2,\chi(u_1)}) \to \mathrm{Gr}^2K_0(\mathcal{N}^{\mathrm{Kra}}_{3,1})$ induced by the closed immersion $i : \mathcal{N}^{\mathrm{Kra}}_{2,\chi(u_1)} \to \mathcal{N}^{\mathrm{Kra}}_{3,1}$. Equation (10.4) reduces the problem of decomposing $^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ in this case to [Shi22, Theorem 4.5] and [HSY23, Theorem 4.1]. We do not make the effort to write the complete result down, but instead look at two basic examples.

Let us begin by the case when $L^\flat$ is unimodular. By (10.4) and either [Shi22, Theorem 4.5] or [HSY23, Theorem 4.1], we have

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{Z}}(L^\flat)^\circ}] \tag{10.5}$$

in the notation of Theorem 4.2.

Next consider $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ with Gram matrix $\mathrm{Diag}(1, -u\pi_0)$ where $u \in \mathcal{O}_{F_0}^\times$. Then $\mathrm{Span}\{\mathbf{x}_1\}^\perp$ is split and $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cong \mathcal{N}_{2,1}^{\mathrm{Kra}}$. By Proposition 9.7 (3), $\mathcal{V}^2(L^\flat)$ consists of two adjacent lattices Λ and Λ'. Moreover by [HSY23, Theorem 4.1] and (10.4), we have

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda\cap\Lambda'}. \tag{10.6}$$

10.2. **The intersection number.** Assume $\Lambda \in \mathcal{V}^2$. For $\mathbf{x} \in \mathbb{V}\backslash\{0\}$, define

$$\mathrm{Int}_\Lambda(\mathbf{x}) := \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}). \tag{10.7}$$

In this subsection we prove the following theorem.

Theorem 10.2. *Let $\Lambda \in \mathcal{V}^2$ and $\mathbf{x} \in \mathbb{V}\backslash\{0\}$. Then*

$$\mathrm{Int}_\Lambda(\mathbf{x}) = 1_\Lambda(\mathbf{x})$$

where 1_Λ is the characteristic function of $\Lambda \subset \mathbb{V}$.

Corollary 10.3. *Assume that $\Lambda_0 \in \mathcal{L}_0$ and $\Lambda \in \mathcal{L}_2$ such that $\Lambda_0 \subset \Lambda$. Then for any $y_0 \in \Lambda_0 \setminus \pi\Lambda_0$ such that $y_0^\perp$ is nonsplit, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}) = 0.$$

Proof. By Proposition 3.2, we know

$$\mathcal{Z}^{\mathrm{Kra}}(y_0) = \tilde{\mathcal{Z}}(y_0) + \mathrm{Exc}_{\Lambda_0}.$$

Now the corollary follows immediately from Theorem 10.2 and Lemma 9.12. □

Proof of Theorem 10.2: We consider three different cases. First if $x \notin \Lambda$ or $\mathrm{v}(\mathbf{x}) < 0$, then by Lemma 9.2, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cap \tilde{\mathcal{N}}_\Lambda = \emptyset$ hence $\mathrm{Int}_\Lambda(\mathbf{x}) = 0$. From now on we assume $x \in \Lambda$ and $\mathrm{v}(\mathbf{x}) \geq 0$. Write $\mathbf{x} = \mathbf{x}_0\pi^n$ with $\mathbf{x}_0 \in \Lambda\backslash\pi\Lambda$ and $n \geq 0$.

Case 1: First we assume $\mathbf{x}_0 \in \Lambda\backslash\Lambda^\sharp$. Choose a basis $\{e_1', e_2', e_3'\}$ of Λ with Gram matrix $\mathcal{H}_{3,1}^1$ such that

$$\mathbf{x}_0 = xe_1' + ye_2' + ze_3'.$$

Then one of x and y is in $\mathcal{O}_F^\times$ as $\Lambda^\sharp = \mathrm{Span}\{\pi e_1', \pi e_2', e_3\}$. Apparently the equation

$$2u - v\bar{v} = h(\mathbf{x}_0, \mathbf{x}_0)$$

has a solution $(u, v) \in \mathcal{O}_{F_0}^2$ with $u \in \mathcal{O}_{F_0}^\times$. Now according to Lemma A.2, we can find a matrix $g \in \mathrm{U}(\mathcal{H}_{3,1}^1)(\mathcal{O}_{F_0})$ such that

$$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pi u \\ 1 \\ v \end{pmatrix}.$$

Now replace the basis $\{e_1', e_2', e_3'\}$ by $\{e_1, e_2, e_3\} = \{e_1', e_2', e_3'\}g^{-1}$, we have

$$\mathbf{x}_0 = \pi u e_1 + e_2 + ve_3$$

where $u \in \mathcal{O}_{F_0}^\times$, $v \in \mathcal{O}_F$.

Define

$$f_1 = \frac{1}{\pi}u^{-1}e_2,\ f_2 = \pi u e_1,\ f_3 = e_3. \tag{10.8}$$

Then $\{f_1, f_2, f_3\}$ has also Gram matrix $\mathcal{H}^1_{3,1}$ and $\Lambda' := \mathrm{Span}\{f_1, f_2, f_3\}$ is a type 2 lattice adjacent to Λ with $\Lambda_c = \Lambda \cap \Lambda' = \mathrm{Span}\{\pi e_1, e_2, e_3\}$ is a type 0 lattice. Now in terms of the basis $\{f_1, f_2, f_3\}$ we have

$$\mathbf{x} = \pi^n(\pi u f_1 + f_2 + v f_3).$$

Define $\theta \in \mathrm{U}(\mathbb{V})$ by taking the basis $\{e_1, e_2, e_3\}$ to $\{f_1, f_2, f_3\}$. Then

$$\theta(\mathbf{x}) = \mathbf{x}, \quad \theta(\Lambda) = \Lambda'.$$

In particular $\theta(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ and

$$\mathrm{Int}_{\Lambda'}(\mathbf{x}) = \mathrm{Int}_{\Lambda}(\mathbf{x}). \tag{10.9}$$

Now let

$$\mathbf{y}_0 = e_3, \quad \mathbf{y}_1 = \pi(-\pi u e_1 + e_2),$$

$L^\flat = \mathrm{Span}\{\mathbf{y}_0, \mathbf{y}_1\}$, and $L = \mathrm{Span}\{\mathbf{y}_0, \mathbf{y}_1, \mathbf{x}\}$. Then by (10.6) and Theorem 4.2 we have

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda_c} + [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}]. \tag{10.10}$$

where $\tilde{\mathcal{Z}}(M^\flat)$ is the quasi canonical-lifting cycle of the lattice

$$M^\flat := \mathrm{Span}\{e_3, -\pi u e_1 + e_2\}.$$

Combining with (10.9), we have

$$\mathrm{Int}(L) = 2 \cdot \mathrm{Int}_\Lambda(\mathbf{x}) + \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot H_{\Lambda_c}) + \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}]). \tag{10.11}$$

Let $\mathbf{x}' = \pi^n(\pi u e_1 + e_2) = \mathbf{x} - \pi^n v e_3$. Then we have

$$\begin{aligned}
\mathrm{Int}(L) &= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_0) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)) \\
&= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_0) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}') \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)) \\
&= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) \\
&\quad + \sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)})
\end{aligned}$$

where we have used linear invariance ([How19, Corollary D]) and Proposition 3.2. Notice that the Gram matrix of $\{\mathbf{x}', \mathbf{y}_1\}$ is $\mathrm{Diag}(2u(-\pi_0)^n, -2u\pi_0)$. By Proposition 2.6 and [HSY23, Theorem 1.1],

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = \begin{cases} 1 & \text{if } n = 0, \\ 1 + n - 2q & \text{if } n \geq 1. \end{cases}$$

By Corollary 3.6 and [HSY23, Lemma 3.15], we know that

$$\sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = |\mathcal{V}^0(L)| = \begin{cases} 1 & \text{if } n = 0, \\ 2q + 1 & \text{if } n \geq 1. \end{cases}$$

Combining the above two equations we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) = n + 2.$$

On the other hand, by Corollary 3.8,

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0} \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = -1.$$

By [Gro86, Proposition 3.3]

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}) = n + 1.$$

Hence we obtain by (10.11)

$$\mathrm{Int}_{\Lambda}(\mathbf{x}) = 1. \tag{10.12}$$

Case 2: Now we Assume $\mathbf{x}_0 \in \Lambda^\sharp \backslash \pi\Lambda$. As in the proof of the previous case, we can find a basis $\{e_1, e_2, e_3\}$ of Λ with Gram matrix $\mathcal{H}_{3,1}^1$ by Lemma A.4 such that

$$\mathbf{x} = \pi^n(ue_3 + \pi e_1).$$

where $u \in \mathcal{O}_{F_0}^\times$. Define

$$\Lambda' = \mathrm{Span}\{\pi e_1, \frac{1}{\pi}e_2, e_3\},\ \Lambda_c = \Lambda \cap \Lambda',$$

then $\mathbf{x}_0 \in \Lambda' \backslash \Lambda'^\sharp$. Also define

$$\mathbf{y}_0 = e_3,\ \mathbf{y}_1 = \pi(-\pi e_1 + e_2),$$

and $L^\flat \coloneqq \mathrm{Span}\{\mathbf{y}_0, \mathbf{y}_1\}$. Then by Theorem 4.2 and (10.6) we have

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda_c} + [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}], \tag{10.13}$$

where $\tilde{\mathcal{Z}}(M^\flat)$ is the quasi-canonical lifting cycle of the lattice

$$M^\flat \coloneqq \mathrm{Span}\{e_3, -\pi e_1 + e_2\}.$$

Let $\mathbf{x}' \coloneqq \pi^{n+1}e_1 = \mathbf{x} - \pi^n u e_3$, then we have

$$\begin{aligned}\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) =& \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) \\ &+ \sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}).\end{aligned}$$

Notice that the Gram matrix of $\{\mathbf{x}', \mathbf{y}_1\}$ is equivalent to $\mathcal{H}_1$ when $n = 0$, and to $\mathrm{Diag}(u_1\pi_0^n, u_2\pi_0)$ for some $u_1, u_2 \in \mathcal{O}_{F_0}^\times$ when $n \geq 1$. Hence by Proposition 2.6 and [HSY23, Theorem 1.1], we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = \begin{cases} -(q-1) & \text{if } n = 0, \\ 1 + n - 2q & \text{if } n \geq 1. \end{cases}$$

By Corollary 3.6 and Lemmas 3.15 and 3.16 of [HSY23], we know that

$$\sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = |\mathcal{V}^0(L)| = \begin{cases} q+1 & \text{if } n = 0, \\ 2q+1 & \text{if } n \geq 1. \end{cases}$$

Hence we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) = n + 2.$$

On the other hand, by Corollary 3.8,

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0} \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = -1.$$

By [Gro86, Proposition 3.3]

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}) = n + 1.$$

Since $\mathbf{x} \in \Lambda' \backslash \Lambda'^\sharp$, by the previous case we also have

$$\mathrm{Int}_{\Lambda'}(\mathbf{x}) = 1.$$

Combining all above, we have by (10.13)

$$\mathrm{Int}_{\Lambda}(\mathbf{x}) = 1. \tag{10.14}$$

This finishes the proof of Theorem 10.2. $\square$

11. Proof of the modified Kudla-Rapoport conjecture: three dimension case

In this section, we will prove Theorem 1.2. We need some preparation.

Proposition 11.1. *Assume that $L \subset \mathbb{V}$ has a Gram matrix $T = \mathrm{Diag}(u_1, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ with $u_i \in \mathcal{O}_{F_0}^\times$ and $0 \le b \le c$. Then*

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

Moreover, for every decomposition $L = L^\flat \oplus \mathrm{Span}\{\mathbf{x}\}$, we have

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}.$$

Proof. Fix a basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ of L such that the Gram matrix of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is $T = \mathrm{Diag}(u_1, T_2)$ where $u_1 \in \mathcal{O}_{F_0}^\times$ and $T_2 \in \mathrm{Herm}_2(\mathcal{O}_F)$. Let $u_1^{-1} \cdot L$ be a lattice represented by $u_1^{-1} \cdot T$. Since $\mathrm{Int}(u_1^{-1} \cdot L) = \mathrm{Int}(L)$ and $\partial\mathrm{Den}(u_1^{-1} \cdot L) = \partial\mathrm{Den}(L)$, we may assume $u_1 = 1$. Let $L^\flat = \mathrm{Span}\{\mathbf{x}_2, \mathbf{x}_3\}$. According to Propositions 2.6, 3.2, and Corollary 3.6, we have

$$\begin{aligned}
\mathrm{Int}(L) - \mathrm{Int}(L^\flat) &= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) - \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\tilde{\mathcal{Z}}(\mathbf{x}_1) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) \\
&= \sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}] \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) \\
&= |\{\mathcal{V}^0(L)\}|.
\end{aligned} \tag{11.1}$$

Now the result we want follows by comparing (11.1) with (8.5), and the identity $\mathrm{Int}(L^\flat) = \partial\mathrm{Den}(L^\flat)$ proved in [Shi22, Theorem 1.3] and [HSY23, Theorem 1.3]. (2) follows from (1) and Theorem 5.6 (2). ☐

Proof of Theorem 1.4: Under the assumption $\mathrm{v}(L^\flat) > 0$, we can decompose $\mathcal{D}(L^\flat)$ in $\mathrm{Gr}^2 K_0(\mathcal{N}^{\mathrm{Kra}})$ as

$$\mathcal{D}(L^\flat) = \sum_{\Lambda_2 \in \mathcal{V}(L^\flat)} m(\mathcal{D}(L^\flat), \Lambda_2)[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\Lambda_0 \in \mathcal{V}(L^\flat)} m(\mathcal{D}(L^\flat), \Lambda_0) H_{\Lambda_0}, \tag{11.2}$$

by (10.3) and Proposition 4.6.

Claim 1: $m(\mathcal{D}(L^\flat), \Lambda_0) = 0$ unless $L^\flat \subset \Lambda_0$. In such a case,

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \begin{cases} q+1 & \text{if } \Lambda_0 \in \mathcal{V}(L^\flat) \setminus \mathcal{B}(L^\flat), \\ 1 & \text{if } \Lambda_0 \in \mathcal{B}(L^\flat). \end{cases} \tag{11.3}$$

Indeed, since Λ_0 is of type 0, we may choose a $y_0 \in \mathbb{V} \setminus L_F^\flat$ such that $\mathrm{Span}\{y_0\}^\perp$ is non-split and $y_0 \in \Lambda_0 \setminus \pi\Lambda_0$. In this case, Proposition 3.2, Corollaries 3.7 and 3.8 imply that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0} \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}]) = 1.$$

So by (11.2), Corollaries 10.3 and 2.7, we have

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}]).$$

Let $(2a, 2b)$ $(b > a)$ be the fundamental invariant of the projection of $L^\flat$ onto $\mathrm{Span}\{y_0\}^\perp$. Let φ be the natural quotient map $\Lambda_0 \to \Lambda_0/\pi\Lambda_0$ and define

$$m := \dim_{\mathbb{F}_q} \varphi(L^\flat) \le 2.$$

Equation (9.1) implies that $m = 0$ if and only if $\Lambda_0 \in \mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)$. First assume $m = 0$, in other words, $L^\flat \subset \pi\Lambda_0$ so $b \ge a \ge 1$. By the definition of $\mathcal{D}(L^\flat)$ and [Shi22, Theorem 1.2], we have

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \mu(a, b) - q\mu(a-1, b) - \mu(a, b-1) + q\mu(a-1, b-1) = q+1,$$

as claimed where
(11.4)

$$\mu(a,b)=\chi(\mathcal{N}^{\mathrm{Kra}},{}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)\cdot[\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}])=\begin{cases}2\sum_{s=0}^{a}q^s(a+b+1-2s)-a-b-2 & \text{if } a\geq 0\\ 0 & \text{if } a<0.\end{cases}$$

Now assume $m=1$, then $\varphi(L^\flat)$ is a line ℓ and $b\geq 1$. By the assumption that $y_0\notin L^\flat_F$, we know ℓ is not in $\mathrm{Span}\{\varphi(y_0)\}$, hence the projection of ℓ onto $\varphi(y_0)^\perp$ is nonzero. Since $\varphi(y_0)^\perp$ is nonsplit, we must have $a=0$. Hence by the definition of $\mathcal{D}(L^\flat)$ and (11.4), we have

$$m(\mathcal{D}(L^\flat),\Lambda_0)=\mu(0,b)-q\mu(-1,b)-\mu(0,b-1)+q\mu(-1,b-1)=1$$

as claimed. Finally, $m=2$ is impossible since $v(L^\flat)>0$. This finshes the proof of Claim 1.

Claim 2: $m(\mathcal{D}(L^\flat),\Lambda_2)=2$ for any $\Lambda_2\in\mathcal{V}^2(L^\flat)$.

Indeed, according to Lemma 3.9, we have $\chi(\mathcal{N}^{\mathrm{Kra}},\mathcal{D}(L^\flat)\cdot[\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}])=0$. On the other hand, Corollary 3.7 and Lemma 9.12 imply that

$$\chi(\mathcal{N}^{\mathrm{Kra}},\mathcal{D}(L^\flat)\cdot[\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}])=\sum_{\Lambda_0\subset\Lambda_2}m(\mathcal{D}(L^\flat),\Lambda_2)-2m(\mathcal{D}(L^\flat),\Lambda_0). \tag{11.5}$$

Combining the above with Claim 1, we have
(11.6)

$$0=\chi(\mathcal{N}^{\mathrm{Kra}},\mathcal{D}(L^\flat)\cdot[\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}])=\begin{cases}\sum_{\Lambda_0\subset\Lambda_2}m(\mathcal{D}(L^\flat),\Lambda_2)-2(q+1) & \text{if } \Lambda_0\in\mathcal{L}(L^\flat)\setminus\mathcal{B}(L^\flat),\\ \sum_{\Lambda_0\subset\Lambda_2}m(\mathcal{D}(L^\flat),\Lambda_2)-2 & \text{if } \Lambda_0\in\mathcal{B}(L^\flat).\end{cases}$$

Recall $\mathcal{S}(L^\flat)$ in Definition 9.8. First assume $\Lambda_2\in\mathcal{L}(L^\flat)\setminus\mathcal{S}(L^\flat)$. If $d(\Lambda_2,\mathcal{B}(L^\flat))$ equals to $\frac{1}{2}$, choose $\Lambda_0\in\mathcal{B}(L^\flat)$ such that $\Lambda_0\subset\Lambda_2$, then Λ_2 is the unique lattice in $\mathcal{V}^2(L^\flat)$ that contains Λ_0, hence (11.6) implies that $m(\mathcal{D}(L^\flat),\Lambda_2)=2$ in this case. Now Corollary 9.10 allows us to show $m(\mathcal{D}(L^\flat),\Lambda_2)=2$ by induction on the distance $d(\Lambda_2,\mathcal{B}(L^\flat))$ for any $\Lambda_2\in\mathcal{L}(L^\flat)\setminus\mathcal{S}(L^\flat)$.

Similarly for $\Lambda_2\in\mathcal{S}(L^\flat)$, we can show $m(\mathcal{D}(L^\flat),\Lambda_2)=2$ by induction on its distance to $\mathcal{S}(L^\flat)\cap\mathcal{B}(L^\flat)$. This finishes the proof of Claim 2.

Notice that for $\Lambda_0\in\mathcal{V}(L^\flat)$

$$\sum_{\Lambda_2\in\mathcal{V}^2(L^\flat)}\sum_{\Lambda_0\subset\Lambda_2}1=\begin{cases}q+1 & \text{if } \Lambda_0\in\mathcal{V}(L^\flat)\setminus\mathcal{B}(L^\flat),\\ 1 & \text{if } \Lambda_0\in\mathcal{B}(L^\flat).\end{cases}$$

This finishes the proof of Theorem 1.4. □

In the following discussion we freely use Theorem 10.2 and Corollary 3.8 without explicitly referring to them.

Proposition 11.2. *Assume $L=L^\flat\obot\mathrm{Span}\{\mathbf{x}\}$ with Gram matrix*

$$T=\mathrm{Diag}(\mathcal{H}_a,u_3(-\pi_0)^c)$$

where a is a positive odd integer, and $c\geq 0$. Then

$$\mathrm{Int}(L)^{(2)}=\partial\mathrm{Den}(L)^{(2)}=\begin{cases}1-q^a & \text{if } a\leq 2c,\\ 1-q^{2c+1} & \text{if } a>2c.\end{cases} \tag{11.7}$$

Proof. By Proposition 8.5, it suffices to prove the identity for $\mathrm{Int}(L)^{(2)}$.

Now we compute $\mathrm{Int}(L)^{(2)}$. We may take $L^\flat=\mathrm{Span}\{\pi^{\frac{a+1}{2}}e_1,\pi^{\frac{a+1}{2}}e_2\}$, where the Gram matrix of $\{e_1,e_2\}$ is $\mathcal{H}$. Let $e_3=\pi^{-c}\mathbf{x}$. Then $\mathcal{L}(L^\flat)$ is centered at $\mathrm{Span}\{e_1,e_2,e_3\}$ of radius $\frac{a}{2}$ by Proposition 9.7.

Assume $a \le 2c$ first. In this case, $\mathcal{L}(L^\flat) \subset \mathcal{L}(\mathbf{x})$. As a result, we have $\mathrm{Int}_{\Lambda_2}(\mathbf{x}) = 1$ and $\mathrm{Int}_{\Lambda_0}(\mathbf{x}) = -1$ for any $\Lambda_2 \in \mathcal{V}^2(L^\flat)$ and $\Lambda_0 \in \mathcal{V}^0(L^\flat)$. Hence by Theorem 1.4, we have

$$\begin{aligned}\mathrm{Int}(L)^{(2)} &= \sum_{\Lambda_2 \in \mathcal{L}(L^\flat)} \chi(\mathcal{N}^{\mathrm{Kra}}, (2[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\Lambda_0 \subset \Lambda_2} H_{\Lambda_0}) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) \\ &= (1-q)|\{\Lambda_2 \mid \Lambda_2 \in \mathcal{L}(L^\flat)\}| \\ &= (1-q)(1 + (1+q)q + (1+q)q^3 + \cdots + (1+q)q^{a-2}) \\ &= 1 - q^a,\end{aligned} \tag{11.8}$$

as claimed.

Now we assume $a > 2c$. We consider the case $c = 0$ first. Recall that $\tilde{\mathcal{Z}}(e_3) \approx \mathcal{N}^{\mathrm{Kra}}_{2,1}$, hence $\mathcal{L}(L^\flat) \cap \mathcal{L}(e_3)$ is a ball of radius $\frac{a}{2}$ in the Bruhat-Tits tree $\mathcal{L}_{2,1}$ of $\mathcal{N}^{\mathrm{Pap}}_{2,1}$ centerd at the vertex lattice corresponding to $\pi^{-\frac{a+1}{2}} \cdot L^\flat$, within which a vertex lattice Λ_0 of type 0 is contained in two vertex lattices of type 2, and a vertex lattice Λ_2 of type 2 contains $q+1$ vertex lattice of type 0. Hence

$$|\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)\}| = 1 + q + (1+q)q + \cdots + (1+q)q^{\frac{a-3}{2}},$$

and

$$|\{\Lambda_0 \mid \Lambda_0 \in \mathcal{B}(L^\flat) \cap \mathcal{L}(e_3)\}| = (1+q)q^{\frac{a-1}{2}}.$$

Moreover, notice that if $e_3 \in \Lambda_0$, then $\mathrm{Int}_{\Lambda_2}(e_3) = 1$ for any Λ_2 such that $\Lambda_0 \subset \Lambda_2$. As a result,

$$\begin{aligned}\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot \mathcal{Z}^{\mathrm{Kra}}(e_3)) &= 2(1 + q \cdot |\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)\}|) \\ &\quad - (q+1)|\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)\}| \\ &\quad - |\{\Lambda_0 \mid \Lambda_0 \in \mathcal{B}(L^\flat) \cap \mathcal{L}(e_3)\}| \\ &= 2 + (q-1)(1 + q + (1+q)q + \cdots + (1+q)q^{\frac{a-3}{2}}) - (1+q)q^{\frac{a-1}{2}} \\ &= 1 - q,\end{aligned}$$

which is compatible with (11.7).

Next we show

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi e_3) - \mathcal{Z}^{\mathrm{Kra}}(e_3))) = q - q^3.$$

According to Proposition 9.6, $\mathcal{V}(\pi e_3) = \{\Lambda \mid d(\Lambda, \mathcal{L}(e_3)) \le 1\}$. Hence, around each $\Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)$, there will be $q(q-1)$ many new vertex lattices of type 0 in $\mathcal{L}(L^\flat) \cap \mathcal{L}(\pi e_3) \setminus \mathcal{L}(L^\flat) \cap \mathcal{L}(e_3)$. Hence,

$$\begin{aligned}&\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi e_3) - \mathcal{Z}^{\mathrm{Kra}}(e_3))) \\ &= 2q \cdot q(q-1)(1 + q + (1+q)q + (1+q)q^2 + \cdots + (1+q)q^{\frac{a-1}{2}-2}) \\ &\quad - q(q-1)(q+1)(1 + q + (1+q)q + (1+q)q^2 + \cdots + (1+q)q^{\frac{a-1}{2}-2}) \\ &\quad - q(q-1)(1+q)q^{\frac{a-1}{2}-1} \\ &= q - q^3.\end{aligned}$$

Continuing in this way, we can show

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi^i e_3) - \mathcal{Z}^{\mathrm{Kra}}(\pi^{i-1} e_3))) = q^{2i-1} - q^{2i+1}$$

for $2i < a$. So

$$\mathrm{Int}(L)^{(2)} = \mathcal{D}(L^\flat) \cdot \mathcal{Z}^{\mathrm{Kra}}(\pi^c e_3) = 1 - q^{2c+1} = \partial\mathrm{Den}^{(2)}(L)$$

as claimed. □

Proposition 11.3. *Assume $L = L^\flat \obot \mathrm{Span}\{\mathbf{x}\}$ with Gram matrix*

$$T = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$$

where $0 < a \leq b \leq c$, then

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)} = 1 + \chi(-u_2u_3)q^a(q^a - q^b) - q^{a+b}.$$

Proof. By Proposition 8.4, it suffices to show

$$\mathrm{Int}(L)^{(2)} = 1 + \chi(-u_2u_3)q^a(q^a - q^b) - q^{a+b}. \tag{11.9}$$

Notice that since $a \leq b \leq c$, we have $\mathcal{L}(L^\flat) \subset \mathcal{L}(\mathbf{x})$ by Propositions 9.6 and 9.7.

First, we assume $\chi(-u_2u_3) = -1$, then (11.9) specializes to

$$\partial\mathrm{Den}(T)^{(2)} = 1 - q^{2a}.$$

On the other hand, $\mathcal{L}(L^\flat)$ is a ball centered at a vertex lattice of type 0 with radius a in this case. One can show $\mathrm{Int}(L)^{(2)} = 1 - q^{2a}$ exactly as in (11.8).

Now we assume $\chi(-u_2u_3) = 1$. In this case, (11.9) specializes to

$$\partial\mathrm{Den}(L)^{(2)} = 1 + q^{2a} - 2q^{a+b}.$$

Let $r = b - a$, and $L^\flat = \mathrm{Span}\{x_1, x_2\}$. Then $\mathcal{L}(\pi^{-a}L^\flat)$ is a ball centered at a vertex lattice of type 0 with radius r in the Bruhat-Tits tree $\mathcal{L}_{2,1}$. Hence,

$$\mathcal{L}(L^\flat) = \{\Lambda \mid \Lambda \in \mathcal{L}_3,\ d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \leq a\}.$$

When $a = 1$, $\mathcal{V}^0(\pi^{-1}L^\flat) = \mathcal{V}^0(L^\flat) \setminus \mathcal{B}(L^\flat)$. Then combining with Theorem 1.4, it is not hard to see

$$\begin{aligned}\mathrm{Int}(L)^{(2)} &= 2(q + 1 + q\cdot 2(q + q^2 + \cdots + q^r)) - (q+1)|\mathcal{V}^0(L^\flat) \setminus \mathcal{B}(L^\flat)| - |\mathcal{B}(L^\flat)| \\ &= 1 + q^2 - 2q^{b+1},\end{aligned}$$

where we use the fact

$$|\mathcal{V}^0(\pi^{-1}L^\flat)| = 1 + 2(q + q^2 + \cdots + q^r),$$

and

$$|\mathcal{B}(L^\flat)| = (q-1)q(1 + 2(q + q^2 + \cdots + q^{r-1})) + 2q^{r+2}.$$

Now assume $a > 1$. Let T be the Hermitian matrix associated with $L^\flat \obot \mathrm{Span}\{\mathbf{x}\}$, then

$$\begin{aligned}&\partial\mathrm{Den}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \partial\mathrm{Den}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} \\ &= 1 + q^{2a+2} - 2q^{r+2a+2} - (1 + q^{2a} - 2q^{r+2a}) \\ &= q^{2a}(q^2 - 1)(1 - q^{2r}) \\ &= q^2\left(\partial\mathrm{Den}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \partial\mathrm{Den}(\pi^{-1}L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)}\right),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Int}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \mathrm{Int}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} \\ &= 2q|\mathcal{B}(L^\flat)| - q|\mathcal{B}(L^\flat)| - |\mathcal{B}(\pi L^\flat)| \\ &= (2q - q - q^2)|\mathcal{B}(L^\flat)| \\ &= q^2\left(\mathrm{Int}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \mathrm{Int}(\pi^{-1}L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)}\right),\end{aligned}$$

where we use the fact $|\mathcal{B}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})| = q^2|\mathcal{B}(L^\flat)|$. Since r is arbitrary, an induction on a gives the result we want. □

Proof of Theorem 1.2: The case $\mathrm{v}(L) < 0$ follows from Proposition 8.1 and the fact that $\mathrm{Int}(L) = 0$ under this condition. Assume $v(L) \geq 0$. There are three cases.

Case 1: When L has a Gram matrix $\mathrm{Diag}(u_1, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ as in Proposition 11.1, it is proved by Proposition 11.1.

Case 2: When L has a basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ whose Gram matrix is $T = \mathrm{Diag}(\mathcal{H}_a, u_3(-\pi_0)^c)$, take $L^\flat = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$ and $\mathbf{x} = \mathbf{x}_3$. By Propositions 11.1, 11.2, and 11.3, we have

$$\mathrm{Int}(L^{\flat,\prime} \oplus \mathrm{Span}\{\mathbf{x}\})^{(2)} = \partial\mathrm{Den}(L^{\flat,\prime} \oplus \mathrm{Span}\{\mathbf{x}\})^{(2)}$$

for any $L^\flat \subset L^{\flat,\prime} \subset L^\flat_F$ (direct sums in the above identity are actually orthogonal direct sums). Thus we have by Theorem 5.6 (1)

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

Case 3: When L has a Gram matrix $\mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ with $0 \leq a \leq b \leq c$), the same argument as Case 2 gives $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$. This finishes the proof of the theorem. $\square$

Theorem 1.2 and Theorem 5.6 imply the following corollary.

Corollary 11.4. *For any lattice $L = L^\flat \oplus \mathcal{O}_F\mathbf{x} \subset \mathbb{V}$ of rank 3, we have*

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}.$$

12. Global Applications

In this section we assume that F is an imaginary quadratic field with discriminant d_F. Denote by $a \mapsto \bar{a}$ the complex conjugation on F. The result in this section can be easily extended to CM number fields and more general level structures at split places, see [LZ22a] and [HLSY22]. We restrict to the imaginary quadratic fields to make the exposition as simple as possible.

12.1. **Unitary Shimura varieties and special cycles.** In this subsection, we briefly review the definition of an integral model of Shimura variety defined in [BHK+20] over $\mathrm{Spec}\,\mathcal{O}_F$. Let

$$\mathcal{M}^{\mathrm{Kra}}_{(1,n-1)} \to \mathrm{Spec}\,\mathcal{O}_F$$

be the algebraic stack which assigns to each $\mathcal{O}_F$-scheme S the groupoid of isomorphism classes of quadruples $(A, \iota, \lambda, \mathcal{F}_A)$ where

(1) $A \to S$ is an abelian scheme of relative dimension n;
(2) $\iota : \mathcal{O}_F \to \mathrm{End}(A)$ is an action satisfying the following determinant condition (the Kottwitz condition of signature $(1, n-1)$)

$$\mathrm{char}(T - \iota(\alpha) \mid \mathrm{Lie}\, A) = (T - s(\alpha))(T - s(\bar{\alpha}))^{n-1} \in \mathcal{O}_S[T],$$

for all $\alpha \in \mathcal{O}_F$ where $s : \mathcal{O}_F \to \mathcal{O}_S$ is the structure morphism;
(3) $\lambda : A \to A^\vee$ is a principal polarization whose Rosati involution satisfies $\iota(\alpha)^* = \iota(\bar{\alpha})$ for all $\alpha \in \mathcal{O}_F$;
(4) $\mathcal{F}_A \subset \mathrm{Lie}\, A$ is an $\mathcal{O}_F$-stable $\mathcal{O}_S$-module local direct summand of rank $n-1$ satisfying the Krämer condition: $\mathcal{O}_F$ acts on $\mathrm{Lie}\, A/\mathcal{F}_A$ by the structure map $s : \mathcal{O}_F \to \mathcal{O}_S$ and acts on $\mathcal{F}_A$ by the complex conjugate of the structure map.

Two objects $(A, \iota, \lambda, \mathcal{F}_A)$ and $(A', \iota', \lambda', \mathcal{F}_{A'})$ in $\mathcal{M}^{\mathrm{Kra}}_{(1,n-1)}(S)$ are isomorphic if there is an $\mathcal{O}_F$-linear isomorphism $f : A \to A'$ of abelian schemes such that $f^*(\lambda') = \lambda$ and $f_*(\mathcal{F}_A) = \mathcal{F}_{A'}$. The stack $\mathcal{M}^{\mathrm{Kra}}_{(1,n-1)}$ is flat of dimension $n-1$ over $\mathrm{Spec}\,\mathcal{O}_F$. It is smooth over $\mathrm{Spec}\,\mathcal{O}_F[\frac{1}{d_F}]$ and has semi-stable reduction over primes of F dividing d_F. This is indicated by the corresponding behaviour of its local model studied in [Krä03]. Analogously one can define $\mathcal{M}_{(0,1)} \to \mathcal{O}_F$ be the algebraic stack which assigns to each $\mathcal{O}_F$-scheme S the groupoid of isomorphism classes of triples (E, ι_0, λ_0) where

(1) $E \to S$ is an abelian scheme of relative dimension 1;

(2) $\iota_0 : \mathcal{O}_F \to \mathrm{End}(E)$ is an action such that its induced action on $\mathrm{Lie}\, E$ agrees with the complex conjugate of the structural map $s : \mathcal{O}_F \to \mathcal{O}_S$.
(3) $\lambda_0 : E \to E^\vee$ is a principal polarization whose Rosati involution satisfies $\iota_0(\alpha)^* = \iota_0(\bar{\alpha})$ for all $\alpha \in \mathcal{O}_F$.

The stack $\mathcal{M}_{(0,1)}$ is smooth of relative dimension 0 over $\mathrm{Spec}\, \mathcal{O}_F$, see for example [How15, Proposition 2.1.2].

Assume that $\mathbb{F}$ is an algebraically closed field of characteristic p over $\mathcal{O}_F$. Let

$$(E_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A) \in (\mathcal{M}_{(0,1)} \times \mathcal{M}^{\mathrm{Kra}}_{(1,n-1)})(\mathrm{Spec}\, \mathbb{F}).$$

For any prime number $\ell \neq p$, we can define a hermitian form $h(x, y)$ on the Tate module

$$T_\ell(E_0, A) := \mathrm{Hom}_{\mathcal{O}_F}(T_\ell(E_0), T_\ell(A)) \tag{12.1}$$

as in [KR14b, Section 2.3] using the polarizations λ_0, λ and Weil pairings on E_0, A.

Fix a hermitian space W over F of signature $(1, n-1)$ that contains a self-dual lattice $\mathfrak{a}$ and a hermitian space W_0 over F of signature $(0, 1)$ that contains a self-dual lattice $\mathfrak{a}_0$. Define

$$V := \mathrm{Hom}_F(W_0, W), \quad L := \mathrm{Hom}_{\mathcal{O}_F}(\mathfrak{a}_0, \mathfrak{a}). \tag{12.2}$$

Here V and L are equipped with hermitian forms coming from the ones on W_0 and W. Define $G := \mathrm{U}(W)$. Also define the group scheme $\mathrm{GU}(W)$ over $\mathbb{Q}$ by

$$\mathrm{GU}(W)(R) = \{g \in \mathrm{GL}_R(W \otimes R) \mid (gv, gw) = c(g)(v, w), \forall v, w \in W \otimes R\}$$

where R is any $\mathbb{Q}$-algebra. Also define $Z := \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m = \mathrm{GU}(W_0)$ and

$$\tilde{G} := Z \times_{\mathbb{G}_m} \mathrm{GU}(W) \tag{12.3}$$

where the maps from the factors on the right hand side to $\mathbb{G}_m$ are $\mathrm{Nm}_{F/\mathbb{Q}}$ and the similitude character $c(g)$ respectively. We have an isomorphism of group schemes

$$\tilde{G} \to Z \times \mathrm{U}(W), (z, g) \mapsto (z, z^{-1} g). \tag{12.4}$$

Let K_G be the compact subgroup of $G(\mathbb{A}_f)$ that stabilizes the lattice $\mathfrak{a} \otimes \hat{\mathbb{Z}}$ and $K_Z = \hat{\mathbb{Z}}^\times \subset Z(\mathbb{A}_f)$. Under the isomorphism (12.4), define

$$K := K_Z \times K_G. \tag{12.5}$$

Now define $\mathcal{M} \subset \mathcal{M}_{(0,1)} \times \mathcal{M}^{\mathrm{Kra}}_{(1,n-1)}$ to be the open and closed substack such that

$$(E_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A) \in \mathcal{M}(S)$$

if and only if there is an isomorphism of hermitian $\mathcal{O}_F \otimes \mathbb{Z}_\ell$-modules

$$T_\ell(E_{0,s}, A_s) \cong L \otimes \mathbb{Z}_\ell \tag{12.6}$$

for any geometric point $s \in S$ and prime ℓ that is not the same as the characteristic of s. Then $\mathcal{M}$ is an integral model of the Shimura variety associated to the group $\tilde{G}$ with level structure defined by K.

Now we review the definition of special cycles. For $(E, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A) \in \mathcal{M}(S)$ where S is an $\mathcal{O}_F$-scheme, consider the projective $\mathcal{O}_F$-module of finite rank

$$V'(E, A) = \mathrm{Hom}_{\mathcal{O}_F}(E, A).$$

On this module there is a hermitian form $h'(x, y)$ defined by

$$h'(x, y) = \iota_0^{-1}(\lambda_0^{-1} \circ y^\vee \circ \lambda \circ x), \tag{12.7}$$

where $y^\vee$ is the dual homomorphism of y. It is proved in [KR14b, Lemma 2.7] that $h'(x, y)$ is positive semi-definite. The following is [KR14b, Definition 2.8].

Definition 12.1. For $T \in \mathrm{Herm}_m(\mathbb{Z})_{>0}$, the special cycle $\mathcal{Z}(T)$ is the stack of collections $(E, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A, \mathbf{x})$ where

$$(E, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A) \in \mathcal{M}(S)$$

and $\mathbf{x} = (x_1, \ldots, x_m) \in \mathrm{Hom}_{\mathcal{O}_F}(E, A)^m$ such that

$$h'(\mathbf{x}, \mathbf{x}) = (h'(x_i, x_j)) = T.$$

When $t \in \mathbb{Z}_{>0}$, each component of $\mathcal{Z}(t)$ is a divisor by [How15, Proposition 3.2.3]. In general, $\mathcal{Z}(T)$ does not necessarily have the expected codimension which is the rank of T.

Let $\mathcal{C} = \{\mathcal{C}_p\}$ be a incoherent collection of local Hermitian spaces of rank n such that $\mathcal{C}_\ell \cong V_\ell$ for all finite ℓ and $\mathcal{C}_\infty$ is positive definite. For a nonsingular Hermitian matrix T of rank n with values in $\mathcal{O}_F$, Let V_T be the Hermitian space with gram matrix T. Define

$$\mathrm{Diff}(T, \mathcal{C}) := \{p \text{ a place of } \mathbb{Q} \mid \mathcal{C}_p \text{ is not isomorphic to } (V_T)_p\}. \tag{12.8}$$

Then $\mathrm{Diff}(T, \mathcal{C})$ is a finite set consisting of places of $\mathbb{Q}$ inert or ramified in F. By [KR14b, Proposition 2.22], $\mathcal{Z}(T)$ is empty if $|\mathrm{Diff}(T, \mathcal{C})| > 1$. If $\mathrm{Diff}(T, \mathcal{C}) = \{p\}$ for a finite prime p inert or ramified in F, it is proved in loc.cit. that the support of $\mathcal{Z}(T)$ is on the supersingular locus of $\mathcal{M}$ over $\mathrm{Spec}\,\bar{\mathbb{F}}_p$. Let e be the ramification index of $F_p/\mathbb{Q}_p$. Define the arithmetic degree

$$\widehat{\deg}_T = \chi(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(t_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_n)}) \cdot \log p^{2/e}, \tag{12.9}$$

where $\otimes^{\mathbb{L}}$ stands for derived tensor product on the category of coherent sheaves on $\mathcal{M}$, χ is Euler-characteristic and t_i $(1 \le i \le n)$ are the diagonal entries of T. When $\mathrm{Diff}(T, \mathcal{C}) = \{\infty\}$, $\mathcal{Z}(T)$ is empty ([KR14b, Lemma 2.7]) and the arithmetic degree $\widehat{\deg}_T(v)$ is the integration of a green current $G(T, v)$ ($v > 0$ is a positive definite Hermitian matrix of order n) defined by Liu ([Liu11]) and Garcia-Sankaran ([GS19]), see for example [LZ22a, Equation (15.3.0.2)].

12.2. **Eisenstein series.** On the analytic side, let $\chi : \mathbb{A}/\mathbb{Q}^\times \to \mathbb{C}^\times$ be the quadratic character associated to the extension $F/\mathbb{Q}$. Fix a character $\eta : \mathbb{A}_F^\times \to \mathbb{C}^\times$ such that $\eta|_{\mathbb{A}^\times} = \chi^n$. We consider an incoherent Eisenstein series $E(z, s, \Phi)$ associated to a section $\Phi = \otimes \Phi_p$ in a degenerate principal series representation $I(s, \eta)$ of $\mathrm{U}(n,n)(\mathbb{A})$ (see [LZ22a, §12] or [KR14b]), where $\tau \in \mathbb{H}_n$ (see (1.18)), $s \in \mathbb{C}$ and Φ_p is given as follows. The section Φ_∞ is the standard weight n section. When $p < \infty$ is unramified in F, Φ_p is the standard section associated to the characteristic function of L_p^n via the map $\lambda : S(\mathcal{C}_p^n) \to I(0, \eta_p)$:

$$\lambda(\varphi)(g) = \omega(g)\varphi(0),$$

where ω is the Weil representation of $\mathrm{U}(n,n)$ associated to the character χ. When p is ramified in F, define

$$\Phi_p = \Phi_p^0 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} A_{p,\epsilon}^i(s) \cdot \Phi_p^i. \tag{12.10}$$

Here Φ_p^0 is the standard section associated to the characteristic function L_p^n, Φ_p^i is the standard section associated to the characteristic function of $(\mathcal{H}_{n,i}^\epsilon)^n$ at p with $\epsilon = -\chi_p(L_p)$, and

$$A_{p,\epsilon}^i(0) = 0, \quad \frac{d}{ds} A_p^i|_{s=0} = \frac{(-1)^n}{p^{2i}} \cdot c_{n,i}^\epsilon \cdot \log p, \tag{12.11}$$

where $c_{n,i}^\epsilon$ are as in (1.9). Let ψ be the standard additive character of $\mathbb{A}/\mathbb{Q}$, i.e.,

$$\psi_\infty(x) = \exp(2\pi i x), \quad \psi_\ell(x) = \exp(-2\pi i \lambda(x)), \tag{12.12}$$

where λ is the canonical map $\mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Let $E_T(\tau, s, \Phi)$ (resp. $E'_T(\tau, \Phi)$) be the T-th Fourier coefficient of $E(\tau, s, \Phi)$ (resp. $E'(\tau, 0, \Phi)$) with respect to ψ. Then for $s \gg 0$ we have the following product formula (see for example [LZ22a, §12.4])

$$E_T(\tau, s, \Phi) = c_\infty \cdot \prod_{p<\infty} W_{T,p}(1, s, \Phi_p) \cdot q^T, \tag{12.13}$$

where c_∞ is a constant independent of T calculated in [Liu11, Proposition 4.5], and $W_{T,p}(1, s, \Phi_p)$ is the local Whittaker integral defined in [KR14b, Equation (10.2)].

12.3. **An equivalent form of Conjecture 1.1.** In this subsection we assume p is a prime of $\mathbb{Q}$ ramified in F. Let $|\cdot|_p$ be the nonarchimedean valuation on F_p normalized so that $|\sqrt{d_F}|_p = \frac{1}{p}$. For $\epsilon = \pm 1$, let V_p^ϵ be the (unique up to isomorphism) $F_p/\mathbb{Q}_p$-Hermitian space of dimension n and sign ϵ. For any lattice M_p of rank n in V_p^ϵ, let $\Phi_{M_p} \in I_p(s, \eta_p)$ be the standard section associated to the characteristic function of M_p^n. By [Shi22, Proposition 9.7], we have

$$W_{T,p}(1, r, \Phi_{M_p}) = \gamma_p(V_p^\epsilon)^n \cdot |\det(M_p)|_p^n \cdot |d_F|_p^f \cdot \alpha_v(M_p, L_T, X)|_{X=p^{-2r}}, \tag{12.14}$$

where $f = \frac{1}{2}n^2 + \frac{1}{4}n(n-1)$, $\alpha_p(\cdot, \cdot, X)$ is the local density polynomial defined in (5.1) at the place p and $\gamma_p(V_p^\epsilon)$ is an 8-th root of unity defined in [KR14b, Equation (10.3)]. By loc. cit., we know

$$\gamma_p(V_p^\epsilon) = -\gamma_p(V_p^{-\epsilon}). \tag{12.15}$$

For $T \in \mathrm{Herm}_n(F)$, choose a lattice L_T in the Hermitian space $V_p^{\chi_p(T)}$ with Gram matrix T. Then equations (12.14) and (12.15) imply that Conjecture 1.1 at the place p is equivalent to the following conjecture.

Conjecture 12.2. *Let $\mathbb{V}$ be the space of special quasi-homomorphisms as in* (1.2) *such that $\chi_p(\mathbb{V}) = \epsilon$. Let $T \in \mathrm{Herm}_n(F)$ such that $\chi_p(T) = \epsilon$ and L_T be a lattice of rank n in $\mathbb{V}$ with Gram matrix T. Then*

$$\mathrm{Int}(L_T) \cdot \log p = \frac{W'_{T,p}(1, 0, \Phi_p)}{W_{I_n^{-\epsilon},p}(1, 0, \Phi_p)},$$

where Φ_p is defined in (12.10).

12.4. **(Global) Arithmetic Siegel-Weil formula.** Similar to [LZ22a, Theorem 1.3.1], we have the following theorem.

Theorem 12.3. *(Arithmetic Siegel-Weil formula for non-singular coefficients) Assume that the fundamental discriminant of F is $d_F \equiv 1 \pmod 8$ and that Conjecture 1.1 holds for every F_p with $p|d_F$. For any non-singular Hermitian matrix T with values in $\mathcal{O}_F$ of size n, we have*

$$E'_T(\tau, 0, \Phi) = C \cdot \widehat{\deg}_T(v) \cdot q^T, \quad q^T = \exp(2\pi i \mathrm{tr}(T\tau)),$$

where C is an explicit constant that only depends on F and L, $\widehat{\deg}_T(v) = \widehat{\deg}_T$ for positive definite T, and $\tau = u + iv$. In particular, the arithmetic Siegel-Weil formula holds for $n = 2, 3$ for non-singular T.

Proof. We sketch the main idea of the proof. When $|\mathrm{Diff}(T, \mathcal{C})| > 1$, both sides are zero. When $\mathrm{Diff}(T, \mathcal{C}) = \{p\}$ for a finite prime $p \neq 2$ (as $d_F \equiv 1 \pmod 8$), then T is positive definite and the support of $\mathcal{Z}(T)$ is on the supersingular locus of $\mathcal{M}$ over $\mathrm{Spec}\, \bar{\mathbb{F}}_p$ although it has higher than expected dimension and needs 'derivation' to make it correct dimensional cycle (we skip it here and just define its degree below). When p is inert in F, the theorem is proved in [LZ22a, Theorem 1.3.1]. When p is ramified in F, the theorem can be proved in a similar fashion assuming Conjecture 1.1.

The key is that by the p-adic uniformization theorem ([RZ96, Chapter 6]), for each component $\mathcal{Z}$ of $\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$, the arithmetic degree of $\mathcal{Z}(T)$ supported on $\mathcal{Z}$

$$\chi(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}(t_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_n)}) \cdot \log p, \tag{12.16}$$

is the same as $\mathrm{Int}(L) \log p$ (L has gram matrix T) in Conjecture 1.1. In particular this number is independent of the choice of $\mathcal{Z}$ and depends only on T. Assuming Conjecture 1.1 (or rather its equivalent form 12.2), this is then equal to $c_{p,1} W'_{T,p}(1, 0, \Phi_p)$ for some constant $c_{p,1} \neq 0$. So $\widehat{\deg}_T$ is this number times the number of components of $\mathcal{Z}(T)(\bar{\mathbb{F}}_p)$, which can be counted via the Siegel-Weil formula. Combining these results together with (12.13), we can prove

$$C_p \cdot \widehat{\deg}_T \cdot q^T = E'_T(\tau, 0, \Phi)$$

for come explicit constant C_p independent of T. Similar argument holds when $\mathrm{Diff}(T, \mathcal{C}) = \{\infty\}$ in which case the theorem is proved in [Liu11] and [GS19]. Finally, one checks that C_p is independent of the choice of p. □

Appendix A. Calculation of primitive local density

In this appendix, we provide the proofs of Propositions 5.9 and 5.10. Throughout this section, M is unimodular of rank $m \geq 2$ unless clearly stated otherwise. Let $\{v_1, \cdots, v_{2k}, v_{2k+1}, \cdots, v_{2k+m}\}$ be a basis of $M^{[k]} = \mathcal{H}^k \obot M$ with Gram matrix $\mathcal{H}^k \obot \mathrm{Diag}(I_{m-1}, \nu)$. Let L be a Hermitian lattice of rank n with Gram matrix T. An isometric embedding $\varphi : L \to M$ is called primitive if its image in $M/\pi M$ has dimension $\mathrm{rank}_{\mathcal{O}_F}(L)$. We call a vector v primitive in M if $\pi^{-1} v \notin M$, or equivalently the natural embedding $\varphi : \mathrm{Span}_{\mathcal{O}_F}\{v\} \hookrightarrow M$ is primitive. For a $v \in M^{[k]}$, we let $\mathrm{Pr}_{\mathcal{H}^k}(w_i)$ be the projection of w_i to $\mathcal{H}^k$.

A.1. **Proof of Proposition 5.9.** The main purpose of this subsection is to prove the first four parts of Proposition 5.9. Part (5) of this proposition follows from Proposition 5.7 and Corollaries A.10 and A.12.

Proof. For (1), choose $M(1) = \frac{t\pi}{2} v_1 + v_2 \in M^{[k]}$ with $q(M(1)) = t$. Then

$$\begin{aligned} M(1)^{\perp} &= \mathrm{Span}_{\mathcal{O}_F}\{\frac{-t\pi}{2} v_1 + v_2, v_3, ..., v_{2k}, v_{2k+1}, \cdots, v_{2k+m}\} \\ &\cong \langle -t \rangle \obot \mathcal{H}^{k-1} \obot M, \end{aligned} \tag{A.1}$$

which is represented by $\mathrm{Diag}(-t, \mathcal{H}^{k-1}, S)$. It is easy to check

$$|M^{[k]} : M(1) \obot M(1)^{\perp}|^{-1} |M(1)^{\vee} : M(1)| = |t\pi|_F |t\pi|_F^{-1} = 1.$$

For (2) and (3), assume first that M is isotropic (and unimodular). In this case, we may choose a basis $\{v'_{2k+1}, \ldots, v'_{2k+m}\}$ of M with Gram matrix $\mathrm{Diag}(\mathcal{H}_0, 1, \ldots, 1, -\nu)$. Choose $M(0) = \frac{t}{2} v'_{2k+1} + v'_{2k+2}$ with $q(M(0)) = t$. Then

$$\begin{aligned} M(0)^{\perp} &= \mathrm{Span}\{v_1, \cdots, v_{2k}, -\frac{t}{2} v'_{2k+1} + v'_{2k+2}, v'_{2k+3}, \cdots, v'_{2k+m}\} \\ &\cong \mathcal{H}^k \obot \mathrm{Span}\{v'_{2k+3}, \cdots, v'_{2k+m}\} \obot \langle -t \rangle. \end{aligned}$$

as claimed. Moreover

$$|M^{[k]} : M(0) \obot M(0)^{\perp}|^{-1} |M(0)^{\vee} : M(0)| = |t|_F |t\pi|_F^{-1} = q.$$

Next, assume that M is anisotropic. In this case, M has rank 2 and has Gram matrix $\mathrm{Diag}(1, \nu)$ with $\chi(M) = \chi(-\nu) = -1$. In this case, $E = F_0(\sqrt{-\nu})$ is a unramified quadratic field extension of F_0, and $N_{E/F_0}\mathcal{O}_E^\times = \mathcal{O}_{F_0}^\times$. When $\mathrm{v}(t)=0$, $t \in N_{E/F_0}\mathcal{O}_E^\times$, i.e., $t = a\bar{a} + b\bar{b}\nu$. Take $M(0) = av_{2k+1} + bv_{2k+2}$. Then $q(M(0)) = t$, and

$$M(0)^\perp = \mathrm{Span}\{v_1, \cdots, v_{2k}, -\nu\bar{b}v_{2k+1} + \bar{a}v_{2k+2}\} = \mathcal{H}^k \obot \langle t\nu\rangle,$$

and

$$|M^{[k]} : M(0) \obot M(0)^\perp|^{-1} |M(0)^\vee : M(0)| = |\pi|_F^{-1} = q.$$

When $\mathrm{v}(t) > 0$, $t \notin N_{E/F_0}\mathcal{O}_E^\times$. So there is no primitive $M(0) \in M$ with $q(M(0)) = t$. This proves (1)—(3) of Proposition 5.9.

The proof of (4) follows from the following 4 lemmas.

Lemma A.1. *For primitive vectors $w_1,\ w_2 \in \mathcal{H}_i$ with $q(w_1) = q(w_2)$, we can find an element $g \in \mathrm{U}(\mathcal{H}_i)$ such that $g(w_1) = w_2$.*

Proof. We treat the case i is odd first. Assume $v = a_1v_1 + a_2v_2$. Then v is primitive implies that a_1 or a_2 is a unit. Without loss of generality, we assume a_2 is a unit and we can further assume $a_2 = 1$ by the action of $\begin{pmatrix} \bar{a}_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix}$. Now notice that $q(v) = (v, v) = (a_1 - \bar{a}_1)\pi^i$. Hence we can write $a_1 = \alpha + \frac{q(v)\pi^{-i}}{2}$, where $\alpha \in \mathcal{O}_{F_0}$. Now let $g = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$, and it is straightforward to check that $g \in \mathrm{U}(\mathcal{H}_i)$ and $g(v) = \frac{q(v)\pi^{-i}}{2}v_1 + v_2$.

Now we deal with the case i is even. Again, we can assume $v = a_1v_1 + v_2$. Then $q(v) = (a_1 + \bar{a}_1)\pi^i$. Hence we can write $a_1 = \frac{q(v)}{2}\pi^{-i} + \beta\pi$, where $\beta \in \mathcal{O}_{F_0}$. Now let $g = \begin{pmatrix} 1 & -\beta\pi \\ 0 & 1 \end{pmatrix}$, and it is straightforward to check that $g \in \mathrm{U}(\mathcal{H}_i)$ and $g(v) = \frac{q(v)\pi^{-i}}{2}v_1 + v_2$. □

Lemma A.2. *Assume M is any lattice such that $\mathrm{v}(M) \geq i$. For $w_1,\ w_2 \in \mathcal{H}_i^k \obot M$, if $\mathrm{Pr}_{\mathcal{H}_i^k}(w_1)$ and $\mathrm{Pr}_{\mathcal{H}_i^k}(w_2)$ are primitive and $q(w_1) = q(w_2)$, then there exists $g \in \mathrm{U}(\mathcal{H}_i^k \obot M)$ with $g(w_1) = w_2$.*

Proof. Choose a basis $\{v_1, ..., v_{2k}\}$ of $\mathcal{H}_i^k$ such that the associated Gram matrix is $\mathcal{H}_i^k$. We also choose a basis $\{v_{2k+1}, ..., v_{2k+m}\}$ of M. Write $w_1 = \sum_{i=1}^{2k+m} a_iv_i$. Since $\mathrm{Pr}_{\mathcal{H}_i^k}(w_1)$ is primitive, a_i is a unit for some $i \in \{1, ..., 2k\}$. Without loss of generality, we may assume $a_1 = 1$. Let $w' = w_1 + \frac{(-1)^{i+1}q(w_1)\pi^{-i}}{2}v_2$, then

$$q(w') = q(w_1) + (w_1, \frac{(-1)^{i+1}q(w_1)\pi^{-i}}{2}v_2) + (\frac{(-1)^{i+1}q(w_1)\pi^{-i}}{2}v_2, w_1) = 0,$$

and $(w', v_2) = (v_1, v_2)$. As a result, $M_1 = \mathrm{Span}_{\mathcal{O}_F}\{w_1, v_2\} = \mathrm{Span}_{\mathcal{O}_F}\{w', v_2\}$ is isometric to $\mathcal{H}_i$. Notice that $\mathrm{val}_\pi(q(w_1)) \geq i$ is guaranteed by the assumption $\mathrm{v}(M) \geq i$.

Similarly, we can show $w_2 \in M_2$ for some M_2 that is isometric to $\mathcal{H}_i$. However, the assumption $\mathrm{v}(M) \geq i$ and [Jac62, Proposition 4.2] imply that there exist $g \in \mathrm{U}(\mathcal{H}_i^k \obot M)$ such that $g(M_1) = M_2$. In particular, $g(w_1) \in M_2$. Since both $g(w_1)$ and w_2 are in M_2, the problem is reduced to Lemma A.1. □

Lemma A.3. *For primitive vectors $w_1,\ w_2 \in M$ with $q(w_1) = q(w_2)$, we can find an element $g \in \mathrm{U}(M)$ such that $g(w_1) = w_2$.*

Proof. Since M is unimodular, we can decompose

$$M = \mathcal{H}_0^k \obot M',$$

where $M' = 0$ or an anisotropic unimodular Hermiatian lattice of rank 1 or 2. If $\mathrm{Pr}_{\mathcal{H}_0^k}(w_1)$ and $\mathrm{Pr}_{\mathcal{H}_0^k}(w_2)$ are primitive, this is Lemma A.2. If $\mathrm{Pr}_{\mathcal{H}_0^k}(w_1)$ is not primitive, then $\mathrm{Pr}_{M'}(w_1)$ is primitive and thus $q(\mathrm{Pr}_{M'}(w_1)) \in \mathcal{O}_F^\times$. This implies that $q(w_2) = q(w_1)$ is a unit, and $M = \mathcal{O}_F w_i \obot (\mathcal{O}_F w_i)^\perp$. Therefore there is some $g \in \mathrm{U}(M)$ with $g(w_1) = w_2$. □

Lemma A.4. *Assume that $w_1,\ w_2 \in M^{[k]}$ are primitive and that $\mathrm{Pr}_{\mathcal{H}^k}(w_1)$ and $\mathrm{Pr}_{\mathcal{H}^k}(w_2)$ are not primitive. Then we can find $g \in \mathrm{U}(M^{[k]})$ such that $g(w_1) = w_2$.*

Proof. Let $\{v_1, ..., v_{2k+m}\}$ be a basis of $\mathcal{H}^k \obot M$, whose Gram matrix is $\mathcal{H}^k \obot \mathrm{Diag}(1, ..., \nu)$ where ν is a unit. Assume $v \in M^{[k]}$ is primitive and $\mathrm{Pr}_{\mathcal{H}^k}(v)$ is not primitive, then we can write $v = \sum_{i=1}^{2k} \pi a_i v_i + \sum_{j=2k+1}^{2k+m} a_j v_j$, where some a_j is a unit for $2k+1 \le j \le 2k+m$. Again, without loss of generality, we may assume $a_{2k+m} = 1$. For $i \le k$, we set

$$v'_{2i-1} = v_{2i-1} + \frac{\bar{a}_{2i}}{\nu} v_{2k+m}, \quad v'_{2i} = v_{2i} + \frac{-\bar{a}_{2i-1}}{\nu} v_{2k+m}.$$

Let $M_v = \mathrm{Span}_{\mathcal{O}_F}\{v'_1, ..., v'_{2k}\}$. Then it is easy to check that M_v is perpendicular to v. Moreover, M_v is isometric to $\mathcal{H}^k$ since $\mathrm{val}_\pi((v'_{2i-1}, v'_{2i})) = -1$ and $0 \le \mathrm{val}_\pi((v'_i, v'_j))$ for other $1 \le i, j \le 2k$. Hence we can find $g_v \in \mathrm{U}(M^{[k]})$ such that $g_v(M_v) = \mathrm{Span}_{\mathcal{O}_\mathrm{F}}\{v_1, ..., v_{2k}\}$, and $g_v(v) \in \mathrm{Span}_{\mathcal{O}_\mathrm{F}}\{v_{2k+1}, ..., v_{2k+m}\} = M$.

Applying the above to w_1 and w_2, we can find $g_{w_1}, g_{w_2} \in \mathrm{U}(M^{[k]})$ such that $g_{w_1}(w_1),\ g_{w_2}(w_2) \in M$. Now the problem is reduced to Lemma A.3, and the lemma is proved. □

According to Lemma A.2 and Lemma A.4, a primitive vector $v \in M^{[k]}$ is either in the same orbit of a vector $M(1) \in \mathcal{H}^k$ or a vector $M(0) \in M$. Lemma A.1 implies that primitive vectors $M(1), M'(1) \in \mathcal{H}^k$ with $q(M(1)) = q(M'(1))$ lie in the same orbit. Lemma A.3 implies the similar result for primitive $M(0), M'(0) \in M$ with $q(M(0)) = q(M'(0))$. A combination of the above proves Part (4) of Proposition 5.9. □

A.2. **Proof of Proposition 5.10.** In this subsection, we prove the first part of Proposition 5.10, which we restate as follows for the convenience of the reader.

Proposition A.5. *Let L be a Hermitian $\mathcal{O}_F$-lattice of rank 2 and $\mathrm{v}(L) > 0$. Let $\varphi : L \to M^{[k]}$ be a primitive isometric embedding. Let $d(\varphi)$ be the dimension of the image of the map*

$$\mathrm{Pr}_{\mathcal{H}^k} \circ \varphi : L \to \mathcal{H}^k$$

in $\mathcal{H}^k/\pi\mathcal{H}^k$. Then

$$\varphi(L)^\perp \cong (-L) \obot \mathcal{H}^{k-d(\varphi)} \obot M^{(d(\varphi))}$$

where $M^{(d(\varphi))}$ is unimodular of rank equal to $(\mathrm{rank}(M) - 2(2-d(\varphi)))$ and $\det M^{(d(\varphi))} = (-1)^{d(\varphi)} \det M$. In particular, if $d(\varphi) = 1$ then $\mathrm{rank}(M) \ge 2$, and if $d(\varphi) = 0$ then $\mathrm{rank}(M) \ge 4$.

Proof. This proposition follows from Lemmas A.6 and A.7 below. □

Lemma A.6. *Let the notation be as in Proposition A.5. If $\mathrm{rank}(M^{[k]}) \le 4$, then*

$$\varphi(L)^\perp \approx -L.$$

In particular, such an φ does not exist if $\chi(M^{[k]}) = -1$ or $\mathrm{rank}(M^{[k]}) < 4$.

Proof. First, assume $M^{[k]} = \mathcal{H}^2$ and $L \approx \mathcal{H}_i$ where $i > 0$. Let $\varphi(L) = \mathrm{Span}_{\mathcal{O}_\mathrm{F}}\{w_1, w_2\}$ such that the Gram matrix of $\{w_1, w_2\}$ is $\mathcal{H}_i$. By Lemma A.1, we may assume $w_1 = v_1$. Then we may write $w_2 = a_1 v_1 + \pi^{i+1} v_2 + a_3 v_3 + a_4 v_4$, and $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$ by assumption. Without loss of generality, we may assume $a_3 = 1$. Now a direct calculation shows that

$$\varphi(L)^\perp = \mathrm{Span}_{\mathcal{O}_\mathrm{F}}\{v_1 + (-\pi)^{i+1} v_4, v_3 + \bar{a}_4 v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & (-\pi)^i \\ \pi^i & (a_4-\bar{a}_4)\pi^{-1} \end{pmatrix} = \begin{pmatrix} 0 & (-\pi)^i \\ \pi^i & -a_1(-\pi)^i - \bar{a}_1\pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix},$$

hence

$$\varphi(L)^\perp \approx -L.$$

Now we treat the case $M^{[k]} = \mathcal{H}^2$ and $L \approx \mathrm{Diag}(u_1(-\pi_0^a), u_2(-\pi_0)^b)$ where $0 < a \leq b$. Again, let $\varphi(L) = \mathrm{Span}_{\mathcal{O}_F}\{w_1, w_2\}$ such that the Gram matrix of $\{w_1, w_2\}$ is $\mathrm{Diag}(u_1(-\pi_0^a), u_2(-\pi_0)^b)$, and we can assume $w_1 = v_1 - \frac{q(w_1)\pi}{2} v_2$ without loss of generality. Then we may write $w_2 = a_1(v_1 + \frac{q(w_1)\pi}{2} v_2) + a_3 v_3 + a_4 v_4$, hence $\min\{v_\pi(a_3), v_\pi(a_4)\} = 0$ by assumption again. We may assume $a_3 = 1$ and a direct calculation shows that

$$\varphi(L)^\perp = \mathrm{Span}_{\mathcal{O}_F}\{v_1 + \frac{q(w_1)\pi}{2} v_2 - \bar{a}_1 q(w_1)\pi v_4, v_3 + \bar{a}_4 v_4\}.$$

Set $v_3' = v_1 + \frac{q(w_1)\pi}{2} v_2 - \bar{a}_1 q(w_1)\pi v_4$ and $v_4' = a_1 v_3' + v_3 + \bar{a}_4 v_4$. Then $\varphi(L)^\perp = \mathrm{Span}_{\mathcal{O}_F}\{v_3', v_4'\}$ and the Gram matrix of $\{v_3', v_4'\}$ is

$$\begin{pmatrix} -q(w_1) & 0 \\ 0 & a_1\bar{a}_1 q(w_1) - (a_3\bar{a}_4 - \bar{a}_3 a_4)\pi^{-1} \end{pmatrix} = \begin{pmatrix} -q(w_1) & 0 \\ 0 & -q(w_2) \end{pmatrix}.$$

Now assume $M^{[k]} = \mathcal{H} \oplus M$, where M is unimodular of rank 2. We only treat the case $L \approx \mathcal{H}_i$ in detail, and the argument for L represented by a diagonal matrix is similar. We assume that $M^{[k]}$ has a basis $\{v_1, \ldots, v_4\}$ with Gram matrix $\mathcal{H} \oplus \mathrm{Diag}(1, \nu)$ where ν is a unit. Let $\varphi(L) = \mathrm{Span}_{\mathcal{O}_F}\{w_1, w_2\}$ where the Gram matrix of $\{w_1, w_2\}$ is $\mathcal{H}_i$. Then one can check that at least one of w_1 and w_2 is primitive in $\mathcal{H}$. By Lemma A.1, we can assume that

$$w_1 := \varphi(m_1) = v_1, w_2 := \varphi(m_2) = a_1 v_1 + \pi^{i+1} v_2 + a_3 v_3 + a_4 v_4$$

and

$$(w_2, w_2) = a_1\pi^i - \bar{a}_1\pi^i + a_3\bar{a}_3 + a_4\bar{a}_4\nu = 0. \tag{A.2}$$

By our assumption we know that $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$. Since we assume $i \geq 1$, (A.2) implies that both a_3 and a_4 are in $\mathcal{O}_{F_0}^\times$. This in turn implies that $-\nu \in \mathrm{Nm}_{F/F_0^\times}(\mathcal{O}_F^\times) = \mathcal{O}_{F_0}^2$. Hence $M^{[k]} \approx \mathcal{H} \oplus \mathcal{H}_0$ and we can instead assume that $\{v_1, v_2, v_3, v_4\}$ has Gram matrix $\mathcal{H} \oplus \mathcal{H}_0$. We can furthur assume that

$$w_1 = v_1, w_2 = a_1 v_1 + \pi^{i+1} v_2 + v_3 + a_4 v_4$$

with

$$(w_2, w_2) = a_1\pi^i - \bar{a}_1\pi^i + a_4 + \bar{a}_4 = 0.$$

By direct calculation, it is easy to see that

$$\varphi(L)^\perp = \mathrm{Span}_{\mathcal{O}_F}\{v_1 - (-\pi)^i v_4, v_3 - \bar{a}_4 v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & -a_4 - \bar{a}_4 \end{pmatrix} = \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & a_1\pi^i - \bar{a}_1\pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix}.$$

Finally, assume $M^{[k]}$ is unimodular of rank 4. We treat the case $L \approx \mathcal{H}_i$ in detail, and the other cases follow from a similar argument. Let $\varphi(L) = \mathrm{Span}_{\mathcal{O}_F}\{w_1, w_2\}$ such that the Gram matrix of $\{w_1, w_2\}$ is $\mathcal{H}_i$. Apparently $M^{[k]}$ contains a $\mathcal{H}_0$. We can assume that $M^{[k]}$ has a basis $\{v_1, v_2, v_3, v_4\}$

with Gram matrix $\mathcal{H}_0 \obot \mathrm{diag}\{1, \epsilon\}$ where $\epsilon \in \mathcal{O}_{F_0}^\times$. By Lemma A.3 we can assume that $w_1 = v_1$. Then we have

$$w_2 = a_1 v_1 + \pi^i v_2 + \sum_{j=3}^{4} a_j v_j,$$

and

$$(w_2, w_2) = a_1(-\pi)^i + \bar{a}_1 \pi^i + a_3 \bar{a}_3 + a_4 \bar{a}_4 \epsilon = 0. \tag{A.3}$$

By our assumption we know that $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$. Since we assume $i \geq 1$, (A.3) implies that both a_3 and a_4 are in $\mathcal{O}_{F_0}^\times$. This in turn implies that $-\epsilon \in \mathrm{Nm}_{F/F_0^\times}(\mathcal{O}_F^\times) = \mathcal{O}_{F_0}^2$. Hence $M^{[k]} = \mathcal{H}_0^2$ and we can instead assume that $\{v_1, v_2, v_3, v_4\}$ has Gram matrix $\mathcal{H}_0 \obot \mathcal{H}_0$. We can furthur assume that

$$w_1 = v_1, w_2 = a_1 v_1 + \pi^i v_2 + v_3 + a_4 v_4$$

with

$$(w_2, w_2) = a_1(-\pi)^i + \bar{a}_1 \pi^i + a_4 + \bar{a}_4 = 0.$$

By a direct calculation, it is easy to see that

$$\varphi(L)^\perp = \mathrm{Span}_{\mathcal{O}_F}\{v_1 - (-\pi)^i v_4, v_3 - \bar{a}_4 v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & -a_4 - \bar{a}_4 \end{pmatrix} = \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & a_1(-\pi)^i + \bar{a}_1 \pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix}.$$

Notice that, as a byproduct of the above argument, we actually also proved that if $\mathrm{rank}(M^{[k]}) < 4$ or M is not split, then no such φ exists. The lemma is proved.

□

Lemma A.7. *Assume* $\mathrm{v}(L) \geq 0$. *Let* $\varphi : L \to M^{[k]}$ *be a primitive isometric embedding. Let* $d(\varphi)$ *be the dimension of* $\mathrm{Pr}_{\mathcal{H}^k}(\varphi(L)) \otimes_{\mathcal{O}_F} \mathbb{F}_q$ *in* $\mathcal{H}^k / \pi \mathcal{H}^k$. *Then there exist a* $g \in \mathrm{U}(M^{[k]})$ *such that*

$$g(\varphi(L)) \subset \mathcal{H}^{d(\varphi)} \obot I_{4-2d(\varphi)} \subset M^{[k]},$$

where $I_{4-2d(\varphi)}$ *is a unimodular sublattice of* $M^{[k]}$ *with rank* $4 - 2d(\varphi)$.

Proof. We prove the case for $L \approx \mathcal{H}_i$ in detail, and the other cases are similar. Let $\{v_1, \ldots, v_{2k+m}\}$ be a basis of $M^{[k]}$ whose Gram matrix is $\mathcal{H}^k \obot \mathrm{diag}\{1, \ldots, 1, \nu\}$ where ν is a unit. Set $\varphi(L) = \mathrm{Span}_{\mathcal{O}_\mathrm{F}}\{w_1, w_2\}$.
Assume $d(\varphi) = 2$. If $i = -1$, then there is nothing to prove. Therefore, we may assume $i > -1$. By Lemma A.2, without loss of generality, we can assume that $w_1 = v_1$. Then

$$w_2 = a_1 v_1 + \pi^{i+1} v_2 + \sum_{j=3}^{2k+m} a_j v_j.$$

By the assumption that $d(\varphi) = 2$, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 3 \leq j \leq 2k\} = 0.$$

Hence applying Lemma A.2 to $\mathcal{H}^{k-1} \obot M$, we can find a $g \in \mathrm{U}(M^{[k]})$ such that

$$g w_1 = v_1, \quad g w_2 \in \mathcal{H}^2$$

where $\mathcal{H}^2$ refers to the first direct summand in the decomposition $\mathcal{H}^k \obot M = \mathcal{H}^2 \obot \mathcal{H}^{k-2} \obot M$.

When $d(\varphi) = 1$, without loss of generality, we can assume $\mathrm{Pr}_{\mathcal{H}^k}(w_1)$ is primitive. By Lemma A.2, we can assume that $w_1 = v_1$. Then

$$w_2 = a_1 v_1 + \pi^{i+1} v_2 + \sum_{j=3}^{2k+m} a_j v_j.$$

By the assumption that $d(\varphi) = 1$, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 3 \leq j \leq 2k\} \geq 1.$$

Since we assume φ is primitive, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 2k+1 \leq j \leq 2k+r\} = 0.$$

Then we are done by applying Lemma A.4 to $\mathcal{H}^{k-1} \oplus M$.

When $d(\varphi) = 0$, without loss of generality, we may assume $w_1 = v_{2k+1} + v_{2k+2}$ by Lemma A.4. Here, we pick v_{2k+i} so that the corresponding Gram matrix is $\mathrm{Diag}(1, -1, 1, \ldots, -\nu)$ (this is possible since we assume $m \geq 4$). Since φ is primitive with $d(\varphi) = 0$, then

$$w_2 = \sum_{i=1}^{2k} \pi a_i v_i + \sum_{i=2k+1}^{2k+m} a_i v_i,$$

and

$$\min\{\mathrm{v}_\pi(a_j) \mid 2k+3 \leq j \leq 2k+r\} = 0.$$

We are done by applying Lemma A.4 to $\mathcal{H}^k \oplus \mathrm{Span}_{\mathcal{O}_\mathrm{F}}\{v_{2k+3}, \ldots, v_{2k+m}\}$. □

A.3. **Calculation of primitive local density.** In this subsection, we compute primitive local density polynomials and prove the formulas in Propositions 5.9 and 5.10. Assume L is represented by a nonsingular Hermitian matrix T of rank $n \leq 2$. We let $\bar{v}$ denote the image of v in $M^{[k]} \otimes_{\mathcal{O}_F} \mathbb{F}_q$. Let

$$(M^{[k]})^n(i) := \{(v_j) \in M_k^{n,(n)} \mid \mathrm{Span}_{\mathbb{F}_q}\{\mathrm{Pr}_{\mathcal{H}^k}(\bar{v}_j),\ 1 \leq j \leq n\} \text{ has rank } i\}$$

where $M^{n,(n)}$ is as in (5.5), and

$$\beta_i(M, L, X) := \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^n(i)} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x}. \tag{A.4}$$

Notice that

$$\sum_{i=0}^{n} \beta_i(M, L, X) = \beta(M, L, X)^{(n)} \tag{A.5}$$

is the primitive local density defined earlier, and we will shorten it as $\beta(M, L, X)$. Notice that if L is of the form $\mathcal{H}^j$, then $\beta(M, L, X) = \beta_n(M, L, X)$.

First, by a variant of [CY20], Chao Li and Yifeng Liu obtained the following formula of $\beta(\mathcal{H}^k, L)$.

Lemma A.8. [LL22, Lemma 2.16] *Let $b_1 \leq \cdots \leq b_n$ be the unique integers such that $L^\vee/L \approx \mathcal{O}_F/(\pi^{b_1}) \oplus \cdots \oplus \mathcal{O}_F/(\pi^{b_n})$. Let $t_o(L)$ be the number of nonzero entries in $(b_1, \cdots, b_n)$. Then*

$$\beta(\mathcal{H}^k, L) = \prod_{k - \frac{n+t_o(L)}{2} < i \leq k} (1 - q^{-2i}).$$

Lemma A.9. *Assume L is of rank n, then*

$$\beta_n(M, L, q^{-2k}) = \beta(\mathcal{H}^k, L).$$

In particular, if L is of the form $\mathcal{H}^j$, then

$$\beta(M, L, q^{-2k}) = \beta_n(M, L, q^{-2k}) = \beta(\mathcal{H}^k, L).$$

Proof. Recall that $T(\mathbf{x}) = (\mathbf{x}, \mathbf{x})$ is the moment matrix of $\mathbf{x} \in (M^{[k]})^n$. For a $\mathbf{x}_2 \in M^n$, let $T'(\mathbf{x}_2) = T - T(\mathbf{x}_2)$. Then

$$\begin{aligned}
\beta_n(M, L, q^{-2k}) &= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T({}^t(\mathbf{x}_1, \mathbf{x}_2)) - T\rangle) d\mathbf{x}_1 d\mathbf{x}_2 \\
&= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) + T(\mathbf{x}_2) - T\rangle) d\mathbf{x}_1 d\mathbf{x}_2 \\
&= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T'(\mathbf{x}_2)\rangle) d\mathbf{x}_1 d\mathbf{x}_2
\end{aligned}$$

Notice that if L and L' are two Hermitian $\mathcal{O}_F$-lattices with moment matrix T and T' such that $T - T' \in \mathrm{Herm}_n(\mathcal{O}_{F_0})$, then $t_o(L) = t_o(L')$. Hence, for any $\mathbf{x}_2 \in M^n$, we have by Lemma A.8

$$\begin{aligned}
\beta(\mathcal{H}^k, T'(\mathbf{x}_2)) &= \int_{\mathrm{Herm}_n(F)} dY \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T'(\mathbf{x}_2)\rangle) d\mathbf{x}_1 \\
&= \int_{\mathrm{Herm}_n(F)} dY \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T\rangle) d\mathbf{x}_1 \\
&= \beta(\mathcal{H}^k, T).
\end{aligned}$$

Therefore

$$\begin{aligned}
\beta_n(M, L, q^{-2k}) &= \mathrm{vol}(M^n, d\mathbf{x}_2) \cdot \int_{\mathrm{Herm}_n(F)} dY \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T\rangle) d\mathbf{x}_1 \\
&= \int_{\mathrm{Herm}_n(F)} dY \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T\rangle) d\mathbf{x}_1 \\
&= \beta(\mathcal{H}^k, L).
\end{aligned}$$

□

Combining the above two lemmas, we have the following.

Corollary A.10.

(1) If L is of rank 1, then we have

$$\beta_1(M, L, X) = 1 - X.$$

(2) If L is of rank 2, then we have

$$\beta_2(M, L, X) = \begin{cases} (1 - X) & \text{if } L = \mathcal{H}, \\ (1 - X)(1 - q^2 X) & \text{otherwise.} \end{cases}$$

Lemma A.11. *For an $\mathcal{O}_F$-Hermitian lattice, let $\bar{L} = L/\pi L$ be its reduction modulo π with resulting quadratic form. Let $r(\overline{M}, \overline{L})$ to be the number of isometries from $\overline{L}$ to $\overline{M}$. Then*

$$\beta_0(M, L, X) = X^n \beta(M, L) = q^{-mn+n^2} r(\overline{M}, \overline{L}) X^n.$$

Proof. The second identity follows from the same proof of [CY20, Theorem 3.12]. Then a similar argument as in the proof of Lemma A.9 gives the first identity. In this case, we need to replace $M^n \obot (\mathcal{H}^k)^{n,(n)}$ in the proof of Lemma A.9 with $M^{n,(n)} \obot (\pi\mathcal{H}^k)^n$. The factor X^n shows up because $\mathrm{vol}((\pi\mathcal{H}^k)^n) = (q^{-2k})^n$. We leave the details to the reader. □

Notice that [LZ22b, Lemma 3.2.1] provides a uniform formula for $|r(\overline{M}, \overline{L})|$. As a result, we obtain the following corollaries.

Corollary A.12. *Assume $L = \mathcal{O}_F\mathbf{x}$ is of rank 1 (we allow $q(\mathbf{x}) = 0$).*

(1) If $\mathrm{v}(L) = 0$, then

$$\beta_0(M, L, X) = \begin{cases} (1 + \chi(M)\chi(L)q^{-\frac{m-1}{2}})X & \text{if } m \text{ is odd,} \\ (1 - \chi(M)q^{-\frac{m}{2}})X & \text{if } m \text{ is even.} \end{cases}$$

(2) If $\mathrm{v}(L) > 0$, then

$$\beta_0(M, L, X) = \begin{cases} (1 - q^{1-m})X & \text{if } m \text{ is odd,} \\ (1 - q^{1-m} + \chi(M)(q-1)q^{-\frac{m}{2}})X & \text{if } m \text{ is even.} \end{cases}$$

Corollary A.13. *Assume L is of rank 2. When $t(L) = 1$, we assume that L has gram matrix $T = \mathrm{Diag}(u_1, u_2(-\pi_0)^b)$ with $b > 0$.*

(1) If m is odd, then

$$\beta_0(M, L, X) = \begin{cases} q(1 - q^{1-m})X^2 & \text{if } t(L) = 0, \\ q(1 + \chi(M)\chi(u_1)q^{\frac{3-m}{2}})(1 - q^{1-m})X^2 & \text{if } t(L) = 1, \\ q(1 - q^{1-m})(1 - q^{3-m})X^2 & \text{if } t(L) = 2. \end{cases}$$

(2) If m is even, then

$$\beta_0(M, L, X) = \begin{cases} q(1 - \chi(L)q^{1-m} + \chi(L)\chi(M)(q - \chi(L))q^{-\frac{m}{2}})X^2 & \text{if } t(L) = 0, \\ q(1 - \chi(M)q^{-\frac{m}{2}})(1 - q^{2-m})X^2 & \text{if } t(L) = 1, \\ q((1 - q^{2-m}) + \chi(M)(q^2 - 1)q^{-\frac{m}{2}})(1 - q^{2-m})X^2 & \text{if } t(L) = 2. \end{cases}$$

Finally, we calculate $\beta_1(M, L, X)$.

Proposition A.14. *Assume L is as in Corollary A.13. Let $\delta_e(m) = 1$ or 0 depending on whether m is even or odd.*

(1) If $t(L) = 2$, then

$$\beta_1(M, L, X) = q(q+1)((1 - q^{1-m}) + \delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}})X(1 - X).$$

(2) If $t(L) = 1$, then

$$\beta_1(M, L, X) = \begin{cases} q(1 + q - q^{1-m} + \chi(M)\chi(u_1)q^{\frac{3-m}{2}})X(1 - X) & \text{if } m \text{ is odd,} \\ q(1 + q - q^{1-m} - \chi(M)q^{-\frac{m}{2}})X(1 - X) & \text{if } m \text{ is even.} \end{cases}$$

(3) If $t(L) = 0$ and $\chi(L) = 1$, i.e. $L \cong \mathcal{H}_0$, then

$$\beta_1(M, L, X) = q(q + 1 - 2q^{1-m} + \delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}})X(1 - X).$$

(4) If $t(L) = 0$ and $\chi(L) = -1$, then

$$\beta_1(M, L, X) = q(q+1)(1 - \delta_e(m)\chi(M)q^{-\frac{m}{2}})X(1 - X).$$

Proof. First we assume $L = \mathcal{H}_i$. We claim that

$$\beta_1(M, \mathcal{H}_i, X) = \begin{cases} q(1-X)\left(2\beta_0(M,0,X) + \sum_{\alpha\in(\mathcal{O}_{F_0}/(\pi_0))^\times} \beta_0(M, \langle -2\alpha\rangle, X)\right) & \text{if } i = 0, \\ q(q+1)(1-X)\beta_0(M,0,X) & \text{if } i \geq 1. \end{cases}$$

Here $\alpha(M, 0, X) = \alpha(M, \mathcal{O}_F\mathbf{x}, X)$ with $q(\mathbf{x}) = 0$ and $\mathbf{x} \neq 0$. Assuming the claim, the proposition for $L = \mathcal{H}_i$ follows from Corollary A.12.

To prove the claim, it suffices to show the identity for $X = q^{-2k}$ for sufficiently many $k \geq 0$. Recall

$$\begin{aligned} I(M^{[k]}, L, d) = \{\phi \in &\mathrm{Hom}_{\mathcal{O}_F}(L/\pi_0^d L, M^{[k]}/\pi_0^d M^{[k]}) \mid \\ &(\phi(x), \phi(y)) \equiv (x, y) \mod \pi^{2d-1},\ \forall x, y \in L\}. \end{aligned}$$

Let

$$J(M^{[k]}, L, d) \coloneqq \{\phi \in I(M^{[k]}, L, d) \mid \dim_{\mathbb{F}_q}\overline{\mathrm{Pr}_{\mathcal{H}^k}(\phi(L))} = \dim_{\mathbb{F}_q}\overline{\mathrm{Pr}_M(\phi(L))} = 1\}.$$

Then

$$\beta_1(M, L, q^{-2k}) = \lim_{d\to\infty} q^{-(4(2k+m)-4)d}|J(M^{[k]}, L, d)|.$$

Let $\{l_1, l_2\}$ be a basis of L with Gram matrix $\mathcal{H}_i$. For $\phi \in J(M^{[k]}, L, d)$, it will be determined by $w_i = \phi(l_i)$. Let $w_{i,\mathcal{H}} = \mathrm{Pr}_{\mathcal{H}^k}(w_i)$, and $w_{i,M} = \mathrm{Pr}_M(w_i)$. Since $\mathrm{rank}_{\mathbb{F}_q}\overline{\mathrm{Pr}_{\mathcal{H}^k}(\phi(L))} = 1$, $\mathrm{rank}_{\mathbb{F}_q}\overline{\mathrm{Pr}_{\mathcal{H}^k}(w_i)} = 1$ for $i = 1$ or 2.

Now we define a partition of $J(M^{[k]}, L, d)$. Assume $\alpha \in \mathcal{O}_{F_0}$. Let

$$\begin{aligned} J_\alpha(M^{[k]}, L, d) &\coloneqq \{\phi \in I(M^{[k]}, L, d) \mid \mathrm{rank}_{\mathbb{F}_q}\overline{w}_{1,\mathcal{H}} = 1,\ \overline{w}_{2,\mathcal{H}} = \alpha\overline{w}_{1,\mathcal{H}}\},\ \text{and} \\ J_\infty(M^{[k]}, L, d) &\coloneqq \{\phi \in I(M^{[k]}, L, d) \mid \mathrm{rank}_{\mathbb{F}_q}\overline{w}_{2,\mathcal{H}} = 1,\ \overline{w}_{1,\mathcal{H}} = 0\}. \end{aligned}$$

Then it is easy to verify

$$J(M^{[k]}, L, d) = \bigcup_{\alpha\in\mathcal{O}_{F_0}/(\pi_0)} J_\alpha(M^{[k]}, L, d) \cup J_\infty(M^{[k]}, L, d).$$

Now we compute $|J_\alpha(M^{[k]}, L, d)|$. To determine a $\phi \in J_\alpha(M^{[k]}, L, d)$, we choose $w_1 = \phi(l_1)$ first. By definition, we have

$$\begin{aligned} \text{(A.6)} \qquad &\lim_{d\to\infty} q^{(2(2k+m)-1)d}\#\{w_1 \in M^{[k]}/\pi_0^d M^{[k]} \mid w_{1,\mathcal{H}} \text{ is primitive, and } q(w_1) \equiv 0 \bmod \pi_0^d\} \\ &= \beta_1(M^{[k]}, 0) = 1 - q^{-2k}. \end{aligned}$$

Given such a w_1, now we find the number of $w_2 = \phi(l_2)$ such that ϕ lies in $J_\alpha(M^{[k]}, L, d)$. By Lemma A.2, we may assume $w_{1,S} = 0$. Let $w_2 = w_{2,M} + \alpha w_1 + \pi w_{\mathcal{H}}$, where $w_{\mathcal{H}} \in \mathcal{H}^k$. Then the corresponding ϕ lies in $J_\alpha(M^{[k]}, L, d)$ if and only if

$$\pi^i \equiv (w_1, w_2) \equiv (w_1, \pi w_{\mathcal{H}}) \mod \pi^{2d-1}$$

and

$$\begin{aligned} 0 \equiv q(w_2) &\equiv \mathrm{tr}((\alpha w_1, \pi w_{\mathcal{H}})) - \pi_0 q(w_{\mathcal{H}}) + q(w_{2,M}) \\ &\equiv \alpha\mathrm{tr}(\pi^i) - \pi_0 q(w_{\mathcal{H}}) + q(w_{2,M}) \mod \pi^{2d-1}. \end{aligned}$$

First,

$$\text{(A.7)} \qquad \lim_{d\to\infty} q^{-2d(2k-1)}\#\{\pi w_{\mathcal{H}} \in \mathcal{H}^k/\pi_0^d\mathcal{H}^k \mid (w_1, \pi w_{\mathcal{H}}) \equiv \pi^i \mod \pi^{2d-1}\} = q^{1-2k}.$$

Second, for each fixed $\pi w_{\mathcal{H}}$ we have

(A.8)

$$\begin{aligned}&\lim_{d\to\infty} q^{(-2m+1)d}\#\{w_{2,M}\in M/\pi_0^d M \mid w_{2,M}\text{ primitive, } q(w_{2,M})\equiv -\alpha\mathrm{tr}(\pi^i)+\pi_0 q(w_{\mathcal{H}}) \mod \pi^{2d-1}\}\\ &=\beta(M,\langle -\alpha\mathrm{tr}(\pi^i)+\pi_0 q(w_{\mathcal{H}})\rangle)\\ &=\begin{cases}\beta(M,\langle -2\alpha\rangle) & \text{if } i=0,\\ \beta(M,0) & \text{if } i>0.\end{cases}\end{aligned}$$

By symmetry, $|J_\infty(M^{[k]},L,d)|=|J_0(M^{[k]},L,d)|$. Now a combination of (A.6), (A.7) and (A.8) implies that

$$\begin{aligned}\beta_1(M,\mathcal{H}_i,q^{-2k}) &= \lim_{d\to\infty} q^{(-4(2k+m)+4)d}\left(\sum_{\alpha\in\mathcal{O}_{F_0}/(\pi_0)}|J_\alpha(M^{[k]},L,d)|+|J_\infty(M^{[k]},L,d)|\right)\\ &=\begin{cases} q(1-X)\left(2\beta_0(M,0,q^{-2k})+\sum_{\alpha\in\mathcal{O}_{F_0}^\times/(\pi_0)}\beta_0(M,-2\alpha,q^{-2k})\right) & \text{if } i=0,\\ q(q+1)(1-X)\beta_0(M,0,q^{-2k}) & \text{if } i\geq 1,\end{cases}\end{aligned}$$

as claimed.

Next, we assume L has a basis $\{l_1,l_2\}$ whose Gram matrix is $\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b)$ with $0\leq a\leq b$. Let $w_i=\phi(l_i)$ as before. Then the number of possible choices for w_1 is given by

$$q^{(2(2k+m)-1)d}\beta_1(M,\langle u_1(-\pi_0)^a\rangle,q^{-2k})$$

for sufficiently large d. We may assume $w_1=w_{1,\mathcal{H}}$ without loss of generality. Let $w_2=w_{2,M}+\alpha w_1+\pi w_{\mathcal{H}}$ as before. Then ϕ lies in $J_\alpha(M^{[k]},L,d)$ if and only if

$$0\equiv (w_1,w_2)\equiv (w_1,\alpha w_1)+(w_1,\pi w_{\mathcal{H}}) \mod \pi^{2d-1}$$

and

$$\begin{aligned}u_2(-\pi_0)^b\equiv q(w_2) &\equiv (w_{2,M}+\alpha w_1+\pi w_{\mathcal{H}},w_2)\\ &\equiv q(w_{2,M})-\alpha^2 q(w_1)-\pi_0 q(w_{\mathcal{H}}) \mod \pi^{2d-1}.\end{aligned}$$

Now

$$\lim_{d\to\infty} q^{(-4k+2)d}\#\{\pi w_{\mathcal{H}}\in \mathcal{H}^k/\pi_0^d\mathcal{H}^k \mid (w_1,\pi w_{\mathcal{H}})\equiv -(w_1,\alpha w_1) \mod \pi^{2d-1}\}=q^{1-2k},$$

and for a fixed $\pi w_{\mathcal{H}}$ we have

$$\begin{aligned}&\lim_{d\to\infty} q^{(-2m+1)d}\#\{w_{2,M}\in L_S/\pi_0^d L_S \mid w_{2,M}\text{ primitive,}\\ &\qquad q(w_{2,M})\equiv u_2(-\pi_0)^b+\alpha^2 q(w_1)+\pi_0 q(w_{\mathcal{H}}) \mod \pi^{2d-1}\}\\ &=\beta(M,\langle u_2(-\pi_0)^b+\alpha^2 q(w_1)+\pi_0 q(w_{\mathcal{H}})\rangle).\end{aligned}$$

Now this proposition follows from a similar argument as before, and we leave the details to the reader. □

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