THE KUDLA-RAPOPORT CONJECTURE AT A RAMIFIED PRIME FOR $U(1,1)$

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Abstract. In this paper, we proved a local arithmetic Siegel-Weil formula for a $U(1,1)$-Shimura variety at a ramified prime, a.k.a. a Kudla-Rapoport conjecture at a ramified case. The formula needs to be modified from the original Kudla-Rapoport conjecture. In the process, we also gives an explicit decomposition of the special divisors of the Rapoport-Zink space of unitary type $(1,1)$ (Krämer model). A key ingredient is to relate the Rapoport-Zink space to the Drinfeld upper plane.

Contents

1. Introduction

The Arithmetic Siegel-Weil formula is part of the Kudla program, revealing a deep relation between the derivative of some Eisenstein series and the arithmetic degree or intersection of arithmetic special cycles on certain Shimura varieties of unitary or orthogonal type. It was started with Kudla's seminal paper in 1997 ([Kud97]), and the first two complete examples were worked out by Kudla, Rapoport, and one of the authors (T.Y.) ([KRY99], [KRY04], and [KRY06]). Afterwards, more progress have been made in other cases, e.g., [HY12], [Liu11], [DY19], [LZ22], [GS19], and [BY21]. In almost all cases, the proof is via term by term comparison on T-th Fourier coefficients. When T is non-singular, the T-th coefficient of the Eisenstein series factors through product of local Wittaker functions while the arithmetic intersection (degree) is often supported on a single prime. The key step is a local identity—local arithmetic Siegel-Weil formula: a deep relation between the derivative of the local Whittaker function and the 'local intersection number' at a special point. In particular, the 'local intersection index' is independent of the choice of 'special points' (depending only on T)—an amazing phenomenon itself. At the infinite prime, it was proved by Yifeng Liu in the unitary case ([Liu11]), inspired

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by Kudla's original proof for Shimura curves([Kud97]). The idea does not seem to extend to orthogonal cases. Two different proofs in the general orthogonal case were given independently by Garcia and Sankaran [GS19] and Bruinier and Yang ([BY21]). At a 'good' finite prime (everything is 'unramified'), Kudla and Rapoport formulated a precise local arithmetic Siegel-Weil formula, the so-called Kudla-Rapoport conjecture. This conjecture was recently proved by Li and Zhang ([LZ22], see also [BY21] for a special case) in the unitary case. More recently, Cho ([Cho22]) considered the unitary $U(n,1)$ case of miniscule parahoric level over an unramified quadratic extension E/F_0 (of local p-adic fields), and made some precise conjectural local arithmetic Siegel-Weil formula. The case $n=1$ was essentially Sankaran's early result ([San17]). In Shi's work ([Shi20]) and this paper, we consider the simplest case $U(1,1)$ over a ramified quadratic local field extensions F/F_0. Ramification causes a lot of technical difficulty, including singularity and so on. The resulting formula also needs modification in general. We hope to extend the result to general $U(n,1)$ in a sequel. We now set up some notations and describe the main results in some detail.

Let $F=F_0(\pi)$ be a ramified quadratic field extension of a p-adic field F_0 with $p>2$ with $\pi_0=\pi^2$ being a uniformizer of F_0. Let $\boldsymbol{k}=\mathbb{F}_q$ be the residue field of F_0 (also F) with the algebraic closure $\bar{\boldsymbol{k}}$. We denote the Galois conjugate of $a\in F$ by $\bar{a}$. Let $(V,(\ ,\ ))$ be a Hermitian space over F of dimension 2 together with an $\mathcal{O}_F$-unimodular lattice L. Up to equivalence, we can describe L explicitly as follows. Let B be a quaternion algebra over F_0 together with an embedding $\mathcal{O}_F\subset\mathcal{O}_B$ (when $B=M_2(F_0)$, we take $\mathcal{O}_B=M_2(\mathcal{O}_{F_0})$). Choose $\delta\in\mathcal{O}_B$ with $\delta x=\bar{x}\delta$ for all $x\in\mathcal{O}_F$ and $\Delta=\delta^2\in\mathcal{O}_{F_0}^\times$. Then we can take $L=\mathcal{O}_F+\mathcal{O}_F\delta$ with $z_i=x_i+y_i\delta$

$$(z_1,z_2)=\operatorname{tr}_F(z_1z_2^\iota)=x_1\bar{x}_2-y_1\bar{y}_2\Delta.$$

Here $\operatorname{tr}_F(x+y\delta)=x\in F$. Notice that $V=B$ in this case.

Recall that a strict formal $\mathcal{O}_{F_0}$-module over an $\mathcal{O}_{F_0}$-scheme S is a formal p-divisible group X over S with an action $\mathcal{O}_{F_0}\to\operatorname{End}(X)$ such that the induced $\mathcal{O}_{F_0}$-action on the Lie algebra $\operatorname{Lie}X$ is via the structural morphism $\mathcal{O}_{F_0}\to\mathcal{O}_S$. Let $\mathbb{Y}=(\mathbb{Y},\iota_{\mathbb{Y}},\lambda_{\mathbb{Y}})$ be the unique supersingular strict formal $\mathcal{O}_{F_0}$-module of dimension 1 and F_0 height 2 over $\bar{\boldsymbol{k}}$ with an $\mathcal{O}_F$-action $\iota_{\mathbb{Y}}$ and a principal polarization $\lambda_{\mathbb{Y}}:\mathbb{Y}\to\mathbb{Y}^\vee=\mathbb{Y}$. Let $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$ be a strict formal $\mathcal{O}_{F_0}$-module of dimension 2 and F_0 height 4 over $\bar{\boldsymbol{k}}$, together with the $\mathcal{O}_F$-action $\iota_{\mathbb{X}}$ and principal polarization $\lambda_{\mathbb{X}}:\mathbb{X}\mapsto\mathbb{X}^\vee$. Let $\mathbb{L}=\operatorname{Hom}_{\mathcal{O}_F}(\mathbb{Y},\mathbb{X})$ be endowed with the Hermitian form

$$h(\mathbf{x},\mathbf{y})=\lambda_{\mathbb{Y}}^{-1}\circ\mathbf{y}^\vee\circ\lambda_{\mathbb{X}}\circ\mathbf{x}\in\operatorname{End}_{\mathcal{O}_F}(\mathbb{Y})\xrightarrow[\sim]{\iota_{\mathbb{Y}}^{-1}}\mathcal{O}_F. \tag{1.1}$$

The Hermitian lattice $(\mathbb{L},h)$ is $\mathcal{O}_F$-unimodular as the polarizations are principal. Let $\mathbb{V}=\mathbb{L}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ be the associated Hermitian space. To describe the Rapoport-Zink (RZ) space for the Shimura curve related to L, we require

$$\det L/\det\mathbb{L}\notin\mathrm{N}_{F/F_0}F^\times, \tag{1.2}$$

i.e., V and $\mathbb{V}$ represent two different non-degenrate Hermitian spaces over F of dimension 2. By [RTW14, Remark 4.2] (also see [BC91, Proposition II.5.2] and the discussion below on the isomorphism between $\mathcal{N}$ and $\breve{\Omega}$), the determinant condition determines $\mathbb{X}$ uniquely up to $\mathcal{O}_F$-linear quasi-isogenies that preserve the polarization, hence determines the corresponding RZ space.

Let $\breve{F}$ be the completion of the maximal unramified extension of F with ring of integers $\mathcal{O}_{\breve{F}}$, and similarly for $\breve{F}_0$. We also denote $W = \mathcal{O}_{\breve{F}_0}$. Let $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$ be the category of $\mathcal{O}_{\breve{F}}$-schemes S such that $\pi \cdot \mathcal{O}_S$ is a locally nilpotent ideal sheaf. Recall the RZ-space $\mathcal{N}_{(1,1)}$ over $\mathcal{O}_{\breve{F}}$ which represents the following moduli problem ([RTW14]): for $S \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$, $\mathcal{N}_{(1,1)}(S)$ is the groupoid of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ given as follows:

(1) X is a strict formal $\mathcal{O}_{F_0}$-module over S of dimension 2 and F_0 height 4;
(2) $\iota : \mathcal{O}_F \to \mathrm{End}(X)$ is an $\mathcal{O}_F$-action on X satisfying Kottwitz condition:
$$\mathrm{char}(\iota(\pi)|\mathrm{Lie}X) = (T-\pi)(T+\pi) = T^2 - \pi_0;$$
(3) $\lambda : X \to X^\vee$ is a principal quasi-polarization whose associated Rosati involution induces on $\mathcal{O}_F$ the nontrivial automorphism over F_0;
(4) Finally, $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, k} \bar{S}$ is a $\mathcal{O}_F$-linear quasi-isogeny of height 0 such that λ and $\rho^*(\lambda_{\mathbb{X}})$ differ locally on $\bar{S}$ by a factor in $\mathcal{O}_{F_0}^\times$.

Although $\mathcal{N}_{(1,1)}$ is normal and Cohen-Macaulay (see [Pap00]), it is not regular. There is a desingularization, called the Krämer model $\mathcal{N}^{\mathrm{Kra}}_{(1,1)}$([Krä03]), which represents the isomorphism classes of quintuples $(X, \iota, \lambda, \rho, \mathcal{F}_X)$, where $(X, \iota, \lambda, \rho) \in \mathcal{N}_{(1,1)}(S)$, and

(1) $\mathcal{F}_X$ is an $\mathcal{O}_F \otimes \mathcal{O}_S$-submodule of $\mathrm{Lie}(X)$ with $\mathcal{O}_S$-rank 1 and is a direct $\mathcal{O}_S$-summand of $\mathrm{Lie}(X)$;
(2) $\mathcal{O}_F$ acts on $\mathcal{F}_X$ via the structure morphism $\mathcal{O}_F \to \mathcal{O}_S$ and acts on $\mathrm{Lie}(X)/\mathcal{F}_X$ via the Galois conjugate of the structure morphism.

By [Shi20, Proposition 2.7], $\mathcal{N}^{\mathrm{Kra}}$ is the blow up of $\mathcal{N}_{(1,1)}$ along its singular locus which consists of "superspecial" points.

Let $\mathcal{N}_{(1,0)}$ be the similar Rapoport-Zink space with the framing object $\mathbb{X}$ replaced by $\mathbb{Y}$. Then $\mathcal{N}_{(1,0)}$ is smooth and has a universal strict formal $\mathcal{O}_{F_0}$-module $\mathcal{Y}$ over $\mathcal{O}_{\breve{F}}$. For every $S \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$, we have $\mathcal{N}_{(1,0)}(S) = \{Y = \mathcal{Y}_S\}$. The RZ-space we considered is $\mathcal{N}^{\mathrm{Kra}} = \mathcal{N}_{(1,0)} \times \mathcal{N}^{\mathrm{Kra}}_{(1,1)}$. Adding $\mathcal{N}_{(1,0)}$ enables us to define cycles naturally. For simplicity, we often identify $\mathcal{N}^{\mathrm{Kra}}$ with $\mathcal{N}^{\mathrm{Kra}}_{(1,1)}$ in this paper (ignore the one element formal scheme $\mathcal{N}_{(1,0)}$).

For a $\mathbf{x} \in \mathbb{V}\backslash\{0\}$, let $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ be the closed (formal) subscheme of $\mathcal{N}^{\mathrm{Kra}}$ consisting of $(Y, X) = (Y, \iota_Y, \lambda_Y, \rho_Y, X, \iota_X, \lambda_X, \rho_X, \mathcal{F}_X) \in \mathcal{N}^{\mathrm{Kra}}$ such that $\rho_X^{-1} \circ \mathbf{x} \circ \rho_Y$ lifts to an $\mathcal{O}_F$ homomorphism $x : Y \to X$. According to [How19, Proposition 4.3][1], $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ is a Cartier divisor. The closed formal subscheme $\mathcal{Z}(\mathbf{x}) \subset \mathcal{N}_{(1,0)} \times \mathcal{N}_{(1,1)}$ can be defined the same way. Notice, however, that $\mathcal{Z}(\mathbf{x})$ is not a Cartier divisor. For a pair of independent vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{L}$, define the intersection number

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}), \tag{1.3}$$

which depends only on the lattice $L_{\mathbf{x}_1,\mathbf{x}_2} \subset \mathbb{V}$ generated by $\mathbf{x}_1$ and $\mathbf{x}_2$. Two Hermitian matrices $T_1, T_2 \in \mathrm{Herm}_2(F)$ are said to be equivalent, denoted by $T_1 \approx T_2$, if there is a non-singular matrix $g \in \mathrm{GL}_2(\mathcal{O}_F)$ such that $g^t T_1 \bar{g} = T_2$. It is known (see [Jac62, Proposition 4.3]) that every Hermitian matrix of order 2 is equivalent to either a diagonal matrix or an anti-diagonal matrix of the form $\begin{pmatrix} 0 & \pi^n \\ (-\pi)^n & 0 \end{pmatrix}$ with n odd.

[1][How19] deals with the case when $F_0 = \mathbb{Q}_p$. Using the relative display theory and its Grothendieck-Messing theory developed in [ACZ16], the results and proofs there extend word by word to general F_0.

The first main result of this paper, together with that in [Shi20], is the following theorem.

Theorem 1.1. *For a pair of independent vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$, let $T(\mathbf{x}_1, \mathbf{x}_2) = (h(\mathbf{x}_i, \mathbf{x}_j))$ be the associated Gram matrix. Then $Int(L_{\mathbf{x}_1,\mathbf{x}_2})$ depends only on the equivalence class of $T = T(\mathbf{x}_1, \mathbf{x}_2)$. More precisely, $Int(L_{\mathbf{x}_1,\mathbf{x}_2}) = 0$ unless $T(\mathbf{x}_1, \mathbf{x}_2)$ is integral. Assume that $T(\mathbf{x}_1, \mathbf{x}_2)$ is integral.*

(1) When $T \approx diag(u_1(-\pi_0)^\alpha, u_2(-\pi_0)^\beta)$ with $u_i \in \mathcal{O}_{F_0}^\times$ and $0 \le \alpha \le \beta$, we have

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = \begin{cases} \alpha+\beta-\frac{2q(q^\alpha-1)}{q-1} & \text{if } \mathbb{V} \text{ is isotropic,} \\ 2\sum_{s=0}^{\alpha} q^s(\alpha+\beta+1-2s)-\alpha-\beta-2 & \text{if } \mathbb{V} \text{ is anisotropic.} \end{cases}$$

(2) When $T \approx \left(\begin{smallmatrix} 0 & \pi^n \\ (-\pi)^n & 0 \end{smallmatrix}\right)$ with n odd (which occurs only when $\mathbb{V}$ is isotropic), we have

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = -\frac{(q+1)(q^{\frac{n+1}{2}}-1)}{q-1} + n + 1.$$

The case that $\mathbb{V}$ is anisotropic was proved by one of the authors (Y.S.) in [Shi20]. This paper was inspired by his work. A key ingredient is to understand the arithmetic divisor $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ and in particular the contribution of the exceptional divisors, which is new in the ramified case. In our special $U(1,1)$-case, we use critically the special facts:

(1) $\mathcal{N}_{(1,1)}$ descends to a **regular** formal scheme $\mathcal{N}$ over $\mathcal{O}_{\breve{F}_0}$;
(2) The formal scheme $\mathcal{N}$ is isomorphic to the RZ-space $\mathcal{M}_{\Gamma_0(\pi_0)}$ (see [RSZ18, Definition 8.1]) in the case $\mathbb{V}$ is anisotropic (i.e., the quaternion algebra B at the beginning of the introduction is a matrix algebra), and is isomorphic to the formal scheme $\mathcal{M}$, which is represented by the formal completion $\breve{\Omega}$ of the Drinfeld half plane over $\mathcal{O}_{\breve{F}_0}$ in the case $\mathbb{V}$ is isotropic (i.e. B is a division algebra)([Dri76]), see Section 2.

In this paper, we consider the case that $\mathbb{V}$ is isotropic. Using properties of $\breve{\Omega}$, we can understand the special fibers of the divisors $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$. Using explicit local equations at 'superspecial points' of $\breve{\Omega}$, we obtain the following decomposition result of independent interest. An $\mathcal{O}_F$-lattice $\Lambda \subset \mathbb{V}$ is called a vertex lattice of type $t = 0$ or 2 if $\pi\Lambda \subset \Lambda^\sharp \subset \Lambda$ with $[\Lambda : \Lambda^\sharp] = t$. Here $\Lambda^\sharp$ is the $\mathcal{O}_F$-dual of Λ with respect to the Hermitian form on $\mathbb{V}$. Associated to each vertex lattice Λ_0 of type 0 is an exceptional divisor $\mathrm{Exc}_{\Lambda_0} \subset \mathcal{N}^{\mathrm{Kra}}/\bar{\boldsymbol{k}}$, and associated to each vertex lattice Λ_2 of type 2 is a vertical projective line $\mathbb{P}_{\bar{\Lambda}_2}/\bar{\boldsymbol{k}} \subset \mathcal{N}^{\mathrm{Kra}}/\bar{\boldsymbol{k}}$. Here and throughout this paper, we write $\bar{\Lambda} = \Lambda \otimes_{\mathcal{O}_F} \bar{\boldsymbol{k}}$ for an $\mathcal{O}_F$-lattice Λ.

Theorem 1.2. *Let $\mathbf{x} \in \mathbb{V}\setminus\{0\}$ with $h(\mathbf{x},\mathbf{x}) \in \mathcal{O}_{F_0}$, and let $\mathcal{T}(\mathbf{x})$ be the set of vertex lattices Λ such that $\mathcal{Z}(\mathbf{x})^{\mathrm{Kra}}(\bar{\boldsymbol{k}})$ intersects with $\mathbb{P}_{\bar{\Lambda}}$ or Exc_Λ, depending on whether Λ is of type 2 or 0. Then*

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(\mathbf{x},\Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(\mathbf{x},\Lambda_0)+1)\mathrm{Exc}_{\Lambda_0} + \mathcal{Z}^h(\mathbf{x}).$$

Here $\mathcal{Z}^h(\mathbf{x})$ is the horizontal component of $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$, which is empty or non-empty irreducible depending on $h(\mathbf{x},\mathbf{x}) = 0$ or not. $n(\mathbf{x},\Lambda)$ is the largest integer n such that $\pi^{-n}\mathbf{x} \in \Lambda$. When $h(\mathbf{x},\mathbf{x}) \notin \mathcal{O}_{F_0}$, $\mathcal{Z}^{Kra}(\mathbf{x}) = 0$.

The proof of this theorem occupies Section 4 with preparation in Section 3. Now proof of Theorem 1.1 becomes explicit computation of the intersection number among the divisors in the decomposition above. This occupies Section 5

To state the last main result of this paper—the Kudla-Rapoport formula, we need to define the local density function. Let $(L,(\ ,\ ))$ be an integral $\mathcal{O}_F$-Hermitian lattice of rank m with a Gram matrix S, and let $T \in \mathrm{Herm}_n(\mathcal{O}_F)$. Let $\mathcal{H} = \mathcal{O}_F^2$ be the Hermitian hyperbolic plane with the standard Hermitian form $(x,y) = \frac{1}{\pi}(x_1\bar{y}_2 - x_2\bar{y}_1)$ with Gram matrix $\frac{1}{\pi}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let $L_r = L \oplus \mathcal{H}^r$ with Gram matrix S_r. Then it can be shown that there is a unique 'local density polynomial' $\alpha(L,T,X)$ of $X = q^{-2r}$ such that

$$\alpha(L,T,q^{-2r}) = \int_{\mathrm{Herm}_n(F)} \int_{L_r^n} \psi(\mathrm{tr}(b(T(x)-T)))dx\,db, \tag{1.4}$$

where $T(x) = ((x_i,x_j))$, dx is the Haar measure on L_r^n with total volume 1, db is the Haar measures on $\mathrm{Herm}_n(\mathcal{O}_F)$ with total volume 1, and ψ is an additive character of F_0 with conductor $\mathcal{O}_{F_0}$. We will denote $\alpha(L,T) = \alpha(L,T,1)$ so that $\alpha(L,T,q^{-2r}) = \alpha(L_r,T)$. We will also denote

$$\alpha'(L,T) = -\frac{\partial}{\partial X}\alpha(L,T,X)|_{X=1}. \tag{1.5}$$

Let ∂_F be the different ideal of F/F_0. Define

$$\mathrm{Herm}_n(\mathcal{O}_F)^\vee := \{T = (t_{ij}) \in \mathrm{Herm}_n(F) \mid \mathrm{ord}_\pi(t_{ii}) \geq 0, \text{ and } \mathrm{ord}_\pi(t_{ij}\partial_F) \geq 0\}.$$

Notice that $\mathrm{Herm}_n(\mathcal{O}_F)^\vee$ is the dual of $\mathrm{Herm}_n(\mathcal{O}_F)$ under the pairing $\psi(\mathrm{tr}(XY))$. Simple calculation gives

$$\alpha(L,T) = q^{-an(2m-n)}|\{X \in M_{m,n}(\mathcal{O}_F/(\pi_0^a)) \mid X^tS\bar{X} - T \in \pi_0^a \cdot \mathrm{Herm}_n(\mathcal{O}_F)^\vee\}| \tag{1.6}$$

for sufficiently large integer $a > 0$. For this reason, we will also denote $\alpha(L,T,X)$ by $\alpha(S,T,X)$ and so on. Now we can state the Kudla-Rapoport formula at the ramified prime as follows, which comes from the comparison between the formula in Theorem 1.1 and the local density formulas in [Shi20] (see Section 6).

Theorem 1.3. *Let L be the unimodular $\mathcal{O}_F$-Hermitian lattice of rank 2 as in the beginning of the introduction with Gram matrix S, and let $T = T(\mathbf{x}_1,\mathbf{x}_2) \in \mathrm{Herm}_2(F)$ with $\mathbf{x}_i \in \mathbb{V}$ being independent. Then $\alpha(L,T) = 0$, and*

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = 2\frac{\alpha'(L,T)}{\alpha(L,S)} - \frac{2q^2}{q^2-1}\frac{\alpha(\mathcal{H},T)}{\alpha(L,S)}.$$

Remark 1.4. *The case that $\mathbb{V}$ is anisotropic was proved in* [Shi20]. *In his case, $\alpha(\mathcal{H},T) = 0$.*

The paper is organized as follows. In Section 2, we collect the definitions of various relevant moduli spaces and discuss the relations among these spaces. In Section 3, we give a description of the special fiber of the moduli spaces and special cycles. In Section 4, we give a decomposition of the special divisor by studying the local equations of special divisors at superspecial points, and prove Theorem 1.2. Section 5 contains a computation of the intersection number between special divisors and the proof of Theorem 1.1. Finally in Section 6, we review Shi's local density formula and prove Theorem 1.3.

For the rest of this paper, we assume that V **is anisotropic and** $\mathbb{V}$ **is isotropic**, i.e., the associated quaternions B being the division algebra, and $\mathbb{B} = M_2(F_0)$.

Recall that $\mathcal{O}_B = \mathcal{O}_F + \mathcal{O}_F\delta$ with $\delta x = \bar{x}\delta$ and $\Delta = \delta^2 \in \mathcal{O}_{F_0}^\times$. So $E = F_0(\delta)$ is the unique unramified quadratic field extension of F_0 with ring of integers $\mathcal{O}_E = \mathcal{O}_{F_0}(\delta)$, which will be viewed as a subfield of $\breve{F}_0$. Let $\sigma \in \mathrm{Gal}(\breve{F}_0/F_0)$ be the Frobennius element lifting $\sigma(x) = x^q$ of $\mathrm{Gal}(\bar{\boldsymbol{k}}/\boldsymbol{k})$. Denote $W = \mathcal{O}_{\breve{F}_0}$. We will also identify

$$B = \{\left(\begin{smallmatrix} a & b\pi_0 \\ b^\sigma & a^\sigma \end{smallmatrix}\right) : a, b \in E\}, \tag{1.7}$$

together with two embeddings

$$\iota : E \hookrightarrow B, a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a^\sigma \end{smallmatrix}\right), \text{ and } \iota : F \hookrightarrow B, a + b\pi \mapsto \left(\begin{smallmatrix} a & b\pi_0 \\ b & a \end{smallmatrix}\right). \tag{1.8}$$

Note that $\pi a = a^\sigma \pi$ for all $a \in E$.

If R is an $\mathcal{O}_E$-algebra, there is a ring isomorphism

$$\mathcal{O}_E \otimes_{\mathcal{O}_{F_0}} R \cong R \times R, \quad a \otimes x \mapsto (ax, \sigma(a)x). \tag{1.9}$$

For any $\mathcal{O}_E \otimes_{\mathcal{O}_{F_0}} R$ module M, the above ring isomorphism induces a $\mathbb{Z}/2\mathbb{Z}$-grading

$$M = M_0 \oplus M_1. \tag{1.10}$$

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2. Preliminary

Following [BC91] and [KR14], we briefly review several moduli functors in this section for later use.

2.1. **Framing objects and their Dieudonné modules.** Throughout the paper, by relative Dieudonné module we mean relative $\mathcal{O}_{F_0}$-Dieudonné module in the sense of [RZ96, 3.56] or [Far06, Appendix B.8], which is a special case of the $\mathcal{O}_{F_0}$-display studied in [ACZ16]. Let $\mathbb{Y}$ be the unique 1 dimensional formal $\mathcal{O}_{F_0}$-module of relative height 2 over $\bar{\boldsymbol{k}}$ as in the introduction. Then its relative Dieudonné module (see page 4 of [KR14] or [RZ96, Proposition 3.56]) $M(\mathbb{Y})$ is a free $\mathcal{O}_{\breve{F}_0}$-module of rank 2. We can choose a basis $\{e, f\}$ of $M(\mathbb{Y})$ such that

$$Fe = f,\ Ff = \pi_0 e,\ Ve = f,\ Vf = \pi_0 e,\ \langle e, f\rangle = \delta. \tag{2.1}$$

With respect to this basis, we have

$$\mathrm{End}(M(\mathbb{Y})) \cong \left\{ \begin{pmatrix} a & b\pi_0 \\ b^\sigma & a^\sigma \end{pmatrix} \mid a, b \in \mathcal{O}_E \right\} = \mathcal{O}_B. \tag{2.2}$$

In particular

$$\iota_{\mathbb{Y}}(\pi) = \begin{pmatrix} 0 & \pi_0 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

The framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ in the introduction can be explicitly realized as follows. Let $\mathbb{X} = \mathbb{Y} \times \mathbb{Y}$ and identify $\mathrm{End}(\mathbb{X})$ with $M_2(\mathcal{O}_B)$. Then we can define an action of $\mathcal{O}_B$ on $\mathbb{X}$ via

$$\iota_{\mathbb{X}}(b) = \begin{pmatrix} b & 0 \\ 0 & \pi b \pi^{-1} \end{pmatrix}. \tag{2.4}$$

Finally, we identify $\mathbb{X}^\vee = \mathbb{X}$ and define the principal polarization

$$\lambda_{\mathbb{X}} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right). \tag{2.5}$$

Then the Rosati involution associated to $\lambda_{\mathbb{X}}$ induces the involution $b \mapsto b^* = \pi b^\iota \pi^{-1}$ on B. This triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is the basic Drinfeld triple defined in [KR14, Proposition

1.1]. We obtain the basic framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ from the above framing object by restricting $\iota_{\mathbb{X}}$ on $\mathcal{O}_F$. One can check that the Hermitian form on $\mathbb{L} = \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X})$ induced by $\lambda_{\mathbb{X}}$ is **isotropic** as required in this paper (see Lemma 2.2).

We choose a basis $\{e_0, f_0, e_1, f_1\}$ of $M(\mathbb{X})$, the relative Dieudonné module of $\mathbb{X}$, such that $\{e_i, f_i\}$ $(i = 0, 1)$ is the basis of $M(\mathbb{Y})$ for the i-th copy of $\mathbb{Y}$. Then we have

$$Fe_i = f_i,\ Ff_i = \pi_0 e_i,\ Ve_i = f_i,\ Vf_i = \pi_0 e_i. \tag{2.6}$$

$$\iota_{\mathbb{X}}(\pi)e_i = f_i,\ \iota_{\mathbb{X}}(\pi)f_i = \pi_0 e_i,\ \iota_{\mathbb{X}}(\delta)e_i = (-1)^i \delta e_i,\ \iota_{\mathbb{X}}(\delta)f_i = (-1)^{i+1}\delta f_i. \tag{2.7}$$

Remark 2.1. *In some literature (e.g.* [BC91] *and* [RTW14]*), the operator $\iota_{\mathbb{X}}(\pi)$ is denoted as Π. In this paper, since the action of $\mathcal{O}_F$ on the relative Dieudonné module (later Cartier module) is unambiguous, we will mostly write $\iota_{\mathbb{X}}(a)$ (later $\iota(a)$) simply as a for $a \in \mathcal{O}_F$.*

We also need to consider the $\mathcal{O}_E$-action on $M(\mathbb{X})$ obtained by restricting $\iota_{\mathbb{X}}$ on $\mathcal{O}_E$. The grading on $M(\mathbb{X})$ as defined in (1.10) is

$$M_0 = M(\mathbb{X})_0 = \mathrm{span}_{\mathcal{O}_{\breve{F}_0}}\{e_0, f_1\},\ M_1 = M(\mathbb{X})_1 = \mathrm{span}_{\mathcal{O}_{\breve{F}_0}}\{e_1, f_0\}. \tag{2.8}$$

Let $N = N(\mathbb{X}) \coloneqq M(\mathbb{X}) \otimes \mathbb{Q}$. The principal polarization $\lambda_{\mathbb{X}}$ induces an alternating form $\langle , \rangle$ on N such that

$$\begin{aligned} \langle Fx, y\rangle &= \langle x, Vy\rangle^{\sigma}, \\ \langle ax, y\rangle &= \langle x, \bar{a}y\rangle, a \in \mathcal{O}_F. \end{aligned} \tag{2.9}$$

In terms of the $\breve{F}_0$-basis $\{e_0, f_0, e_1, f_1\}$, we have

$$\langle e_0, f_1\rangle = \langle e_1, f_0\rangle = \delta, \tag{2.10}$$

and all the other pairings between the basis vectors are 0. Following [KR14], we can define a Hermitian form $(,)$ on N by

$$(x, y) = \delta[\langle \pi x, y\rangle + \pi\langle x, y\rangle]. \tag{2.11}$$

Let $\tau = \pi V^{-1}$. Then $C \coloneqq N^{\tau} = \mathrm{span}_F\{e_0, e_1\}$ is a Hermitian F-vector space with $C \otimes_F \breve{F} = N$. Moreover,

$$(e_0, e_1) = -\delta^2,\ (e_0, e_0) = (e_1, e_1) = 0. \tag{2.12}$$

So $(C, (,))$ is isotropic, satisfying the assumption of this paper. There is a similarly defined Hermitian form $(,)_{\mathbb{Y}}$ on $N(\mathbb{Y})^{\tau}$ given by

$$(x, y)_{\mathbb{Y}} = \delta[\langle \pi x, y\rangle + \pi\langle x, y\rangle]. \tag{2.13}$$

Here $\langle e, f\rangle = \delta$ as in (2.1).

Recall $\mathbb{L} = \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X}) \cong \mathrm{Hom}_{\mathcal{O}_F}(M(\mathbb{Y}), M(\mathbb{X}))$ and $\mathbb{V} = \mathbb{L} \otimes_{\mathcal{O}_F} F$. For $\mathbf{x} \in \mathbb{L}$, we will also use $\mathbf{x}$ to denote the corresponding homomorphism between relative Dieudonné modules when there is no confusion. Now $\mathbf{x}(e) \in M(\mathbb{X})^{\tau}$ since $\tau(e) = e$ by (2.2), (2.3) and $\mathbf{x}$ commutes with π and V. This way, we obtain a map from $\mathbb{L}$ to $M(\mathbb{X})^{\tau}$ by sending $\mathbf{x} \in \mathbb{V}$ to $\mathbf{x}(e)$.

Lemma 2.2. *(*[Shi18, Lemma 3.3]*) We can identity $\mathbb{L}$ with $M(\mathbb{X})^{\tau}$ by sending $\mathbf{x} \in \mathbb{L}$ to $\mathbf{x}(e)$. Moreover,*

$$h(\mathbf{x}, \mathbf{x})(e, e)_{\mathbb{Y}} = (\mathbf{x}(e), \mathbf{x}(e)), \tag{2.14}$$

where $h(,)$ is defined in (1.1). According to (2.13), $(e, e)_{\mathbb{Y}} = -\delta^2$. In particular, $h(,)$ is also isotropic since $(,)$ is isotropic as we showed previously.

Because of Lemma 2.2, we will often identify $\mathbb{V}$ with C, and $\mathbb{L}$ with $M(\mathbb{X})^\tau$, via $\mathbf{x} \mapsto \mathbf{x}(e)$. The Hermitian form is shifted by a factor $-\delta^2$.

2.2. **Moduli space $\mathcal{M}$.** We first recall the following result of Kramer ([Krä03]).

Proposition 2.3. *$\mathcal{N}^{\mathrm{Kra}}$ is representable by a formal scheme over* $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ *which is regular and has semi-stable reduction. The natural forgetful map*

$$\Phi : \mathcal{N}^{\mathrm{Kra}} \to \mathcal{N}_{(1,1)}$$
$$(X, \iota, \lambda, \rho, \mathcal{F}_X) \mapsto (X, \iota, \lambda, \rho)$$

is an isomorphism outside the singular locus Sing *of $\mathcal{N}_{(1,1)}$, and $\Phi^{-1}(x)$ for $x \in$ Sing is an exceptional divisor which is isomoprhic to the projective line $\mathbb{P}^1_{\bar{\boldsymbol{k}}}$.*

By [Shi20, Proposition 2.7], $\mathcal{N}^{\mathrm{Kra}}$ is the blow up of $\mathcal{N}_{(1,1)}$ along Sing.

As mentioned in the introduction, the formal scheme $\mathcal{N}_{(1,1)}$ descends to a formal scheme $\mathcal{N}$ over $\mathcal{O}_{\breve{F}_0}$ (since the Kottwitz determinant condition is defined over $\mathcal{O}_{\breve{F}_0}$). A beautiful result of Kudla and Rapoport ([KR14]) asserts that $\mathcal{N}$ is actually isomorphic to the modular functor $\mathcal{M}$ which is represented by the base change $\breve{\Omega}$ of the formal completion of the Drinfeld's upper half plane. The observation is a key ingredient in obtaining the main results of this paper. We now briefly review $\mathcal{M}$ and $\breve{\Omega}$ in this subsection and next one.

A formal $\mathcal{O}_B$-module over an $\mathcal{O}_{F_0}$-algebra R is a formal $\mathcal{O}_{F_0}$-module X over R with an action $\iota_B : \mathcal{O}_B \to \mathrm{End}(X)$ extending the action of $\mathcal{O}_{F_0}$. X is called **special** if the action of $\mathcal{O}_E \subset \mathcal{O}_B$ makes $\mathrm{Lie}X$ a free $R \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E$-module of rank one. The R-module $\mathrm{Lie}(X)$ is $\mathbb{Z}/2\mathbb{Z}$-graded by the action of $\mathcal{O}_E$:

$$(\mathrm{Lie}X)_0 = \{x \in \mathrm{Lie}(X) \mid \iota_{\mathcal{O}_B}(a)m = am \text{ for all } a \in \mathcal{O}_E\},$$
$$(\mathrm{Lie}X)_1 = \{x \in \mathrm{Lie}(X) \mid \iota_{\mathcal{O}_B}(a)m = \sigma(a)m \text{ for all } a \in \mathcal{O}_E\}.$$

Then X is special if $\mathrm{Lie}(X)_i$ is a free R-module of rank one. Over $\bar{\boldsymbol{k}}$, there is a unique special formal $\mathcal{O}_B$-module up to $\mathcal{O}_B$-linear isogeny, which is $(\mathbb{X}, \iota_{\mathbb{X}})$ in Section 2.1.

Definition 2.4. *We define a moduli functor $\mathcal{M}$ on* $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}_0}}$ *that associates S with the set of isomorphism classes of triples (X, ι_B, ρ), where*

- *(X, ι_B) is a special formal $\mathcal{O}_B$-module of dimension 2 and relative height 4 over S.*
- *ρ is a $\mathcal{O}_B$-linear quasi-isogeny*

$$\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\,\bar{\boldsymbol{k}}} \bar{S}$$

of height 0. Here $\bar{S}$ is the special fiber of S.

The following is a result of Drinfeld that shows the automatic existence of polarizations on special formal $\mathcal{O}_B$-modules (see Proposition 1.1 of [KR14]).

Proposition 2.5 (Drinfeld)**.** *Let $\pi \in \mathcal{O}_B$ be a uniformizer such that $\pi^2 = \pi_0$, and consider the involution $b \to b^* = \pi b' \pi^{-1}$ of B, where $b \to b'$ denotes the main involution.*

(i) On $\mathbb{X}$ there exists a principal polarization $\lambda^0_{\mathbb{X}} : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ with associated Rosati involution $b \to b^*$. Furthermore, $\lambda^0_{\mathbb{X}}$ is unique up to a factor in $\mathcal{O}^\times_{F_0}$.

(ii) Fix $\lambda^0_{\mathbb{X}}$ as in (i). Let $(X,\iota,\rho)\in\mathcal{M}(S)$, where $S\in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}_0}}$. On X there exists a unique principal polarization $\lambda^0_X : X\xrightarrow{\sim} X^\vee$ making the following diagram commutative,

$$\begin{array}{ccc} X\times_S \bar{S} & \xrightarrow{\lambda^0_X} & X^\vee\times_S \bar{S} \\ \downarrow{\scriptstyle \rho} & & {\scriptstyle \rho^\vee}\uparrow \\ \mathbb{X}\times_{\mathrm{Spec}\,\bar{\boldsymbol{k}}} \bar{S} & \xrightarrow{\lambda^0_{\mathbb{X}}} & \mathbb{X}^\vee\times_{\mathrm{Spec}\,\bar{\boldsymbol{k}}} \bar{S} \end{array}$$

Theorem 2.6. *(*[KR14, Theorem 1.2]*) Assume that $p\neq 2$. The morphism of functors on $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}_0}}$ given by $(X,\iota_B,\rho)\to(X,\iota,\lambda^0_X,\rho)$ induces an isomorphism of formal schemes*

$$\eta_F : \mathcal{M}\xrightarrow{\sim}\mathcal{N}.$$

Here ι is the restriction of ι_B to $\mathcal{O}_F$ and λ^0_X is the principal polarization given by Proposition 2.5.

2.3. **Drinfeld upper half plane and its formal completion.** For convenience of the reader and to set up notation for the rest of the paper, we briefly review the well-known facts about the Drinfeld upper half place Ω, its formal completion $\widehat{\Omega}$, and its base change $\breve{\Omega}$ to $W=\mathcal{O}_{\breve{F}_0}$, following [BC91].

Recall that the Bruhat-Tits tree $\mathcal{B}(\mathrm{PGL}_2(F_0))$ consists of vertices and edges. The vertices are given by the homothety classes $[\Lambda]$ of $\mathcal{O}_{F_0}$-lattices in F_0^2, and the edges are given by pairs $([\Lambda],[\Lambda'])$ of the homothety classes such that $\pi_0\Lambda'\subset\Lambda\subset\Lambda'$ for suitable choices of lattices Λ and Λ' in their homothety classes. We then say Λ and Λ' are adjacent. We use $\mathbb{P}_{[\Lambda]}$ to denote the projective line over $\mathcal{O}_{F_0}$ associated to Λ depending on its homothety class. Let

$$\Omega_{[\Lambda]}=\mathbb{P}_{[\Lambda]}-\mathbb{P}_{[\Lambda]}(\boldsymbol{k})$$

be the projective line with the $q+1$ rational $\boldsymbol{k}$-points removed. When $([\Lambda],[\Lambda'])$ is an edge, $\Lambda\pmod{\pi_0}$ gives a $\boldsymbol{k}$-rational point in $\mathbb{P}_{[\Lambda']}$. We write $\mathbb{P}_{[\Lambda,\Lambda']}$ for the blow up of $\mathbb{P}_{[\Lambda']}$ at this point, which is isomorphic to the blow up of $\mathbb{P}_{[\Lambda]}$ at the rational point determined by Λ'. We write $\Omega_{[\Lambda,\Lambda']}$ for the complement of the nonsingular rational points of the special fiber of $\mathbb{P}_{[\Lambda,\Lambda']}$. There is an open embedding $\Omega_{[\Lambda]}\hookrightarrow\Omega_{[\Lambda,\Lambda']}$.

Define $\widehat{\Omega}_{[\Lambda]}$ and $\widehat{\Omega}_{[\Lambda,\Lambda']}$ to be the formal completion of $\Omega_{[\Lambda]}$ and $\Omega_{[\Lambda,\Lambda']}$ along their special fibers respectively. For two different edges with a common vertex $[\Lambda]$, we can glue $\widehat{\Omega}_{[\Lambda,\Lambda']}$ and $\widehat{\Omega}_{[\Lambda,\Lambda'']}$ along $\widehat{\Omega}_{[\Lambda]}$. This gives the formal scheme $\widehat{\Omega}$ by glueing over $\mathcal{B}(\mathrm{PGL}_2(F_0))$. The generic fiber of $\widehat{\Omega}$ is the Drinfeld p-adic half space $\Omega=\mathbb{P}^1(\mathbb{C}_p)-\mathbb{P}^1(F_0)$, where $\mathbb{C}_p$ is the completion of $\bar{F}_0$. Define $\breve{\Omega}=\widehat{\Omega}\times_{\mathrm{Spf}\,\mathcal{O}_{F_0}}\mathrm{Spf}\,\mathcal{O}_{\breve{F}_0}$. Similarly define $\breve{\Omega}_{[\Lambda]}$ and $\breve{\Omega}_{[\Lambda,\Lambda']}$. Drinfeld's well-known result (see [Dri76] and [BC91]) asserts that $\mathcal{M}$ is represented by $\breve{\Omega}$. The following proposition will be used in Section 4.

Proposition 2.7. *(Deligne functor) (*[BC91, Propositions 4.2, 4.4]*)*

(1) $\widehat{\Omega}_\Lambda$ represents the functor that associates an $\mathcal{O}_{F_0}$-algebra $R\in\mathrm{Nilp}_{\mathcal{O}_{F_0}}$ the collection of isomorphism classes of pairs $(\mathcal{L},\alpha)$, where $\mathcal{L}$ is a free R-module of rank 1 and $\alpha:\Lambda\to\mathcal{L}$ is a homomorphism of $\mathcal{O}_{F_0}$-modules satisfying the condition:

- *for all* $x \in \mathrm{Spec}\,(R/\pi_0 R)$, *the map* $\alpha(x) : \Lambda/\pi_0\Lambda \to \mathcal{L} \otimes_R k(x)$ *is injective.*

(2) $\widehat{\Omega}_{[\Lambda,\Lambda']}$ *represents the functor that associates an* $\mathcal{O}_{F_0}$*-algebra* $R \in \mathrm{Nilp}_{\mathcal{O}_{F_0}}$ *the collection of isomorphism classes of commutative diagrams:*

$$\begin{array}{ccccc} \pi_0\Lambda' & \hookrightarrow & \Lambda & \hookrightarrow & \Lambda' \\ \downarrow{\scriptstyle \alpha'/\pi_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ \mathcal{L}' & \xrightarrow{c'} & \mathcal{L} & \xrightarrow{c} & \mathcal{L}' \end{array}$$

where $\mathcal{L}$ *and* $\mathcal{L}'$ *are free* R*-modules of rank 1,* α *and* α' *are the homomorphisms of* $\mathcal{O}_{F_0}$*-modules,* c *and* c' *are homomorphisms of* R*-modules, satisfying the conditions: for all* $x \in \mathrm{Spec}\,(R/\pi_0 R)$*, one has*

- $ker(\alpha'(x)\colon \Lambda'/\pi_0\Lambda' \to \mathcal{L}' \otimes_R k(x)) \subset \Lambda/\pi_0\Lambda'$,
- $ker(\alpha(x)\colon \Lambda/\pi_0\Lambda \to \mathcal{L} \otimes_R k(x)) \subset \pi_0\Lambda'/\pi_0\Lambda$.

We now describe the explicit equations of $\widehat{\Omega}_{[\Lambda]}$ and $\widehat{\Omega}_{[\Lambda,\Lambda']}$. Let $\{e_1, e_2\}$ be a basis of Λ. Then we have

$$\widehat{\Omega}_{[\Lambda]} = (\mathbb{P}_{[\Lambda]} - \mathbb{P}_{[\Lambda]}(k))^\wedge = \mathrm{Spf}\,\mathcal{O}_{F_0}[T, (T^q - T)^{-1}]^\wedge, \tag{2.15}$$

where $T = \frac{X_0}{X_1}$ and X_i is the coordinate of $\mathbb{P}_{[\Lambda]}$ with respect to the basis $\{e_1, e_2\}$ and "hat" indicates completion along the special fiber.

Without loss of generality, we can assume Λ' has a basis $\{e_1, \pi_0^{-1}e_2\}$. Then we have

$$\widehat{\Omega}_{[\Lambda,\Lambda']} = \mathrm{Spf}\,\mathcal{O}_{F_0}[T_0, T_1, (T_0^{q-1} - 1)^{-1}, (T_1^{q-1} - 1)^{-1}]^\wedge/(T_0T_1 - \pi_0). \tag{2.16}$$

In this case, the open immersions $\widehat{\Omega}_{[\Lambda]} \hookrightarrow \widehat{\Omega}_{[\Lambda,\Lambda']}$ and $\widehat{\Omega}_{[\Lambda']} \hookrightarrow \widehat{\Omega}_{[\Lambda,\Lambda']}$ are induced by

$$T_0 \mapsto T,\ T_1 \mapsto \pi_0 \cdot T^{-1}, \tag{2.17}$$

and

$$T_0 \mapsto \pi_0 \cdot T^{-1},\ T_1 \mapsto T. \tag{2.18}$$

3. Special fiber of special cycle

In this section, we study the support of the special cycles. Essentially we only need the Pappas model $\mathcal{N}_{(1,1)}$ (see Proposition 3.6).

3.1. **Special fiber of $\mathcal{N}_{(1,1)}$.** Since $\mathcal{O}_F/(\pi) = \mathcal{O}_{F_0}/(\pi_0) = \boldsymbol{k}$, we have $\mathcal{N}_{(1,1)}/\bar{\boldsymbol{k}} \cong \mathcal{N}/\bar{\boldsymbol{k}}$. We briefly review the structure of $\mathcal{N}_{(1,1)}/\bar{\boldsymbol{k}}$ following [KR14].

Recall that $C = N^\tau, \tau = \pi V^{-1}$. For an $\mathcal{O}_F$-lattice Λ_F in C, set

$$\Lambda_F^\sharp = \{x \in C \mid (x, \Lambda_F) \subset \mathcal{O}_F\}.$$

Similarly, for a lattice $\mathcal{O}_{\breve{F}}$-lattice $M \subset N$, set

$$M^\sharp = \{x \in N \mid (x, M) \subset \mathcal{O}_{\breve{F}}\}.$$

Definition 3.1. *For an* $\mathcal{O}_F$*-lattice* Λ *in* C*,* Λ *is called a vertex lattice of type* t *if* $\pi\Lambda \subset \Lambda^\sharp \overset{t}{\subset} \Lambda$*, meaning* $[\Lambda : \Lambda^\sharp] = t$*. In our case* $t = 0$ *or* 2 *by Lemma 3.2 of* [RTW14].

Remark 3.2. *In the rest of the paper we will often use the subscript* 0 *or* 2 *to indicate the type of a vertex* $\mathcal{O}_F$*-lattice. For example* Λ_0 *often stands for a vertex lattice of type* 0.

Let $\mathcal{B} = \mathcal{B}(\mathrm{PU}(C))$ be the Bruhat-Tits tree of $\mathrm{PU}(C)$—the projective unitary group of C. Its vertices correspond to vertex $\mathcal{O}_F$-lattices Λ_2 of type 2, and its edges correspond to vertex $\mathcal{O}_F$-lattices Λ_0 of type 0. Two vertices Λ_2 and Λ_2' are connected by an edge Λ_0, or adjacent, if and only if $\Lambda_2 \cap \Lambda_2' = \Lambda_0$. The following lemma is easy to check (recall that C is isotropic) and is left to the reader.

Lemma 3.3. *Let Λ_0 be a vertex $\mathcal{O}_F$-lattice of type 0. There is an $\mathcal{O}_F$-basis $\{w_0, w_1\}$ of Λ_0 with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. There are exactly two vertex $\mathcal{O}_F$-lattices of type 2 containing Λ_0:*

$$\Lambda_2 = \mathcal{O}_F\pi^{-1}w_0 + \mathcal{O}_F w_1, \quad \text{and} \quad \Lambda_2' = \mathcal{O}_F w_0 + \mathcal{O}_F\pi^{-1}w_1.$$

There are $q+1$ adjacent (type 2) vertices of Λ_2 in $\mathcal{B}$, and they are

$$\Lambda_\infty = \mathcal{O}_F\pi^{-2}w_0 + \mathcal{O}_F\pi w_1, \quad \Lambda_k = \mathcal{O}_F w_0 + \mathcal{O}_F\pi^{-1}(k\pi^{-1}w_0 + w_1)$$

where k runs through the representatives of $\mathcal{O}_{F_0}/(\pi_0)$.

Recall the following results.

Proposition 3.4. *(*[RTW14, Proposition 2.2] *and* [KR14, Lemma 3.2]*) Let $\mathcal{L}(N)$ be the set of $\mathcal{O}_{\breve{F}}$-lattices*

$$\{M \subset N \mid \pi_0 M \subset VM \overset{2}{\subset} M, \quad M^\sharp = M, \quad \dim_{\bar{\boldsymbol{k}}} VM/(VM \cap \pi M) \leq 1, \}.$$

Then the map

$$\mathcal{N}_{(1,1)}(\bar{\boldsymbol{k}}) \to \mathcal{L}(N), \quad x = (X, \iota, \lambda, \rho) \mapsto M(x) := \rho(M(X)) \subset N$$

is a bijection. Moreover, for $M = M(x) \in \mathcal{L}(N)$, we have

(i) If M is τ-stable, then M is of the form $M = \Lambda_0 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$ for some vertex $\mathcal{O}_F$-lattice Λ_0 of type 0 in C,

(ii) If M is not τ-stable, then

$$M + \tau M = \Lambda_2 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$$

for some vertex $\mathcal{O}_F$-lattice Λ_2 of type 2 in C.

Proposition 3.5. *As in the introduction, we use $\mathbb{P}_{\bar{\Lambda}_2}$ to denote the projective line over $\bar{\boldsymbol{k}}$ associated to $\Lambda_2 \otimes_{\mathcal{O}_F} \bar{\boldsymbol{k}}$.*

(1) For every vertex $\mathcal{O}_F$-lattice Λ_2 of type 2, there is a closed immersion over $\bar{\boldsymbol{k}}$

$$i_{\bar{\Lambda}_2} : \mathbb{P}_{\bar{\Lambda}_2} \to \mathcal{N}_{(1,1)}/\bar{\boldsymbol{k}},$$

which is given on $\bar{\boldsymbol{k}}$-points as follows: it sends a line $l \subset \Lambda_2 \otimes_{\mathcal{O}_F} \bar{\boldsymbol{k}}$ to its preimage under the projection $\Lambda_2 \otimes \mathcal{O}_{\breve{F}} \to (\Lambda_2/\pi\Lambda_2) \otimes_{\mathcal{O}_F} \bar{\boldsymbol{k}}$. Moreover, $i_{\Lambda_2}(l)$ is τ-invariant if and only if l is $\boldsymbol{k}$-rational, i.e., l is the extension of a line in $\Lambda_2/\pi\Lambda_2$.

(2) Let $\mathbb{P}_{\bar{\Lambda}_2}$ denote also its image in $\mathcal{N}_{(1,1)}/\bar{\boldsymbol{k}}$ under i_{Λ_2}. Then

$$\mathcal{N}_{(1,1)}/\bar{\boldsymbol{k}} = \bigcup_{\Lambda_2} \mathbb{P}_{\bar{\Lambda}_2},$$

where the union is over all vertex lattices of type 2. Two such lines $\mathbb{P}_{\bar{\Lambda}_2}$ and $\mathbb{P}_{\bar{\Lambda}_2'}$ intersect if and only if Λ_2 and Λ_2' are connected by an edge, i.e., $\Lambda_2 \cap \Lambda_2' = \Lambda_0$ is a vertex $\mathcal{O}_F$-lattice of type 0. In such a case, the two projective lines intersect at exactly one $\boldsymbol{k}$-rational point, denoted by pt_{Λ_0}. Such a point is called a superspecial point in $\mathcal{N}_{(1,1)}$.

(3) The singular locus Sing *of $\mathcal{N}_{(1,1)}$ consists of all the superspecial pt_{Λ_0} with Λ_0 running through all vertex $\mathcal{O}_F$-lattices of type 0.*

Proof. For the proof of (1), see [KR14, Lemma 3.3, Proposition 3.4]. For (2), if a point x lies in the intersection between $\mathbb{P}_{\bar{\Lambda}_2}$ and $\mathbb{P}_{\bar{\Lambda}_2'}$, then $M(x) \subset \Lambda_2 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}} \cap \Lambda_2' \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$ by the description of $i_{\bar{\Lambda}_2}$ in (1). Together with the fact $M(x) = M(x)^\sharp$, this implies $\Lambda_2 \cap \Lambda_2' = \Lambda_0$, where Λ_0 is some vertex lattice of type 0. The converse is straightforward. Part (3) is immediate from part (2) and the description of the singular locus of $\mathcal{N}_{(1,1)}$ by local model (see for example [Pap00, Theorem 4.5]). □

Recall that the blow up $\Phi : \mathcal{N}^{\mathrm{Kra}} \to \mathcal{N}_{(1,1)}$ is an isomorphism outside the singular locus Sing.

Proposition 3.6. *(1) For a type* 2 *lattice* Λ_2*, the Zariski closure of* $\Phi^{-1}(\mathbb{P}_{\bar{\Lambda}_2} \setminus Sing)$ *is a projective line over* $\bar{\boldsymbol{k}}$ *which we still denote by* $\mathbb{P}_{\bar{\Lambda}_2}$*.*

(2) For a vertex lattice Λ_0 *of type* 0*,* $\Phi^{-1}(pt_{\Lambda_0})$ *is a projective line over* $\bar{\boldsymbol{k}}$ *which we denote by* Exc_{Λ_0}*.*

(3) On the special fiber of $\mathcal{N}^{\mathrm{Kra}}$*, two different lines* $\mathbb{P}_{\bar{\Lambda}_2}$ *and* $\mathbb{P}_{\bar{\Lambda}_2'}$ *never intersect, and two different lines* Exc_{Λ_0} *and* $\mathrm{Exc}_{\Lambda_0'}$ *never intersect.* $\mathbb{P}_{\bar{\Lambda}_2}$ *and* Exc_{Λ_0} *intersect at one point if* $\Lambda_0 \subset \Lambda_2$*, otherwise they do not intersect.*

Proof. The above facts will be clear after the discussion in Section 4.3. □

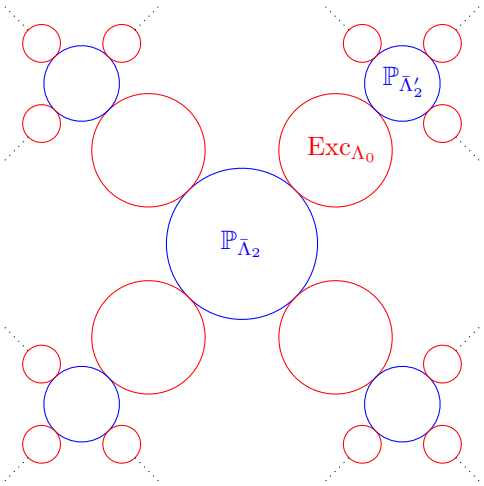

FIGURE 1. Special fiber of $\mathcal{N}^{\mathrm{Kra}}$ when $q = 3$.

A superspecial point pt_{Λ_0} belongs to $\mathcal{Z}(\mathbf{x})$ if and only if $\mathrm{Exc}_{\Lambda_0} \subseteq \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$. So in the rest of the section, we only need to study $\mathcal{Z}(\mathbf{x})$.

3.2. **Bruhat-Tits trees of special cycles: rank** 1 **case.** For an $\mathbf{x} \in \mathbb{V}$, let $\mathcal{T}(\mathbf{x})$ be as in Theorem 1.2. Alternatively, in terms of $\mathcal{Z}(\mathbf{x})$ (instead of $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$), we have

$$\mathcal{T}(\mathbf{x}) = \{\Lambda_2 \text{ is of type } 2 \mid \mathbb{P}_{\bar{\Lambda}_2} \cap \mathcal{Z}(\mathbf{x}) \neq \emptyset\} \cup \{\Lambda_0 \text{ is of type } 0 \mid pt_{\Lambda_0} \in \mathcal{Z}(\mathbf{x})\}.$$

We will also view a lattice in $\mathcal{T}(\mathbf{x})$ as a vertex or an edge in $\mathcal{B}$ depending on whether it is of type 2 or 0. We will see that $\mathcal{T}(\mathbf{x})$ is a tree by Corollaries 3.12 and 3.14.

Lemma 3.7. *Let* $\mathbf{x} \in \mathbb{V}$. *Then*

$$\mathcal{Z}(\mathbf{x})(\bar{\boldsymbol{k}}) \cap \mathbb{P}_{\bar{\Lambda}_2}(\bar{\boldsymbol{k}}) = \begin{cases} \mathbb{P}_{\bar{\Lambda}_2}(\bar{\boldsymbol{k}}), & \text{if } \mathbf{x}(e) \in \pi\Lambda_2, \\ \text{a single point}, & \text{if } \mathbf{x}(e) \in \Lambda_2 \setminus \pi\Lambda_2, \\ \emptyset, & \text{if } \mathbf{x}(e) \notin \Lambda_2. \end{cases}$$

In particular, $\Lambda_2 \in \mathcal{T}(\mathbf{x})$ *if and only if* $\mathbf{x}(e) \in \Lambda_2$.

Proof. Let $x = (X, \iota, \lambda, \rho) \in \mathcal{N}_{(1,1)}(\bar{\boldsymbol{k}})$, let $M(x) = \rho(M(X)) \subset N$ as in Proposition 3.4. Then

$$\begin{aligned} x \in \mathcal{Z}(\mathbf{x})(\bar{\boldsymbol{k}}) &\iff \mathbf{x}(M(\mathbb{Y})) \subset M(x) \\ &\iff \mathbf{x}(e) \in M(x),\ \mathbf{x}(f) \in M(x) \\ &\iff \mathbf{x}(e) \in M(x) \end{aligned}$$

since $\mathbf{x}(e) \in M(x)$ implies $\mathbf{x}(f) = \mathbf{x}(Ve) \in VM(x) \subset M(x)$.
On the other hand,

$$x \in \mathbb{P}_{\bar{\Lambda}_2}(\bar{\boldsymbol{k}}) \iff \pi\Lambda_2 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}} \subset M(x) \subset \Lambda_2 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}.$$

If $\mathbf{x}(e) \in \pi\Lambda_2$, then $\mathbf{x}(e) \in M(x)$ for any x such that $\pi\Lambda_2 \subset M(x) \subset \Lambda_2$. This implies $\mathbb{P}_{\bar{\Lambda}_2}(\bar{\boldsymbol{k}}) \subset \mathcal{Z}(\mathbf{x})(\bar{\boldsymbol{k}})$.

If $\mathbf{x}(e) \in \Lambda_2 \setminus \pi\Lambda_2$, then the image of $\mathbf{x}(e)$ is contained in exactly one line in $\bar{\Lambda}_{F,2}$ and thus gives a single point in $\mathbb{P}^1_{\bar{\Lambda}_2}(\bar{\boldsymbol{k}})$.

Finally, if $\mathbf{x}(e) \notin \Lambda_2$, it can not lie in any sub-lattice of $\Lambda_2 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$.

□

From now on, fix $0 \neq \mathbf{x} \in \mathbb{V}$, and let $b = \mathbf{x}(e),\ q(b) = (b,b) = -\delta^2 h(\mathbf{x},\mathbf{x})$.

Lemma 3.8. *Assume* $q(b) \neq 0$. *Then there exist a unique vertex* $\mathcal{O}_F$*-lattice* Λ_b *of type* 0 *such that* $\pi^{-\operatorname{ord}_{\pi_0}(q(b))} b \in \Lambda_b \setminus \pi\Lambda_b$.

Proof. Replacing b by $\pi^{-\operatorname{ord}_{\pi_0}(q(b))} b$ if necessary, we may assume $\operatorname{ord}_{\pi_0}(q(b)) = 0$, i.e., $(b,b) = u_0 \in \mathcal{O}_{F_0}^\times$. Write $(Fb)^\perp = Fc$, then $C = Fb \oplus Fc$. Since $q(b)$ is a unit, by [Jac62, Proposition 4.2], every vertex lattice of type 0 containing b has the orthogonal decomposition

$$\Lambda = \mathcal{O}_F b + \Lambda_1, \quad \Lambda_1 = \Lambda \cap Fc = \mathcal{O}_F b'$$

for some $b' = rc$. Λ is a vertex lattice of type 0 if and only if $q(b') = r\bar{r}q(c)$ is a unit. Such an r exists and is unique up to a unit in $\mathcal{O}_F^\times$. So the vertex $\mathcal{O}_F$-lattice Λ_b of type 0 such that $b \in \Lambda_b \setminus \pi\Lambda_b$ exists and is unique.

□

Definition 3.9. *Assume that each edge in* $\mathcal{B}$ *has length* 1. *Let* Λ *and* Λ' *be two vertex* $\mathcal{O}_F$*-lattices. The distance* $d(\Lambda, \Lambda')$ *is defined to be*

1. *the distance between the two vertices in* $\mathcal{B}$ *if both are of type* 2,
2. *the distance between the vertex* Λ *and the midpoint of the edge* Λ' *in* $\mathcal{B}$ *if* Λ *is of type* 2 *and* Λ' *is of type* 0,
3. *the distance between the midpoints of the edges* Λ *and* Λ' *in* $\mathcal{B}$ *if both are of type* 0.

For a vertex lattice Λ, we define

$$n(b, \Lambda) := \max\{n \in \mathbb{Z} \mid \pi^{-n} b \in \Lambda\}. \tag{3.1}$$

It is easy to check that for $\Lambda_2 \cap \Lambda_2' = \Lambda_0$,

$$n(b, \Lambda_0) = \min(n(b, \Lambda_2), n(b, \Lambda_2')). \tag{3.2}$$

We have the following reformulation of Lemma 3.7.

Lemma 3.10. *If $b = \mathbf{x}(e)$, $n(b, \Lambda) \geq 0$ if and only if $\Lambda \in \mathcal{T}(\mathbf{x})$.*

Lemma 3.11. *Assume $q(b) \neq 0$.*

(1) If Λ is a vertex $\mathcal{O}_F$-lattice of type 2, then

$$n(b, \Lambda) = \mathrm{ord}_{\pi_0}(q(b)) - d(\Lambda, \Lambda_b) + \frac{1}{2}.$$

(2) If Λ is a vertex $\mathcal{O}_F$-lattice of type 0, then

$$n(b, \Lambda) = \mathrm{ord}_{\pi_0}(q(b)) - d(\Lambda, \Lambda_b).$$

Proof. Claim (2) follows from Claim (1) and (3.2). Now we prove Claim (1). Without lost of generality we can assume $\mathrm{ord}_{\pi_0}(q(b)) = 0$. We prove the lemma by induction on $d(\Lambda, \Lambda_b)$. Let us treat the case $d(\Lambda, \Lambda_b) = \frac{1}{2}$ first, i.e. $\Lambda_b \subset \Lambda$. We have by Lemma 3.3

$$\Lambda_b = \mathcal{O}_F w_0 + \mathcal{O}_F w_1, \quad \Lambda = \mathcal{O}_F \pi^{-1} w_0 + \mathcal{O}_F w_1.$$

Write $b = xw_0 + yw_1$ with $q(b) = x\bar{y} + y\bar{x} \in \mathcal{O}_{F_0}^\times$. Then $x, y \in \mathcal{O}_F^\times$, and $b \in \Lambda \setminus \pi\Lambda$. Hence $n(b, \Lambda) = 0$.

Now we assume the assertion holds for all Λ such that $\frac{1}{2} \leq d(\Lambda, \Lambda_b) \leq d + \frac{1}{2}$. For a Λ such that $d(\Lambda, \Lambda_b) = d + \frac{1}{2} \neq \frac{1}{2}$, which satisfies the formula $n := n(b, \Lambda) = -d$, it suffices to show that all its adjacent vertices also satisfy the formula in the lemma.

Choose a basis $\{v_0, v_1\} = \{\pi^{-1}w_0, w_1\}$ of Λ with Gram matrix $\pi^{-1}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. By Lemma 3.3, the $q+1$ neighbors of Λ in the Bruhat-Tits tree are

$$\Lambda_\infty = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}v_0, \pi v_1\}, \quad \Lambda_k = \mathrm{span}_{\mathcal{O}_F}\{\pi v_0, \pi^{-1}(kv_0 + v_1)\},$$

where k runs through the representatives of $\mathcal{O}_{F_0}/(\pi_0)$. We want to show the claim holds for all the neighbors. One of the neighbors will be closer to Λ_b than other neighbors. Without loss of generality, we can assume Λ_∞ is closer to Λ_b than all the other Λ_k. By induction, we know

$$n(b, \Lambda_\infty) = n + 1.$$

By the definition of $n(b, \Lambda)$, we can write

$$b = \pi^n(a_0v_0 + a_1v_1) = \pi^{n+1}(\pi^{-1}a_0v_0 + \pi^{-1}a_1v_1),$$

where $\mathrm{ord}_\pi(a_i) \geq 0$ and $\min\{\mathrm{ord}_\pi(a_0), \mathrm{ord}_\pi(a_1)\} = 0$. Since $n(b, \Lambda_\infty) = n + 1$, $\pi^{-1}a_0v_0 + \pi^{-1}a_1v_1 \in \Lambda_\infty$ implies $\mathrm{ord}_\pi(a_1) \geq 2$. Hence $\mathrm{ord}_\pi(a_0) = 0$. Claim:

$$\mathrm{ord}_\pi(a_1) \geq 2 \text{ and } \mathrm{ord}_\pi(a_0) = 0 \implies a_0v_0 + a_1v_1 \notin \Lambda_k.$$

To prove this, assume $a_0v_0 + a_1v_1 \in \Lambda_k$ which implies that we can find $a_0', a_1' \in \mathcal{O}_F$ such that

$$a_0v_0 + a_1v_1 = a_0'\pi v_0 + a_1'\pi^{-1}(kv_0 + v_1) = (a_0'\pi + a_1'\pi^{-1}k)v_0 + a_1'\pi^{-1}v_1.$$

As a result,

$$\mathrm{ord}_\pi(a_0'\pi + a_1'\pi^{-1}k) = \mathrm{ord}_\pi(a_0'\pi + a_1k) \geq \min\{\mathrm{ord}_\pi(a_0'\pi), \mathrm{ord}_\pi(a_1)\} \geq 1.$$

But then $a_0'\pi + a_1'\pi^{-1}k$ can not equal to a_0 since we know $\mathrm{ord}_\pi(a_0) = 0$, which is a contradiction.

The claim gives us that $n(b,\Lambda_k)\leq n-1$. we can easily check $\pi(a_0v_0+a_1v_1)\in\Lambda_k$. Therefore $n(b,\Lambda_k)=n-1$ as claimed. □

Combining Lemma 3.10 with Lemma 3.11, we have the following:

Corollary 3.12. *Assume $q(b)\neq 0$, $\mathcal{T}(\mathbf{x})$ is a ball centered at the midpoint of Λ_b with radius $\mathrm{ord}_{\pi_0}(q(b))+\frac{1}{2}$. In particular, $\mathcal{T}(\mathbf{x})$ is empty if and only $q(b)\notin\mathcal{O}_{F_0}$.*

Lemma 3.13. *Assume that $b\neq 0$ and $q(b)=0$. Let Λ be a vertex lattice of type 2 and assume $n(b,\Lambda)=n$. Then there exists a unique type 2 adjacent lattice Λ_+ of Λ such that $n(b,\Lambda_+)=n+1$. For any other type 2 adjacent vertex lattice Λ' of Λ, we have $n(b,\Lambda')=n-1$.*

Proof. By scaling b by a power of π we can assume that $n=0$. By Lemma 3.3, the lattice $\Lambda\approx\mathcal{H}$ with Gram matrix $\left(\begin{smallmatrix}0&\pi^{-1}\\-\pi^{-1}&0\end{smallmatrix}\right)$ (as in the introduction). So there is some $b_1\in\Lambda$ such that $(b,b_1)=\pi^{-1}$. Define $b'=b_1+\pi ab$ where $a=\frac{1}{2}(b_1,b_1)$. Since $(b_1,b_1)\in\pi^{-1}\mathcal{O}_{F_0}$, then $\{b,b'\}$ is a basis of Λ with Gram matrix $\pi^{-1}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. The adjacent type 2 vertex lattices of Λ are

$$\Lambda_\infty=\mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}b,\pi b'\},\ \Lambda_k=\mathrm{span}_{\mathcal{O}_F}\{\pi b,\pi^{-1}(kb+b')\},$$

where k runs through the representatives of $\mathcal{O}_{F_0}/(\pi_0)$. Then it is obvious that $n(b,\Lambda_\infty)=1$ and $n(b,\Lambda_k)=-1$ for any k. The lemma follows. □

Corollary 3.14. *Assume that $b\neq 0$ and $q(b)=0$. Then $\mathcal{T}(\mathbf{x})$ is a cone with the boundary consisting of vertex lattices Λ of type 2 with $n(b,\Lambda)=0$. Starting with such a vertex Λ, there is a unique (half) geodesic such that the number $n(b,\Lambda)$ increases along the geodesic. We call such a geodesic an ascending geodesic starting with Λ. Any two ascending geodesics coincide after finitely many steps. An ascending geodesic can be thought of as an 'axis' of the cone $\mathcal{T}(\mathbf{x})$.*

3.3. **Bruhat-Tits trees of special cycles: rank 2 case.** The following is an analogue of Lemma 2.11 from [San17].

Lemma 3.15. *Let $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{V}$, $b_i=\mathbf{x_i}(e)$, $n_i=\mathrm{ord}_{\pi_0}(q(b_i))$ and assume $q(b_i)\neq 0$, for $i\in\{1,2\}$. Assume $n_1\leq n_2$. Suppose that $\mathcal{T}(\mathbf{x}_1)\cap\mathcal{T}(\mathbf{x}_2)\neq\emptyset$. Let Λ_{b_1,b_2} be the vertex lattice such that*

$$d(\Lambda_{b_1},\Lambda_{b_1,b_2})=n_1+\frac{1}{2}-r\ \text{ and }\ d(\Lambda_{b_1},\Lambda_{b_1,b_2})+d(\Lambda_{b_2},\Lambda_{b_1,b_2})=d(\Lambda_{b_1},\Lambda_{b_2}).$$

Here

$$r=\min(\frac{n_1+n_2+1-d(\Lambda_b,\Lambda_{b'})}{2},n_1+\frac{1}{2}).$$

Then $\mathcal{T}(\mathbf{x}_1)\cap\mathcal{T}(\mathbf{x}_2)\subset\mathcal{B}$ is the ball of radius r centered at either the vertex Λ_{b_1,b_2} or the midpoint of the edge Λ_{b_1,b_2}, depending on whether r is an integer or not.

Proof. It follows from Corollary 3.12 and the fact that $\mathcal{B}$ is a tree. □

Lemma 3.16. *For $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{V}$, let $b_1=\mathbf{x}_1(e),b_2=\mathbf{x}_2(e)$. Assume*

$$T=\begin{pmatrix}(b_1,b_1)&(b_1,b_2)\\(b_2,b_1)&(b_2,b_2)\end{pmatrix}=\begin{pmatrix}0&\pi^n\\(-\pi)^n&0\end{pmatrix},\ n\geq 0.$$

Then $\mathcal{T}(\mathbf{x}_1)\cap\mathcal{T}(\mathbf{x}_2)$ is a ball with center $\Lambda=\mathrm{span}_{\mathcal{O}_F}\{\pi^{-r}b_1,\pi^{-r}b_2\}$ and radius r, where $r=[\frac{n+1}{2}]$ is the integral part of $\frac{n+1}{2}$.

Proof. We assume that $n = 2r-1$ ($r \geq 1$) is odd, the case n is even can be proved similarly. Write $b_i = \pi^r v_i$ for $i = 1, 2$. Define

$$\Lambda_{k,0} = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-k}v_1, \pi^{k+1}v_2\}, \qquad \Lambda_{k,2} = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-k}v_1, \pi^{k}v_2\},\ k \in \mathbb{Z}.$$

Then $\Lambda_{k,0}$ are vertex lattices of type 0 and $\Lambda_{k,2}$ are vertex lattices of type 2. It is easy to check that

$$\begin{aligned} n(b_1, \Lambda_{k,0}) &= r + k, & n(b_1, \Lambda_{k,2}) &= r + k, \\ n(b_2, \Lambda_{k,0}) &= r - k - 1, & n(b_2, \Lambda_{k,2}) &= r - k. \end{aligned}$$

So $\{\Lambda_{k,0}, \Lambda_{k,2} \mid k \geq -r\}$ is the ascending geodesic associated to b_1 starting at $\Lambda_{-r,2}$ and $\{\Lambda_{k,0} \mid k \leq r-1\} \cup \{\Lambda_{k,2} \mid k \leq r\}$ is (the inverse of) the ascending geodesic associated to b_2 ending at $\Lambda_{r,2}$. Let $\mathcal{C}$ be the intersection of the above two half geodesics, namely, the line segment joining $\Lambda_{-r,2}$ and $\Lambda_{r,2}$. By Lemma 3.13, both $n(b_1, \Lambda)$ and $n(b_2, \Lambda)$ are decreasing along a half geodesic starting from any vertex on $\mathcal{C}$ other than $\mathcal{C}$ itself. Combine the above facts, it is easy to see that $\mathcal{T}(\mathbf{x}_1) \cap \mathcal{T}(\mathbf{x}_2)$ is a ball centered at $\Lambda_{0,2}$ with radius r. □

The following lemma is an analogue of [San17, Lemma 2.15]. Recall that wo Hermitian matrices $T_1, T_2 \in \mathrm{Herm}_2(\mathcal{O}_F)$ are said to be equivalent, denoted by $T_1 \approx T_2$ if there is a non-singular matrix $g \in \mathrm{GL}_2(\mathcal{O}_F)$ such that $g^t T_1 \bar{g} = T_2$.

Lemma 3.17. *Let*

$$T = \begin{pmatrix} (b_1, b_1) & (b_1, b_2) \\ (b_2, b_1) & (b_2, b_2) \end{pmatrix}$$

where $b_i = \mathbf{x}_i(e)$ for $\mathbf{x}_i \in \mathbb{V}$. Assume $q(b_i) \neq 0$, $i \in \{1, 2\}$, and T is non-singular. Let $n_i = \mathrm{ord}_{\pi_0}(q(b_i))$ and assume $n_1 \leq n_2$. Set $d = d(\Lambda_{b_1}, \Lambda_{b_2})$, then

(a) If $\mathcal{T}(\mathbf{x_1}) \subset \mathcal{T}(\mathbf{x_2})$ and $\Lambda_{b_1} \neq \Lambda_{b_2}$, then $T \approx \begin{pmatrix} u_1(-\pi_0)^\alpha & 0 \\ 0 & u_2(-\pi_0)^\beta \end{pmatrix}$ where $u_1, u_2 \in \mathcal{O}_{F_0}^\times$ and $-u_1u_2 \in \mathrm{Nm}(F^\times)$ with

$$\alpha = n_1, \quad \beta = n_2 - d.$$

(b) If $\mathcal{T}(\mathbf{x_1}) \not\subset \mathcal{T}(\mathbf{x_2})$, and $\mathcal{T}(\mathbf{x_1}) \cap \mathcal{T}(\mathbf{x_2}) \neq \emptyset$, then $T \approx \begin{pmatrix} 0 & \pi^\alpha \\ (-\pi)^\alpha & 0 \end{pmatrix}$ with

$$\alpha = n_1 + n_2 - d.$$

(c) If $\Lambda_{b_1} = \Lambda_{b_2}$, then $T \approx \begin{pmatrix} u_1(-\pi_0)^\alpha & 0 \\ 0 & u_2(-\pi_0)^\beta \end{pmatrix}$. Here u_1, u_2 satisfy the same conditions as in (a) and

$$\alpha = n_1, \quad \beta = n_2 + 2\,\mathrm{ord}_\pi((c_2, c_1')),$$

where $c_i = \pi^{-n_i} b_i$, and $c_1' \in \Lambda_{b_1} \setminus \pi\Lambda_{b_1}$ such that $(c_1, c_1') = 0$.

Proof. If $n_1 < 0$, the lemma holds trivially. So we assume $n_1 \geq 0$.
We treat the case $\Lambda_{b_1} \neq \Lambda_{b_2}$ first. Assume $\Lambda \cap \Lambda' = \Lambda_{b_1}$ where Λ and Λ' are vertex lattices of type 2. We pick a basis $\{v_0, v_1\}$ of Λ such that

$$\begin{pmatrix} (v_0, v_0) & (v_0, v_1) \\ (v_1, v_0) & (v_1, v_1) \end{pmatrix} = \begin{pmatrix} 0 & \pi^{-1} \\ -\pi^{-1} & 0 \end{pmatrix}.$$

Without loss of generality, we can assume

$$\Lambda' = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}v_0, \pi v_1\}, \text{ hence } \Lambda_{b_1} = \mathrm{span}_{\mathcal{O}_F}\{v_0, \pi v_1\}.$$

By the symmetry of $\mathcal{B}$, we can also assume

$$\Lambda_{b_2} = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-d}v_0, \pi^{d+1}v_1\} \text{ where } d = d(\Lambda_{b_1}, \Lambda_{b_2}).$$

By Lemma 3.11, we can write

$$b_1 = \pi^{n_1}(\alpha_0 v_0 + \alpha_1(\pi v_1)),\ b_2 = \pi^{n_2}(\alpha_0'(\pi^{-d}v_0) + \alpha_1'(\pi^{d+1}v_1)),\ \alpha_i, \alpha_i' \in \mathcal{O}_F.$$

Note that $(b_1, b_1) = (-\pi_0)^{n_1}(-\alpha_0\bar{\alpha}_1 - \alpha_1\bar{\alpha}_0)$. Since $\mathrm{ord}_\pi(-\alpha_0\bar{\alpha}_1 - \alpha_1\bar{\alpha}_0) = 0$, we conclude that $\mathrm{ord}_\pi(\alpha_0) = \mathrm{ord}_\pi(\alpha_1) = 0$. Similarly, $\mathrm{ord}_\pi(\alpha_0') = \mathrm{ord}_\pi(\alpha_1') = 0$. Then a short computation shows:

$$T = \begin{pmatrix}(b_1,b_1) & (b_1,b_2)\\ (b_2,b_1) & (b_2,b_2)\end{pmatrix} = \begin{pmatrix}\pi^{2n_1}\cdot(\text{unit}) & \pi^{n_1+n_2-d}\cdot(\text{unit})\\ (-\pi)^{n_1+n_2-d}\cdot(\text{unit}) & \pi^{2n_2}\cdot(\text{unit})\end{pmatrix}.$$

Note that

$$\mathcal{T}(\mathbf{x}_1) \subset \mathcal{T}(\mathbf{x}_2) \iff n_1 \le n_2 - d \iff n_1 \le \frac{n_1+n_2-d}{2}.$$

Now for the proof of (a), observe that if $\mathcal{T}(\mathbf{x}_1) \subset \mathcal{T}(\mathbf{x}_2)$ then $2n_1 = \min\{\mathrm{ord}_\pi(T_{ij})\}$. This implies

$$T \approx \begin{pmatrix}u_1(-\pi_0)^{n_1} & 0\\ 0 & u_2(-\pi_0)^{n_2-d}\end{pmatrix}.$$

For (b), the assumption implies that $n_1 > n_2 - d$. Then $n_1 + n_2 - d = \min\{\mathrm{ord}_\pi(T_{ij})\}$, which implies

$$T \approx \begin{pmatrix}0 & \pi^{n_1+n_2-d}\\ (-\pi)^{n_1+n_2-d} & 0\end{pmatrix}.$$

The proof for the case $\Lambda_{b_1} = \Lambda_{b_2}$ is essentially the same as the proof in [San17, Lemma 2.15], and is left to the reader. □

Corollary 3.18. *Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{V}^2$ and $b_i = \mathbf{x}_i(e)$ for $i = 1, 2$. Assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. Then the naive intersection $\mathcal{Z}(\mathbf{x}) = \mathcal{Z}(\mathbf{x}_1) \cap \mathcal{Z}(\mathbf{x}_2)$ (resp. $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$) is supported in finitely many irreducible components of the special fiber of $\mathcal{N}_{(1,1)}$ (resp. $\mathcal{N}^{\mathrm{Kra}}$).*

Proof. This follows easily from Lemma 3.15 and Lemma 3.16. □

4. Decomposition of special divisors in the Krämer model

This section is dedicated to the proof of the following theorem, which is slight refinement of Theorem 1.2 with $\mathbf{x}$ replaced by its image $b = \mathbf{x}(e)$ in C.

Theorem 4.1. *For a vector $\mathbf{x} \in \mathbb{V} \setminus \{0\}$, set $b = \mathbf{x}(e)$ as before. Then $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = 0$ unless $q(b) \in \mathcal{O}_{F_0}$.*

(1) If $q(b) \neq 0$ and $q(b) \in \mathcal{O}_{F_0}$, then we have the following decomposition of special divisor

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b, \Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b, \Lambda_0) + 1)\mathrm{Exc}_{\Lambda_0} + \mathcal{Z}^h(\mathbf{x}), \tag{4.1}$$

where the two summations are over type 2 and type 0 vertex lattices respectively and $\mathcal{Z}^h(\mathbf{x}) \cong \mathrm{Spf}\mathcal{O}_F$ is a horizontal divisor meeting the special fiber at Exc_{Λ_b}. Recall that Λ_b is the unique vertex lattice of type 0 such that $\pi^{-\mathrm{ord}_{\pi_0}(q(b))}b \in \Lambda_b \setminus \pi\Lambda_b$.

(2) If $q(b)=0$, then we have the following decomposition of special divisor

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})=\sum_{\Lambda_2\in\mathcal{T}(\mathbf{x})} n(b,\Lambda_2)\mathbb{P}_{\check{\Lambda}_2}+\sum_{\Lambda_0\in\mathcal{T}(\mathbf{x})}(n(b,\Lambda_0)+1)\mathrm{Exc}_{\Lambda_0}, \tag{4.2}$$

where the two summations are over type 2 and type 0 vertex lattices respectively.

4.1. **The Horizontal Component.** We say a formal scheme over $\operatorname{Spf}\mathcal{O}_{\check{F}}$ is horizontal if π is not locally nilpotent in its structure sheaf. We say a divisor in $\mathcal{N}^{\mathrm{Kra}}$ is irreducible if it is connected and is an irreducible Cartier divisor in every local ring of $\mathcal{N}^{\mathrm{Kra}}$. Let $\mathcal{Y}_s$ be the quasi-canonical lifting of $\mathbb{Y}$ of level s over $\mathcal{O}_{\check{F}_0}$ considered by [Gro86] (with $\mathcal{O}_s=\mathcal{O}_{\check{F}_0}+\pi^s\mathcal{O}_{\check{F}}$ action). In particular, $\mathcal{Y}=\mathcal{Y}_0$ is the canonical lifting. We show that all horizontal cycles in $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ comes from canonical lifting.

Theorem 4.2. *Let $\mathcal{Z}$ be an irreducible horizontal component of $\mathcal{Z}(\mathbf{x})$, then $\mathcal{Z}\cong\operatorname{Spf}\mathcal{O}_{\check{F}}$. Moreover $\mathcal{Z}$ intersects with the special fiber of $\mathcal{N}_{(1,1)}$ at a superspecial point.*

Proof. By assumption $\mathcal{Z}=\operatorname{Spf}R$ where R is a finite extension of $\mathcal{O}_{\check{F}_0}$. Let X be the strict formal $\mathcal{O}_{F_0}$-module over $\mathcal{Z}$ that is the pullback from the universal strict formal $\mathcal{O}_{F_0}$-module over $\mathcal{N}_{(1,1)}$ which by Theorem 2.6 carries an $\mathcal{O}_B$ action ι_B. By the definition of $\mathcal{Z}(\mathbf{x})$, $\mathbf{x}$ lifts to a homomorphism $\mathbf{x}:\mathcal{Y}\to X$. We now define a morphism $\phi:\mathcal{Y}\oplus\mathcal{Y}\to X$ by

$$\phi(p_1,p_2)=x(p_1)+\iota_B(\delta)\circ x(p_2),$$

where $p_1,p_2\in\mathcal{Y}(S)$ for an R-algebra S. We give $\mathcal{Y}\oplus\mathcal{Y}$ an $\mathcal{O}_B$ action $\iota:\mathcal{O}_B\to\mathrm{End}_{\mathcal{O}_{F_0}}(\mathcal{Y}\oplus\mathcal{Y})\cong\mathrm{M}_{2\times2}(\mathcal{O}_F)$ defined by

$$\iota(\pi)=\begin{pmatrix}\pi&0\\0&-\pi\end{pmatrix},\ \iota(\delta)=\begin{pmatrix}0&1\\\delta^2&0\end{pmatrix}. \tag{4.3}$$

Then ϕ becomes an $\mathcal{O}_B$-linear homomorphism.

We claim that ϕ is an isogeny. By [Tat67, Proposition 1], the category of connected $\mathcal{O}_F$ modules and the category of divisible commutative formal Lie groups with $\mathcal{O}_F$ action are equivalent. Let $A=R[[T_1,T_2]]$ considered as the structure ring of $\mathcal{Y}\oplus\mathcal{Y}$ and B be the structure ring of X. It suffices to show that the induced map $\phi^\sharp:B\to A$ is injective. If $I=\mathrm{Ker}(\phi^\sharp)$ is nontrivial, then ϕ factors through the sub formal group scheme $X'=\operatorname{Spf}B/I$ of X. Since A has characteristic zero, so does B/I. By base change to the fraction field $F(R)$ of R we can apply a theorem of Cariter [Car62] and conclude that $X'\otimes_R F(R)$ is a one dimensional formal group law over $F(R)$. But by assumption, $X'\otimes_R F(R)$ has an $\mathcal{O}_B$ action which is impossible. Hence $I=\{0\}$. This proves the claim.

Our goal is to show that X is in fact isomorphic to $\mathcal{Y}\oplus\mathcal{Y}$. We define the Tate module for a $\mathcal{O}_F$-module G by

$$T(G)=\varprojlim_n G[\pi^n].$$

We can identify $T=T(\mathcal{Y}\oplus\mathcal{Y})$ with $(\mathcal{O}_F)^2$ and further with $\mathcal{O}_B$ using the $\mathcal{O}_B$ action on $T(\mathcal{Y}\oplus\mathcal{Y})$. To be more specific the element $t=\begin{pmatrix}1\\0\end{pmatrix}\in T$ is a generator of T under the $\mathcal{O}_B$ action given by (4.3). Then the set of $\mathcal{O}_B$-linear isogenies

$\phi : \mathcal{Y} \oplus \mathcal{Y} \to X$ has a one-to-one correspondence with $\mathcal{O}_B$ stable lattices T' such that

$$T \subseteq T' \subset T^0 = T \otimes_{\mathcal{O}_F} F. \tag{4.4}$$

Since T is a free $\mathcal{O}_B$ module of rank 1, T' must be of the form $\pi^{-n} \cdot T$ for some $n \geq 0$. This shows that X is in fact isomorphic to $\mathcal{Y} \oplus \mathcal{Y}$. Identifying $X = \mathcal{Y} \oplus \mathcal{Y}$, we see from (4.3) that $j = \mathrm{diag}(\pi, \pi) \in \mathrm{End}_{\mathcal{O}_B}(X)$. Since $\mathrm{tr}_{F/F_0} \mathrm{tr}\, j = 0$, this implies that $X \in \mathcal{Z}(j) \subset \mathcal{M}$—the special divisor defined in [KR00, Definition 2.1]. So $\mathcal{Z} \subset Z(j)$ under the isomorphism $\mathcal{N} \cong \mathcal{M}$. Now we conclude from case (iii) of [KR00, Proposition 4.5] that $\mathcal{Z} \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$.

By construction we know that the relative Dieudonné module $M(\bar{X}) = M(\mathbb{Y}) \oplus M(\mathbb{Y})$ is fixed by $\tau = \pi V^{-1}$ where $\bar{X} = X \otimes \bar{\boldsymbol{k}}$. By [KR14, Lemma 3.2], $\mathcal{Z}$ intersects with the special fiber of $\mathcal{N}_{(1,1)}$ at a superspecial point. □

4.2. **Hodge filtration and equation of special cycles in $\mathcal{N}_{(1,1)}$.** Now we begin to study the equations of special divisors at a superspecial point. We will use Grothendieck-Messing theory to determine the equations, which in turn requires a description of the Hodge filtrations of the relevant relative Dieudonné crystals. In the following we use the Deligne functor to obtain such a description.

In the rest of this section, for $x = (X, \iota, \lambda, \rho) \in \mathcal{N}_{(1,1)}(\bar{\boldsymbol{k}})$, we denote $\rho(M(X)) \subset N$ by $M(x)$. Then identification $\mathcal{N}_{(1,1)}/\bar{\boldsymbol{k}} \cong \mathcal{M}/\bar{\boldsymbol{k}}$ induces an $\mathcal{O}_B$ and thus an $\mathcal{O}_E$-action on $M(x)$. This makes $M(x)$ an $\mathcal{O}_E \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_{\breve{F}_0}$-module and induces a $\mathbb{Z}/2\mathbb{Z}$ grading (see (1.10))

$$M(x) = M(x)_0 \oplus M(x)_1.$$

Now let x be a superspecial point pt_{Λ_0} of $\mathcal{N}_{(1,1)}$, which implies that $M(x)$ is τ-invariant and that $\Lambda_0 = M(x)^\tau$ is a vertex $\mathcal{O}_F$-lattice of type 0 with $M(x) = \Lambda_0 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$. As explained in Section 2.1 we can choose an $\mathcal{O}_{F_0}$-basis $\{e_0, f_1\}$ for $M_0(x)^\tau$ and $\{f_0, e_1\}$ for $M_1(x)^\tau$ such that

$$\pi e_i = f_i,\ \pi f_i = \pi_0 e_i,\ V e_i = f_i,\ V f_i = \pi_0 e_i,\ i = 0,\ 1. \tag{4.5}$$

and

$$(e_0, e_0) = (e_1, e_1) = 0, (e_0, e_1) = -\delta^2. \tag{4.6}$$

Then $\Lambda_0 = \mathrm{Span}_{\mathcal{O}_F}\{e_0, e_1\}$, and the vertex lattices of type 2 containing Λ_0 are

$$\Lambda_2 = \mathrm{Span}_{\mathcal{O}_F}\{\pi^{-1}e_0, e_1\}, \qquad \Lambda_2' = \mathrm{Span}_{\mathcal{O}_F}\{e_0, \pi^{-1}e_1\}. \tag{4.7}$$

Therefore, $x \in \mathbb{P}_{\bar{\Lambda}_2} \cap \mathbb{P}_{\bar{\Lambda}_2'}$. Since $\mathcal{N} \cong \mathcal{M}$, we have $\mathcal{N}_{(1,1)}(\bar{\boldsymbol{k}}) = \mathcal{M}(\bar{\boldsymbol{k}})$ and there should exist homothety classes of rank 2 $\mathcal{O}_{F_0}$-lattices $[\Lambda]$ and $[\Lambda']$ such that $x \in \mathbb{P}_{[\Lambda]} \cap \mathbb{P}_{[\Lambda']}$. By [San13, Remark 3.4], we can take

$$\begin{aligned}
\Lambda &= M_0(x)^\tau = \mathrm{span}_{\mathcal{O}_{F_0}}\{e_0, f_1\}, \\
\Lambda' &= \pi^{-1} M_1(x)^\tau = \mathrm{span}_{\mathcal{O}_{F_0}}\{\pi^{-1}f_0, \pi^{-1}e_1\} = \mathrm{span}_{\mathcal{O}_{F_0}}\{e_0, \pi_0^{-1}f_1\}.
\end{aligned}$$

Another way to relate these different types of lattices are the following equations

$$\Lambda = ((\Lambda_2 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E)_0)^\tau,\ \Lambda' = ((\Lambda_2' \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_E)_0)^\tau. \tag{4.8}$$

In particular we can identify $\mathbb{P}_{[\Lambda]}(\bar{\boldsymbol{k}})$ with $\mathbb{P}_{\bar{\Lambda}_2}(\bar{\boldsymbol{k}})$ and $\mathbb{P}_{[\Lambda']}(\bar{\boldsymbol{k}})$ with $\mathbb{P}_{\bar{\Lambda}_2'}(\bar{\boldsymbol{k}})$.

Consider the Deligne functor $F_{[\Lambda,\Lambda']}$ (see [BC91]):

$$\begin{array}{ccccc} \pi_0\Lambda' & \longrightarrow & \Lambda & \longrightarrow & \Lambda' \\ \downarrow{\scriptstyle \alpha'/\pi_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ \mathcal{L}' & \xrightarrow{c'} & \mathcal{L} & \xrightarrow{c} & \mathcal{L}'. \end{array}$$

For an $\mathcal{O}_{\breve{F}_0}$-algebra $R \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}_0}}$, the conditions of α and α' in Proposition 2.7 imply that $\alpha(e_0)$ generates $\mathcal{L}$ and $\alpha'(\pi^{-1}e_1)$ generates $\mathcal{L}'$. We identify $\mathcal{L}$ and $\mathcal{L}'$ with R by setting

$$\alpha(e_0) = 1, \quad \text{and} \quad \alpha'(\pi^{-1}e_1) = \alpha'(\pi_0^{-1}f_1) = 1.$$

Let

$$t_0 = \alpha(f_1) \in \mathcal{L} = R, \quad \text{and} \quad t_1 = \alpha'(e_0) = \alpha'(\pi^{-1}f_0) \in \mathcal{L}' = R.$$

Then c is simply the multiplication by t_0 and c' is the multiplication by t_1. So by commutativity of the above diagram, we have

$$t_0 t_1 = \pi_0. \tag{4.9}$$

Consider

$$\alpha \otimes 1 : \Lambda \otimes_{\mathcal{O}_{F_0}} R \to \mathcal{L} \quad \text{and} \quad \alpha' \otimes 1 : \Lambda' \otimes_{\mathcal{O}_{F_0}} R \to \mathcal{L}'. \tag{4.10}$$

We have

$$\text{kernel of } \alpha \otimes 1 = \mathrm{span}_R\{f_1 \otimes 1 - e_0 \otimes t_0\}, \tag{4.11}$$

$$\text{kernel of } \alpha' \otimes 1 = \mathrm{span}_R\{\pi^{-1}f_0 \otimes 1 - \pi^{-1}e_1 \otimes t_1\}. \tag{4.12}$$

For a strict formal $\mathcal{O}_{F_0}$-module X over R, let $\mathbb{D}(X/R)$ be its relative (to $\mathcal{O}_{F_0}$) Dieudonné crystal with Hodge filtration $\mathrm{Fil}\mathbb{D}(X/R)$, see for example [ACZ16, Section 3]). Then we have the exact sequence:

$$0 \to \mathrm{Fil}\mathbb{D}(X/R) \to \mathbb{D}(X/R) \to \mathrm{Lie}(X/R) \to 0. \tag{4.13}$$

Proposition 4.3. *Let $x = pt_{\Lambda_0}$ be a superspecial point. For an $\mathcal{O}_{\mathcal{N}_{(1,1)},x}$-algebra $R \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}_0}}$ where $\mathcal{O}_{\mathcal{N}_{(1,1)},x}$ is the local ring of $\mathcal{N}_{(1,1)}$ at x, let $t_0, t_1 \in R$ be the image of T_0, T_1 under the structure morphism, and X_{t_0,t_1} be the corresponding strict $\mathcal{O}_{F_0}$ module over R. Then we have the following identifications*

$$\mathbb{D}(X_{t_0,t_1}/R)_0 = \mathrm{span}_{\mathcal{O}_{F_0}}\{e_0, f_1\} \otimes_{\mathcal{O}_{F_0}} R, \quad \mathrm{Fil}\mathbb{D}(X_{t_0,t_1}/R)_0 = \mathrm{span}_R\{f_1 \otimes 1 - e_0 \otimes t_0\},$$

and

$$\mathbb{D}(X_{t_0,t_1}/R)_1 = \mathrm{span}_{\mathcal{O}_{F_0}}\{e_1, f_0\} \otimes_{\mathcal{O}_{F_0}} R, \quad \mathrm{Fil}\mathbb{D}(X_{t_0,t_1}/R)_1 = \mathrm{span}_R\{f_0 \otimes 1 - e_1 \otimes t_1\}.$$

Proof. Since x is a superspecial point, both 0 and 1 are critical indices. Hence $M(x)$ is τ invariant and $M(x)^\tau = \mathrm{span}_{\mathcal{O}_{F_0}}\{e_0, f_0, e_1, f_1\}$. According to the constructions in [BC91], especially how Deligne's functor is related with the moduli funtor of special formal $\mathcal{O}_B$-module, we know that

$$\mathbb{D}(X_{t_0,t_1}/R)_0 = M_0(x)^\tau \otimes_{\mathcal{O}_{F_0}} R = \mathrm{span}_{\mathcal{O}_{F_0}}\{e_0, f_1\} \otimes_{\mathcal{O}_{F_0}} R,$$
$$\mathbb{D}(X_{t_0,t_1}/R)_1 = M_1(x)^\tau \otimes_{\mathcal{O}_{F_0}} R = \mathrm{span}_{\mathcal{O}_{F_0}}\{e_1, f_0\} \otimes_{\mathcal{O}_{F_0}} R.$$

Under these identifications, the map $\alpha \otimes 1$ in ((4.10)) is the natural quotient map from $\mathbb{D}(X_{t_0,t_1}/R)_0$ to $\mathrm{Lie}(X_{t_0,t_1})_0$, and the maps $\pi\alpha' \otimes 1$ is the natural quotient map from $\mathbb{D}(X_{t_0,t_1}/R)_1$ to $\mathrm{Lie}(X_{t_0,t_1})_1$. Hence $\mathrm{Fil}\mathbb{D}(X_{t_0,t_1}/R)_0$ is the kernel of $\alpha \otimes 1$

and $\mathrm{Fil}\mathbb{D}(X_{t_0,t_1}/R)_1$ is the kernel of $\pi\alpha'\otimes 1$. By (4.11) and (4.12), we obtain the proposition. □

Similarly for the universal object $\mathcal{Y}$ over $\mathcal{N}_{(1,0)}$ and an $\mathcal{O}_{\breve{F}}$-algebra $R\in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$, it is easy to see that

$$\mathbb{D}(\mathcal{Y}_R)=\mathrm{Span}_R\{e\otimes 1, f\otimes 1\},\quad \mathrm{Fil}\mathbb{D}(\mathcal{Y}_R)=\mathrm{Span}_R\{f\otimes 1-e\otimes\pi\}, \tag{4.14}$$

since the $\mathcal{O}_F$-action coincides with the structure action $\mathcal{O}_F\to R$ on $\mathrm{Lie}(\mathcal{Y}_R)$: $\pi e\otimes 1=e\otimes\pi$. Here the tensor is over $\mathcal{O}_{F_0}$.

Proposition 4.4. *Let $x\in\mathcal{Z}(\mathbf{x})(\bar{\boldsymbol{k}})$ be a superspecial point pt_{Λ_0}. Choose a basis $\{e_0,e_1\}$ of Λ_0 such that (4.5) and (4.6) are satisfied. Assume $b=\mathbf{x}(e)=\alpha_0e_0+\alpha_1e_1\in\Lambda_0$ where $\alpha_i\in\mathcal{O}_F$. Recall a neighborhood of x in $\mathcal{N}_{(1,1)}$ is*

$$\breve{\Omega}_{[\Lambda,\Lambda']_{\breve{F}}}=\mathrm{Spf}(\mathcal{O}_{\breve{F}}[T_0,T_1,(T_0^{q-1}-1)^{-1},(T_1^{q-1}-1)^{-1}]/(T_0T_1-\pi_0))^\wedge. \tag{4.15}$$

Then the equations for $\mathcal{Z}(\mathbf{x})$ in the local ring $\mathcal{O}_{\mathcal{N}_{(1,1)},x}$ are given by (here $\bar\alpha$ stands for the Galois conjugate of α):

$$\bar\alpha_1T_0-\pi\bar\alpha_0=0,\quad \textit{and } \bar\alpha_0T_1-\pi\bar\alpha_1=0. \tag{4.16}$$

Proof. First, Lemma 3.7 implies $b=\mathbf{x}(e)\in\Lambda_0$. Let

$$b'\coloneqq\mathbf{x}(f)=\pi\mathbf{x}(e)=\alpha_0f_0+\alpha_1f_1.$$

for some $\alpha_i\in\mathcal{O}_F$. Consider the local ring $A\coloneqq\mathcal{O}_{\mathcal{N}_{(1,1)},x}$. Let I denote the ideal corresponding to $\mathcal{Z}(\mathbf{x})$. Set

$$A_n\coloneqq A/\pi^nA,\quad I_n\coloneqq(I+\pi^nA)/\pi^nA\subset A_n.$$

Since A is Noetherian, I is π-adic complete. Hence to prove the proposition, we just need to prove I_n are generated by the images of $\bar\alpha_1T_0-\pi\bar\alpha_0$ and $\bar\alpha_0T_1-\pi\bar\alpha_1$ in A_n for all n. Let $\mathfrak{m}_n$ denote the maximal ideal of A_n. Set

$$B\coloneqq A_n/\mathfrak{m}_nI_n,\quad B'\coloneqq A_n/I_n.$$

Let $J\coloneqq I_n/\mathfrak{m}_nI_n$, which is the kernel of of the projection $B\to B'$. Note that $J^2=0$, so it has a PD structure. By Nakayama's lemma, it suffices to show J is generated by images of $\bar\alpha_1T_0-\pi\bar\alpha_0$ and $\bar\alpha_0T_1-\pi\bar\alpha_1$ in B.

Let $\mathcal{X}$ be the universal strict formal $\mathcal{O}_{F_0}$-module over $\mathcal{N}_{(1,1)}$. The natural map

$$\mathcal{O}_{\breve{\Omega}_{[\Lambda,\Lambda']_{\breve{F}}}}\to A\to A_n\to B\to B'\to\bar{\boldsymbol{k}}$$

induces the strict formal $\mathcal{O}_{F_0}$-modules $\mathcal{X}_B$, $\mathcal{X}_{B'}$ and $\mathcal{X}_{\bar{\boldsymbol{k}}}=X$ with X being associated to $x\in\mathcal{Z}(\mathbf{x})(\bar{\boldsymbol{k}})$. Since I is the definition ideal of $\mathcal{Z}(\mathbf{x})$, B' is a quotient of A/I, the quasi-morphism

$$\rho^{-1}\circ\mathbf{x}:\mathbb{Y}\to X$$

lifts to a morphism

$$\mathbf{x}_{B'}:\mathcal{Y}_{B'}\to\mathcal{X}_{B'}.$$

The associated morphism $\mathbb{D}(\mathbf{x}'_B)$ lifts to a morphism

$$\mathbb{D}(\mathbf{x}_R):\mathbb{D}(\mathcal{Y}_R)\to\mathbb{D}(\mathcal{X}_R)$$

for any ring $R=B/\mathfrak{b}$ with $\mathfrak{b}\subset J$, i.e., $B\twoheadrightarrow R\twoheadrightarrow B'$ as $J^2=0$. By Grothebdieck-Messing theory, $\mathbf{x}$ lifts to a momorphism

$$\mathbf{x}_R:\mathcal{Y}_R\to\mathcal{X}_R \tag{4.17}$$

if and only if

$$\mathbb{D}(\mathbf{x}_R)(f\otimes 1-e\otimes\pi)=b'\otimes 1-b\otimes\pi\in \mathrm{Fil}\mathbb{D}(\mathcal{X}_R).$$

Write $\alpha_0=a_0+b_0\pi,\ \alpha_1=a_1+b_1\pi$. Then

$$\begin{aligned}
&b'\otimes 1-b\otimes\pi\\
&=(\alpha_0f_0+\alpha_1f_1)\otimes 1-(\alpha_0e_0+\alpha_1e_1)\otimes\pi\\
&=((a_0+b_0\pi)f_0+(a_1+b_1\pi)f_1)\otimes 1-((a_0+b_0\pi)e_0+(a_1+b_1\pi)e_1)\otimes\pi\\
&=(a_0f_0+b_0\pi_0e_0+a_1f_1+b_1\pi_0e_1)\otimes 1-(a_0e_0+b_0f_0+a_1e_1+b_1f_1)\otimes\pi\\
&=f_1\otimes(a_1-\pi b_1)-e_0\otimes\pi(a_0-\pi b_0)+f_0\otimes(a_0-\pi b_0)-e_1\otimes\pi(a_1-\pi b_1)\\
&=f_1\otimes\bar\alpha_1-e_0\otimes\pi\bar\alpha_0+f_0\otimes\bar\alpha_0-e_1\otimes\pi\bar\alpha_1.
\end{aligned}$$

Combining this with Proposition 4.3, we see that the lifting (4.17) exists if and only if

$$\bar\alpha_1T_0-\pi\bar\alpha_0=0,\ \text{and}\ \bar\alpha_0T_1-\pi\bar\alpha_1=0\ \text{in R},$$

i.e.,

$$\bar\alpha_1T_0-\pi\bar\alpha_0,\ \text{and}\ \bar\alpha_0T_1-\pi\bar\alpha_1\in\mathfrak{b}.$$

Here we identify T_i with their images in R via $A\to A_n\to B\to R$. Since I is the ideal of $\mathcal{Z}(\mathbf{x})$, the lifting (4.17) exists only when $\mathfrak{b}=J$. So J is generated by $\bar\alpha_1T_0-\pi\bar\alpha_0$ and $\bar\alpha_0T_1-\pi\bar\alpha_1$ as claimed.

□

4.3. **Local coordinate charts in the Krämer model.** Now we describe the local equation of a special divisor in the Krämer model and use it to give a decomposition of special divisor. Locally around the superspecial point $x\in\mathcal{N}_{(1,1)}(\bar{\boldsymbol{k}})$ corresponding to $\Lambda_0=\Lambda_2\cap\Lambda_2'$, we have (4.15) and x corresponds to the maximal ideal $\mathfrak{m}_x=(T_0,T_1,\pi)$. We need to blow it up to get the exceptional divisor of $\mathcal{N}^{\mathrm{Kra}}=\mathcal{N}^{\mathrm{Kra}}_{(1,1)}$.

For simplicity, consider (ignoring the other unimportant restrictions of $\breve\Omega_{[\Lambda,\Lambda']_,F}$)

$$D\coloneqq \mathrm{Spf}(\mathcal{O}_{\breve F}[T_0,T_1]/(T_0T_1-\pi_0))^\wedge.$$

Let Bl_xD denote the blow-up of D at $\mathfrak{m}_x$, which has three charts. Over the first chart D_1, we have

$$\pi S_{\frac{T_0}{\pi}}=T_0,\ \pi S_{\frac{T_1}{\pi}}=T_1, \tag{4.18}$$

where we regard $S_{\frac{x}{\pi}}$ as element in $\mathrm{Frac}(\mathcal{O}_{\breve F}[T_0,T_1])$ and

$$\begin{aligned}
D_1&=\mathrm{Spf}\mathcal{O}_{\breve F}([T_0,T_1,S_{\frac{T_0}{\pi}},S_{\frac{T_1}{\pi}}]/(\pi S_{\frac{T_0}{\pi}}-T_0,\pi S_{\frac{T_1}{\pi}}-T_1,S_{\frac{T_0}{\pi}}S_{\frac{T_1}{\pi}}-1))^\wedge\\
&=\mathrm{Spf}\mathcal{O}_{\breve F}([S_{\frac{T_0}{\pi}},S_{\frac{T_1}{\pi}}]/(S_{\frac{T_0}{\pi}}S_{\frac{T_1}{\pi}}-1))^\wedge. \qquad (4.19)
\end{aligned}$$

Over the second chart D_2, we have:

$$T_0S_{\frac{T_1}{T_0}}=T_1,\ T_0S_{\frac{\pi}{T_0}}=\pi, \tag{4.20}$$

and

$$D_2=\mathrm{Spf}\mathcal{O}_{\breve F}([T_0,S_{\frac{\pi}{T_0}}]/(T_0S_{\frac{\pi}{T_0}}-\pi))^\wedge. \tag{4.21}$$

Over the third chart D_3, we have:

$$T_1S_{\frac{T_0}{T_1}}=T_0,\ T_1S_{\frac{\pi}{T_1}}=\pi, \tag{4.22}$$

and

$$D_3 = \mathrm{Spf}\mathcal{O}_{\breve{F}}([T_1, S_{\frac{\pi}{T_1}}]/(T_1 S_{\frac{\pi}{T_1}} - \pi))^\wedge \tag{4.23}$$

by symmetry. D_1, D_2 and D_3 are glued in the obvious way, and it is easy to see that D_1, D_2 and D_3 are all regular. Let Exc denote the exceptional divisor of $\mathcal{N}^{\mathrm{Kra}}$, then

$$\begin{aligned}
\mathrm{Exc} \cap D_1 &= \mathrm{Spf}\bar{\boldsymbol{k}}([S_{\frac{T_0}{\pi}}, S_{\frac{T_1}{\pi}}]/(S_{\frac{T_0}{\pi}} S_{\frac{T_1}{\pi}} - 1))^\wedge, \\
\mathrm{Exc} \cap D_2 &= \mathrm{Spf}\mathcal{O}_{\breve{F}}([T_0, S_{\frac{\pi}{T_0}}]/(T_0, T_0 S_{\frac{\pi}{T_0}} - \pi))^\wedge = \mathrm{Spf}\bar{\boldsymbol{k}}([S_{\frac{\pi}{T_0}}])^\wedge, \\
\mathrm{Exc} \cap D_3 &= \mathrm{Spf}\mathcal{O}_{\breve{F}}([T_1, S_{\frac{\pi}{T_1}}]/(T_1, T_1 S_{\frac{\pi}{T_1}} - \pi))^\wedge = \mathrm{Spf}\bar{\boldsymbol{k}}([S_{\frac{\pi}{T_1}}])^\wedge.
\end{aligned} \tag{4.24}$$

$\mathrm{Exc} \cap D_2$ glues with $\mathrm{Exc} \cap D_3$ over $\mathrm{Exc} \cap D_1$ by $S_{\frac{\pi}{T_0}} = \frac{1}{S_{\frac{\pi}{T_1}}}$, so Exc is isomorphic to $\mathbb{P}^1_{\bar{\boldsymbol{k}}}$. The projective line $\mathbb{P}_{\bar{\Lambda}_2}$ only intersects the second chart and is defined by the equation

$$S_{\frac{\pi}{T_0}} = 0. \tag{4.25}$$

Similarly $\mathbb{P}_{\bar{\Lambda}'_2}$ only intersects the third chart and is defined by the equation

$$S_{\frac{\pi}{T_1}} = 0. \tag{4.26}$$

We refer to Example 8.3.53 of [Liu06] for more details about the blow up considered here.

Now we explain how the coordinates of blow up are related with the moduli problem locally around a superspecial point x. Since blow up commutes with flat base change, we have

$$\mathcal{N}_x^{\mathrm{Kra}} := \mathrm{Bl}_x D \times_{\mathcal{N}_{(1,1)}} \mathrm{Spf}\, \mathcal{O}_{\mathcal{N}_{(1,1)},x} = \mathrm{Bl}_x \mathrm{Spf}\, \mathcal{O}_{\mathcal{N}_{(1,1)},x}, \tag{4.27}$$

and let $D_{i,x}$, $i = 1, 2, 3$ be the three charts for $\mathcal{N}_x^{\mathrm{Kra}}$ coming from D_i.

Let $R \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}_0}}$ be an $\mathcal{O}_{D_{1,x}}$-algebra, and $s_{\frac{T_0}{\pi}}, s_{\frac{T_1}{\pi}} \in R$ be the image of $S_{\frac{T_0}{\pi}}, S_{\frac{T_1}{\pi}}$ under the structure morphism, and let t_0 and t_1 be given by (4.18). Then R determines a point $(X_{t_0,t_1}, \mathcal{F}) \in D_{1,x}(R)$ where X_{t_0,t_1} is described in Proposition 4.3, and $\mathcal{F} = \mathrm{Span}_R\{e_0 \otimes 1 + e_1 \otimes s_{\frac{T_1}{\pi}}\} \subset \mathrm{Lie}(X_{t_0,t_1})$ is the filtration of the Krämer moduli problem.

Let $R \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}_0}}$ be an $\mathcal{O}_{D_{2,x}}$-algebra, and $t_0, s_{\frac{\pi}{T_0}} \in R$ be the image of $T_0, S_{\frac{\pi}{T_0}}$ under the structure morphism, and let t_1 be given by (4.20). Then R determines a point $(X_{t_0,t_1}, \mathcal{F}) \in D_{2,x}(R)$ where X_{t_0,t_1} is as before and $\mathcal{F} = \mathrm{Span}_R\{e_0 \otimes 1 + e_1 \otimes s_{\frac{\pi}{T_0}}\} \subset \mathrm{Lie}(X_{t_0,t_1})$. The corresponding description for an $\mathcal{O}_{D_{3,x}}$-algebra is similar.

4.4. **Proof of Theorem 4.1.**

Proof of Theorem 4.1. In the following, we use $m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathcal{Z})$ to denote the multiplicity of $\mathcal{Z}$ in $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$, where $\mathcal{Z}$ is an irreducible component of $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$. According to Theorem 4.2, the horizontal component of $\mathcal{Z}(\mathbf{x})$ intersects with the special fiber of $\mathcal{N}_{(1,1)}$ at some superspecial points. Moreover, each irreducible component of the special fiber of $\mathcal{N}_{(1,1)}$ also passes through some superspecial points. Therefore to determine the multiplicity of each irreducible component, it is enough to consider the equations of special divisors at superspecial points and their pullbacks to the Krämer model.

Write $b = \alpha_0 e_0 + \alpha_1 e_1$ as in Proposition 4.4. As before we fix a superspecial point $x = pt_{\Lambda_0} \in \mathcal{N}_{(1,1)}$ for a vertex lattice Λ_0 of type 0. Recall that the equations of $\mathcal{Z}(\mathbf{x})$ in $\mathcal{O}_{\mathcal{N}_{(1,1)},x}$ are:

$$\bar{\alpha}_1 T_0 - \pi\bar{\alpha}_0 = 0, \quad \bar{\alpha}_0 T_1 - \pi\bar{\alpha}_1 = 0, \ \alpha_i \in \mathcal{O}_F. \tag{4.28}$$

When $\mathrm{ord}_\pi(\alpha_0) > \mathrm{ord}_\pi(\alpha_1)$, the equations of $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ in $\mathcal{N}_x^{\mathrm{Kra}}$ are

$$\begin{cases} \bar{\alpha}_1\pi(S_{\frac{T_0}{\pi}} - \frac{\bar{\alpha}_0}{\alpha_1}) = 0, \quad \bar{\alpha}_1\pi(\frac{\bar{\alpha}_0}{\alpha_1}S_{\frac{T_1}{\pi}} - 1) = 0, & \text{in the first chart,} \\ \bar{\alpha}_1 T_0(1 - \frac{\bar{\alpha}_0}{\bar{\alpha}_1}S_{\frac{\pi}{T_0}}) = 0, \quad \bar{\alpha}_1 T_0 S_{\frac{\pi}{T_0}}(\frac{\bar{\alpha}_0}{\bar{\alpha}_1}S_{\frac{\pi}{T_0}} - 1) = 0, & \text{in the second chart,} \\ \bar{\alpha}_1 T_1 S^2_{\frac{\pi}{T_1}}(1 - \frac{\bar{\alpha}_0}{\bar{\alpha}_1}\frac{1}{S_{\frac{\pi}{T_1}}}) = 0, \quad \bar{\alpha}_1 T_1 S_{\frac{\pi}{T_1}}(\frac{\bar{\alpha}_0}{\bar{\alpha}_1}\frac{1}{S_{\frac{\pi}{T_1}}} - 1) = 0, & \text{in the third chart.} \end{cases}$$

Notice that $(\frac{\bar{\alpha}_0}{\bar{\alpha}_1}S_{\frac{T_1}{\pi}} - 1)$ and $(1 - \frac{\bar{\alpha}_0}{\bar{\alpha}_1}S_{\frac{\pi}{T_0}})$ are units in coordinate ring of $D_{1,x}$ and coordinate ring of $D_{2,x}$. In the third chart, we have $T_1 S_{\frac{\pi}{T_1}} = \pi$, which implies that

$$\frac{\bar{\alpha}_0}{\bar{\alpha}_1}\frac{1}{S_{\frac{\pi}{T_1}}} - 1 = \alpha T_1 - 1$$

is a unit in $D_{3,x}$ with $\alpha = \frac{\bar{\alpha}_0}{\bar{\alpha}_1\pi} \in \mathcal{O}_F$. So the above equations simplify to

$$\begin{cases} \pi^{\mathrm{ord}_\pi(\alpha_1)+1} = 0, & \text{in the first chart,} \\ \pi^{\mathrm{ord}_\pi(\alpha_1)}T_0 = T_0^{\mathrm{ord}_\pi(\alpha_1)+1}(S_{\frac{\pi}{T_0}})^{\mathrm{ord}_\pi(\alpha_1)} = 0, & \text{in the second chart,} \\ \pi^{\mathrm{ord}_\pi(\alpha_1)+1} = T_1^{\mathrm{ord}_\pi(\alpha_1)+1}(S_{\frac{\pi}{T_1}})^{\mathrm{ord}_\pi(\alpha_1)+1} = 0, & \text{in the third chart.} \end{cases}$$

Therefore, it has no horizonal component, and we have by (4.24), (4.25) and (4.26)

$$\begin{aligned} m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2}) &= \mathrm{ord}_\pi(\alpha_1), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2'}) &= \mathrm{ord}_\pi(\alpha_1) + 1, \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathrm{Exc}_{\Lambda_0}) &= \mathrm{ord}_\pi(\alpha_1) + 1. \end{aligned} \tag{4.29}$$

Similarly if $\mathrm{ord}_\pi(\alpha_0) < \mathrm{ord}_\pi(\alpha_1)$, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ has no horizonal component, and

$$\begin{aligned} m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2}) &= \mathrm{ord}_\pi(\alpha_0) + 1, \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2'}) &= \mathrm{ord}_\pi(\alpha_0), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathrm{Exc}_{\Lambda_0}) &= \mathrm{ord}_\pi(\alpha_0) + 1. \end{aligned} \tag{4.30}$$

When $\mathrm{ord}_\pi(\alpha_0) = \mathrm{ord}_\pi(\alpha_1)$(possible only when $q(b) \neq 0$), the equations of $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ in $\mathcal{N}_x^{\mathrm{Kra}}$ are

$$\begin{cases} \pi^{\mathrm{ord}_\pi(\alpha_0)+1}(S_{\frac{T_1}{\pi}} - \frac{\bar{\alpha}_1}{\bar{\alpha}_0}) = 0, & \text{in the first chart,} \\ T_0^{\mathrm{ord}_\pi(\alpha_0)+1}(S_{\frac{\pi}{T_0}})^{\mathrm{ord}_\pi(\alpha_0)}(S_{\frac{\pi}{T_0}} - \frac{\bar{\alpha}_1}{\bar{\alpha}_0}) = 0, & \text{in the second chart,} \\ T_1^{\mathrm{ord}_\pi(\alpha_0)+1}(S_{\frac{\pi}{T_1}})^{\mathrm{ord}_\pi(\alpha_0)}(S_{\frac{\pi}{T_1}} - \frac{\bar{\alpha}_0}{\bar{\alpha}_1}) = 0, & \text{in the third chart,} \end{cases}$$

which implies by (4.24), (4.25) and (4.26)

$$\begin{aligned} m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2}) &= \mathrm{ord}_\pi(\alpha_0), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2'}) &= \mathrm{ord}_\pi(\alpha_0), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathrm{Exc}_{\Lambda_0}) &= \mathrm{ord}_\pi(\alpha_0) + 1. \end{aligned} \tag{4.31}$$

In addition, it has a horizontal component given by

$$\begin{cases} S_{\frac{T_1}{\pi}} - \frac{\bar{\alpha}_1}{\bar{\alpha}_0} = 0, & \text{in the first chart,} \\ S_{\frac{\pi}{T_0}} - \frac{\bar{\alpha}_1}{\bar{\alpha}_0} = 0, & \text{in the second chart,} \\ S_{\frac{\pi}{T_1}} - \frac{\bar{\alpha}_0}{\bar{\alpha}_1} = 0, & \text{in the third chart.} \end{cases} \tag{4.32}$$

This is the local equation of $\mathcal{Z}^h(\mathbf{x})$ along the superspecial point $x = pt_{\Lambda_0}$. In this case, one has to have $\Lambda_0 = \Lambda_b$. From the equation, one can also see that the horizontal component is irreducible when $q(b) \neq 0$, and it intersects with Exc_{Λ_b} at one point (e.g., the image of $S_{\frac{T_1}{\pi}} = \frac{\bar{\alpha}_1}{\bar{\alpha}_0}$ in $\bar{\boldsymbol{k}}$ via the first chart).

Recall $b = \mathbf{x}(e)$. Let $n = n(b, \Lambda_0) = \min\{\mathrm{ord}_\pi(\alpha_0), \mathrm{ord}_\pi(\alpha_1)\}$. Note that

$$n(b, \Lambda_2) = \begin{cases} n+1, & \mathrm{ord}_\pi(\alpha_1) > \mathrm{ord}_\pi(\alpha_0), \\ n, & \text{otherwise,} \end{cases}$$

and

$$n(b, \Lambda_2') = \begin{cases} n+1, & \mathrm{ord}_\pi(\alpha_1) < \mathrm{ord}_\pi(\alpha_0), \\ n, & \text{otherwise.} \end{cases}$$

Hence in all three cases we have

$$\begin{aligned} m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2}) &= n(b, \Lambda_2), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2'}) &= n(b, \Lambda_2'), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathrm{Exc}_{\Lambda_0}) &= n(b, \Lambda_0) + 1. \end{aligned} \tag{4.33}$$

The above discussion holds for any $\Lambda \in \mathcal{T}(\mathbf{x})$. So we have

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b, \Lambda_2) \mathbb{P}_{\bar{\Lambda}_2} + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b, \Lambda_0) + 1) \mathrm{Exc}_{\Lambda_0} + \delta_b \mathcal{Z}^h(\mathbf{x}), \tag{4.34}$$

where $\delta_b = 0$ or 1 depends on whether $q(b) = 0$ or not. In the latter case, $\mathcal{Z}^h(\mathbf{x}) \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ is a horizontal divisor meeting the special fiber at Exc_{Λ_b}. □

The proof (in particular (4.32)) gives the following corollary, which will be used in next section.

Corollary 4.5. *Let $0 \neq \mathbf{x} \in \mathbb{V}$ and $b = \mathbf{x}(e)$. When $q(\mathbf{x}) = 0$, $\mathcal{Z}^h(\mathbf{x})$ is empty. When $q(\mathbf{x}) \neq 0$ with $b \in \Lambda_b$, $\mathcal{Z}^h(\mathbf{x}) = \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ is irreducible, and intersects with the special fiber of $\mathcal{N}^{\mathrm{Kra}}$ at one point on the exceptional divisor Exc_{Λ_b} and is given by image of (4.32) modulo π. More precisely, write*

$$\pi^{-\mathrm{ord}_{\pi_0} q(b)} b = \alpha_0 e_0 + \alpha_1 e_1,$$

where $\{e_0, e_1\}$ is a basis of Λ_b given in Proposition 4.4. Then $\alpha_0, \alpha_1 \in \mathcal{O}_F^\times$, and the intersection point of $\mathcal{Z}^h(\mathbf{x})$ and $\mathcal{N}^{\mathrm{Kra}}/\bar{\boldsymbol{k}}$ is $S_{\frac{T_1}{\pi}} = \frac{\bar{\alpha}_1}{\bar{\alpha}_0} \pmod{\pi}$ in the first affine chart of the neighborhood of $\mathcal{N}_x^{\mathrm{Kra}}$ where $x = pt_{\Lambda_b}$.

5. Intersection between special divisors

In this section we establish a series of lemmas and prove Theorems 5.7 and 5.9, which are reformulation of Theorem 1.1.

First of all, for two divisors $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of $\mathcal{N}^{\mathrm{Kra}}$ such that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is supported on finitely many irrreducible components of the special fiber of $\mathcal{N}^{\mathrm{Kra}}$, we define their intersection number to be

$$\mathcal{Z}_1 \cdot \mathcal{Z}_2 \coloneqq \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}_1} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}_2}), \tag{5.1}$$

where $\mathcal{O}_{\mathcal{Z}_i}$ is the structure sheaf of $\mathcal{Z}_i$, $\otimes^{\mathbb{L}}$ is the derived tensor product on the coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and χ is the Euler-Poincaré characteristic.

For a full rank lattice $L_{\mathbf{x}_1,\mathbf{x}_2} \subset \mathbb{V}$ with a basis $\{\mathbf{x}_1, \mathbf{x}_2\}$, let

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = \mathcal{Z}(\mathbf{x}_1) \cdot \mathcal{Z}(\mathbf{x}_2).$$

According to [How19, Corollary D], this intersection number does not depend on the choice of the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$. First, we recall the following well-known fact.

Proposition 5.1. *([Liu06, Proposition 9.1.21]) Assume that X is a regular scheme of dimension 2, S is a Dedekind scheme of dimension 1 and we have a flat proper morphism $X \to S$. Let $s \in S$ be a closed point. The following properties are true:*

(a) For any $E \in \mathrm{Div}_s(X)$, we have $E \cdot X_s = 0$. Here $\mathrm{Div}_s(X)$ is the set of Cartier divisors of X supported on the special fiber X_s.
(b) Let $\Gamma_1, \ldots \Gamma_r$ be the irreducible components of X_s of respective multiplicities $d_1, \ldots, d_r$. Then for any $i \le r$, we have

$$\Gamma_i^2 = -\frac{1}{d_i} \sum_{j \neq i} d_j \Gamma_j \cdot \Gamma_i.$$

Lemma 5.2. *Let Λ_0 be a fixed vertex $\mathcal{O}_F$-lattice of type 0. Then*

(a) $\mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2} = \begin{cases} 1, & \text{if } \Lambda_0 \subset \Lambda_2, \\ 0, & \text{otherwise.} \end{cases}$
(b) $\mathrm{Exc}_{\Lambda_0} \cdot \mathrm{Exc}_{\Lambda_0'} = -2\delta_{\Lambda_0,\Lambda_0'}$ *for any type 0 vertex lattice Λ_0'.*
(c) $\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^h(\mathbf{x}) = \delta_{\Lambda_0,\Lambda_b} = 1$ *or* 0 *depending on whether $\Lambda_0 = \Lambda_b$ or not.*

Proof. By (4.24), (4.25) and (4.26), we can see that

$$\mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2} = \mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2'} = 1 \tag{5.2}$$

if $\Lambda_0 = \Lambda_2 \cap \Lambda_2'$. For a Λ_2'' s.t. $\Lambda_0 \not\subset \Lambda_2''$, then clearly $\mathrm{Exc}_{\Lambda_0} \cap \mathbb{P}_{\bar{\Lambda}_2''} = \emptyset$. Hence (a) is proved. Part (b) follows from Proposition 5.1 and part (a). Part (c) follows from Theorem 4.2 and (4.32). □

Lemma 5.3. *Let Λ_2 be a fixed vertex $\mathcal{O}_F$-lattice of type 2. Then*

(a) $\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathbb{P}_{\bar{\Lambda}_2'} = \begin{cases} -(q+1), & \text{if } \Lambda_2 = \Lambda_2', \\ 0, & \text{otherwise.} \end{cases}$
(b) $\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^h(\mathbf{x}) = 0.$

Proof. Note that $\mathbb{P}_{\bar{\Lambda}_2} \cap \mathbb{P}_{\bar{\Lambda}_2'} = \emptyset$ if $\Lambda_2 \neq \Lambda_2'$. Moreover, there are $q+1$ exceptional divisors intersecting with $\mathbb{P}_{\bar{\Lambda}_2}$. Then part (b) of Proposition 5.1 and equation (5.2) imply that $\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathbb{P}_{\bar{\Lambda}_2} + (q+1) = 0$. So part (a) follows. It is clear from (4.24), (4.25) and (4.32) that $\mathbb{P}_{\bar{\Lambda}_2}$ and $\mathcal{Z}^h(\mathbf{x})$ do not intersect. Hence part (b) follows. □

Lemma 5.4. $\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \begin{cases} 1, & \text{if } \Lambda_2 \in \mathcal{T}(\mathbf{x}), \\ 0, & \text{otherwise.} \end{cases}$

Proof. When $\Lambda_2 \notin \mathcal{T}(\mathbf{x})$, the intersection number is obviously 0. When $\Lambda_2 \in \mathcal{T}(\mathbf{x})$. We have by Theorem 4.1, Lemma 5.2 and 5.3,

$$\begin{aligned}\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) =& \mathbb{P}_{\bar{\Lambda}_2} \cdot [\sum_{\Lambda_2' \in \mathcal{T}(\mathbf{x})} n(b, \Lambda_2') \mathbb{P}_{\bar{\Lambda}_2'} + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b,\Lambda_0)+1)\mathrm{Exc}_{\Lambda_0}] \\ =& -(q+1)n(b,\Lambda_2) + \sum_{\Lambda_0 \subset \Lambda_2} (n(b,\Lambda_0)+1) \\ =& \sum_{\Lambda_0 \subset \Lambda_2} (n(b,\Lambda_0)+1-n(b,\Lambda_2)).\end{aligned}$$

Now combining the information from (3.2) with Lemmas 3.11 and 3.13 we see that there is exactly one vertex lattice Λ' of type 0 in Λ_2 such that $n(b,\Lambda') = n(b,\Lambda_2)$ and for any other vertex lattice Λ of type 0 in Λ_2 we have $n(b,\Lambda) = n(b,\Lambda_2) - 1$. Hence we have

$$\sum_{\Lambda_0 \subset \Lambda_2} (n(b,\Lambda_0)+1-n(b,\Lambda_2)) = 1.$$

The lemma follows. □

Lemma 5.5. $\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \begin{cases} -1, & \Lambda_0 \in \mathcal{T}(\mathbf{x}), \\ 0, & \textit{otherwise.} \end{cases}$

Proof. Assume Λ_2 and Λ_2' are the two vertex lattices of type 2 that contain Λ_0. We treat the cases $\Lambda_0 \neq \Lambda_b$ first. According to (3.2), Lemma 3.11 and 3.13, we can without loss of generality assume that

$$n(b,\Lambda_2) = n(b,\Lambda_0)+1, \text{ and } n(b,\Lambda_2') = n(b,\Lambda_0).$$

Then by Theorem 4.1 and Lemma 5.2, we have

$$\begin{aligned}\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) =& \mathrm{Exc}_{\Lambda_0} \cdot [\sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b,\Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} + \sum_{\Lambda_0' \in \mathcal{T}(\mathbf{x})} (n(b,\Lambda_0')+1)\mathrm{Exc}_{\Lambda_0'}] \\ =& n(b,\Lambda_2)\mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2} + n(b,\Lambda_2')\mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2'} + (n(b,\Lambda_0)+1)\mathrm{Exc}_{\Lambda_0} \cdot \mathrm{Exc}_{\Lambda_0} \\ =& (n(b,\Lambda_0)+1) + n(b,\Lambda_0) - 2(n(b,\Lambda_0)+1) \\ =& -1.\end{aligned}$$

Now assume $\Lambda_0 = \Lambda_b$, which occurs only when $q(b) \neq 0$. Then notice that $n(b,\Lambda_2) = n(b,\Lambda_2') = n(b,\Lambda_0)$, and $\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^h(\mathbf{x}) = 1$. The rest of the proof is the same as the previous case. □

Lemma 5.6. *Write $b_i = \mathbf{x}_i(e)$, and assume that $(b_1,b_2) = 0$ and $q(b_i) \neq 0$. Then $\mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^h(\mathbf{x}_2) = 0$, and*

$$\mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) = n(b_2, \Lambda_{b_1}) + 1.$$

Proof. Define $\beta_i = \pi^{-\operatorname{ord}_{\pi_0} q(b_i)} b_i$ for $i = 1,2$. The assumption implies

$$\Lambda_{b_1} = \Lambda_{b_2} = \mathcal{O}_F\beta_1 + \mathcal{O}_F\beta_2,$$

and the Gram matrix of $\{\beta_1,\beta_2\}$ is diagonal with diagonal entries in $\mathcal{O}_{F_0}^\times$. Let $\{e_0,e_1\}$ be an $\mathcal{O}_F$-basis of Λ_{b_1} as in Proposition 4.4 and write

$$\beta_1 = \alpha_0 e_0 + \alpha_1 e_1, \text{ and } \beta_2 = \alpha_0' e_0 + \alpha_1' e_1.$$

By comparing the determinants of the Gram matrices of $\{e_0, e_1\}$ and $\{\beta_1, \beta_2\}$, we know that the transition matrix $\begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_0' & \alpha_1' \end{pmatrix}$ has determinant $\alpha_0\alpha_1' - \alpha_1\alpha_0' = u \in \mathcal{O}_F^\times$.

On the other hand, if $\mathcal{Z}^h(\mathbf{x}_1)$ and $\mathcal{Z}^h(\mathbf{x}_2)$ intersect, the intersection points would be in the special fiber. By Corollary 4.5, they intersect at one point, which is also the intersection between them and the exceptional divisor $\mathrm{Exc}_{\Lambda_{b_i}}$. The same corollary asserts that the point is given by (say in first Chart)

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_0} \equiv \frac{\bar{\alpha}_1'}{\bar{\alpha}_0'} \pmod{\pi}.$$

This implies $\alpha_0\alpha_1' - \alpha_1\alpha_0' \in \pi\mathcal{O}_F$, a contradiction. So $\mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^h(\mathbf{x}_2) = 0$. Now we have by Theorem 4.1, Lemmas 5.2 and 5.3,

$$\begin{aligned} \mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) &= \mathcal{Z}^h(\mathbf{x}_1) \cdot (n(b_2, \Lambda_{b_1}) + 1)\mathrm{Exc}_{\Lambda_{b_1}} \\ &= n(b_2, \Lambda_{b_1}) + 1. \end{aligned}$$

□

Theorem 5.7. *Let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be a basis of a full rank lattice $L_{\mathbf{x}_1,\mathbf{x}_2} \subset \mathbb{V}$, and $b_1 = \mathbf{x}_1(e), b_2 = \mathbf{x}_2(e)$ as before. Assume*

$$T = \begin{pmatrix} (b_1, b_1) & (b_1, b_2) \\ (b_2, b_1) & (b_2, b_2) \end{pmatrix} \approx \begin{pmatrix} u_1(-\pi_0)^\alpha & 0 \\ 0 & u_2(-\pi_0)^\beta \end{pmatrix}$$

where $\alpha \le \beta$, $u_1, u_2 \in \mathcal{O}_{F_0}^\times$ and $-u_1u_2 \in \mathrm{Nm}(F^\times)$. When $\alpha \ge 0$, we have

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = \alpha + \beta - \frac{2q(q^\alpha - 1)}{q-1}.$$

When $\alpha < 0$, we have $\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = 0$.

Proof. We may assume $T = \mathrm{Diag}(u_1(-\pi_0)^\alpha, u_2(-\pi_0)^\beta)$. In such a case, we see that $\mathrm{span}_{\mathcal{O}_F}\{\pi^{-\alpha}\mathbf{x}_1(e), \pi^{-\beta}\mathbf{x}_2(e)\} = \Lambda_{b_1} = \Lambda_{b_2}$. Moreover $\mathcal{T}(\mathbf{x}_1) \subset \mathcal{T}(\mathbf{x}_2)$ and $\mathcal{T}(\mathbf{x}_1)$ is a ball of radius $\alpha + \frac{1}{2}$ centered at Λ_{b_1} by Corollary 3.12. We have by Theorem 1.2 and Lemmas 5.4, 5.5, 5.6, and 3.11,

$$\begin{aligned} &\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) \\ =& \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x}_1)} n(b_1, \Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x}_1)} (n(b_1, \Lambda_0) + 1)\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) \\ &+ \mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) \\ =& \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x}_1)} n(b_1, \Lambda_2) - \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x}_1)} (n(b_1, \Lambda_0) + 1) + (n(b_2, \Lambda_{b_1}) + 1) \\ =& 2(\sum_{i=0}^{\alpha}(\alpha - i)q^i) - (\alpha + 1 + 2(\sum_{i=1}^{\alpha}(\alpha + 1 - i)q^i)) + (\beta + 1) \\ =& \alpha + \beta - 2\sum_{i=1}^{\alpha} q^i \\ =& \alpha + \beta - \frac{2q(q^\alpha - 1)}{q-1}. \end{aligned}$$

□

Remark 5.8. *As we mentioned before, there is a similar result for intersection product between special divisors on a similarly defined RZ space $\mathcal{N}_E$ obtained by Sankaran in* [San17]. *Here E is an unramified quadratic field extension of F_0.*

Theorem 5.9. *For $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$, let $b_1 = \mathbf{x}_1(e), b_2 = \mathbf{x}_2(e)$ as before. Assume*

$$T = \begin{pmatrix} (b_1,b_1) & (b_1,b_2) \\ (b_2,b_1) & (b_2,b_2) \end{pmatrix} \approx \begin{pmatrix} 0 & \pi^n \\ (-\pi)^n & 0 \end{pmatrix},$$

with n odd. Write $r = \frac{n+1}{2}$. Then $\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = 0$ unless $n \geq 0$. In such a case,

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = -\frac{(q+1)(q^r-1)}{q-1} + 2r.$$

Proof. We may assume that $T = \left(\begin{smallmatrix} 0 & \pi^n \\ (-\pi)^n & 0 \end{smallmatrix}\right)$. By Lemmas 5.4, 5.5 and Theorem 4.1, we have

$$\begin{aligned} &\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) \\ &= \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x}_1)} n(b_1,\Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x}_1)} (n(b_1,\Lambda_0)+1)\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) \\ &= \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b_1,\Lambda_2) - \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b_1,\Lambda_0)+1), \end{aligned}$$

where $\mathcal{T}(\mathbf{x}) = \mathcal{T}(\mathbf{x}_1) \cap \mathcal{T}(\mathbf{x}_2)$.

Write $b_i = \pi^r v_i$ for $i = 1, 2$. Define

$$\Lambda_{k,2} = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-k}v_1, \pi^k v_2\},\ \Lambda_{k,0} = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-k}v_1, \pi^{k+1}v_2\},\ k \in \mathbb{Z}.$$

where $\Lambda_{k,2}$ is a vertex lattice of type 2 and $\Lambda_{k,0}$ is a vertex lattice of type 0. It is immediate that

$$n(b_1,\Lambda_{k,0}) = n(b_1,\Lambda_{k,2}) = r + k.$$

By Lemma 3.16, $\mathcal{T}(\mathbf{x})$ is a ball of radius r centered at $\Lambda_{0,2}$. We define $\mathcal{C} = \mathcal{C}(\Lambda_{0,2}, \Lambda_{r,2})$ to be the geodesic segment joining $\Lambda_{0,2}$ and $\Lambda_{r,2}$.

Now we divide $\mathcal{T}(\mathbf{x})$ into $r+1$ parts $\mathcal{L}_k$ using $\mathcal{C}$ as follows: for $0 \leq k \leq r$, let $\mathcal{L}_k$ be the part of $\mathcal{T}(\mathbf{x})$ such that $\Lambda \in \mathcal{L}_k$ if and only if $\Lambda_{k,2}$ is the first vertex lattice of type 2 that the geodesic from Λ to $\Lambda_{0,2}$ encounters on $\mathcal{C}$. In other words, if we set $\mathcal{L}'_k$ to be the subtree of $\mathcal{T}(\mathbf{x}) \setminus \mathcal{C}$ that starts from $\Lambda_{k,2}$. Then $\mathcal{L}_k = \mathcal{L}'_k \cup \Lambda_{k,0}$ if $\Lambda_{k,0} \in \mathcal{T}(\mathbf{x})$. In particular, $\mathcal{L}_r = \{\Lambda_{r,2}\}$. Then we have

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) = \sum_{k=0}^{r} (S(k) - S'(k))$$

where

$$S(k) = \sum_{\Lambda_2 \in \mathcal{L}_k} n(b_1,\Lambda_2), \text{ and } S'(k) = \sum_{\Lambda_0 \in \mathcal{L}_k} (n(b_1,\Lambda_0)+1)$$

For $k = 0$ we have

$$S(0) = \sum_{i=0}^{r} (r-i)q^i,\ S'(0) = \sum_{i=0}^{r} (r+1-i)q^i.$$

For $1 \leq k \leq r-1$ we have

$$S(k) = r + k + \sum_{i=0}^{r-k-1} (r+k-i-1)(q-1)q^i,$$

$$S'(k) = r + k + 1 + \sum_{i=0}^{r-k-1} (r+k-i)(q-1)q^i.$$

For $k = r$ we have

$$S(r) = 2r,\ \ S'(r) = 0.$$

Summing these terms up, we obtain

$$\begin{aligned}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)\cdot\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) =& (S(0)-S'(0)) + \sum_{k=1}^{r-1}(S(k)-S'(k)) + S(r)\\ =& -\sum_{i=0}^{r} q^i - \sum_{k=1}^{r-1} q^{r-k} + 2r\\ =& -\frac{q^{r+1}-1}{q-1} - \frac{q(q^{r-1}-1)}{q-1} + 2r\\ =& -\frac{(q+1)(q^r-1)}{q-1} + 2r.\end{aligned}$$

as claimed.

□

By Theorems 5.7 and 5.9, we see that $\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2})$ depends only on the Gram matrix $T(b_1,b_2) = -\delta^2 T(\mathbf{x}_1,\mathbf{x}_2)$. Since $-\delta^2 \in \mathcal{O}_{F_0}^\times$, we see that the formulas in both theorems are not affected when we change $T(b_1,b_2)$ to $T(\mathbf{x}_1,\mathbf{x}_2)$. This proves Theorem 1.1.

6. Local densities and the Kudla-Rapoport Conjecture

In this section we record basic results on local density and prove Theorem 1.3. Let L be an integral $\mathcal{O}_F$-lattice with Gram matrix S (unique up to equivalence), and let T be an $n\times n$ invertible Hermitan matrix over T. Recall the local density polynomial $\alpha(L,T,X) = \alpha(S,T,X)$ defined in the introduction. the following explicit formulas are special cases considered in ([Shi20]).

Theorem 6.1. *([Shi20, Theorem 6.2]) Let L be an $\mathcal{O}_F$ Hermitian lattice with Gram matrix $S = diag(v,1)$ with $v \in \mathcal{O}_{F_0}^\times$, and let $\mathcal{H}$ be the Hermitian hyperbolic $\mathcal{O}_F$-plane defined in the introduction. Set $\epsilon_2 = \chi(-v)$, where χ be the quadratic character of $F_0^\times$ associated to the extension of F/F_0.*

(1) Assume that $T \approx \begin{pmatrix} u_1(-\pi_0)^\alpha & 0\\ 0 & u_2(-\pi_0)^\beta\end{pmatrix}$ with $\alpha\le\beta$ and $u_i \in \mathcal{O}_{F_0}^\times$. Set $\epsilon_1 = \chi(-u_1u_2)$. Then $\alpha(L,T,X) = \alpha(\mathcal{H},T,X) = 0$ unless $\alpha \ge 0$. Assume $\alpha\ge 0$, we have

$$\begin{aligned}\alpha(L,T,X) =& (1-X)(1+\epsilon_2+q\epsilon_2)\sum_{e=0}^{\alpha}(qX)^e - \epsilon_1 q^{\alpha+1}X^{\beta+1}(1-X)\sum_{e=0}^{\alpha}(q^{-1}X)^e\\ & -\epsilon_1(1+q)(X^{\alpha+\beta+2}+\epsilon_1\epsilon_2) + (1+\epsilon_2)q^{\alpha+1}X^{\alpha+1}(1+\epsilon_1X^{\beta-\alpha}),\end{aligned}$$

and

$$\alpha(\mathcal{H},T,X) = \left(1-q^{-2}X\right)\left(\sum_{e=0}^{\alpha}(qX)^e + \epsilon_1\sum_{e=\beta+1}^{\alpha+\beta+1} q^{\alpha+\beta+1-e}X^e\right)$$

In particular,

$$\alpha(L,T)=-\epsilon_1(1+\epsilon_1\epsilon_2)(1+q)+(1+\epsilon_1)(1+\epsilon_2)q^{\alpha+1},$$

which is zero if and only if $\epsilon_1\epsilon_2=-1$. *Similarly*

$$\alpha(\mathcal{H},T)=(1+\epsilon_1)(1+q^{-1})(q^{\alpha}-q^{-1}),$$

which is zero if and only if $\epsilon_1=-1$.

(2) *Assume that* $T=\begin{pmatrix}0&\pi^n\\(-\pi)^n&0\end{pmatrix}$ *where* n *is odd. Then* $\alpha(L,T,X)=\alpha(\mathcal{H},T,X)=0$ *unless* $n\geq -1$. *When* $n\geq -1$, *we have*

$$\begin{aligned}\alpha(L,T,X)=&-q^{n+2}(1-X)\sum_{e=\frac{n+1}{2}+1}^{n+1}(q^{-1}X)^e+(1-X)(1+\epsilon_2+\epsilon_2 q)\sum_{e=0}^{\frac{n+1}{2}}(qX)^e\\&-(q+1)(\epsilon_2+X^{n+2})+(1+\epsilon_2)(q+1)q^{\frac{n+1}{2}}X^{\frac{n+1}{2}+1},\end{aligned}$$

and

$$\alpha(\mathcal{H},T,X)=\left(1-q^{-2}X\right)\left(\sum_{e=0}^{\frac{n+1}{2}}(qX)^e+\sum_{\frac{n+3}{2}}^{n+1}q^{n+1-e}X^e\right).$$

In particular,

$$\alpha(L,T)=-(1+\epsilon_2)(q+1)(1-q^{\frac{n+1}{2}}),$$

which is zero if and only if $\epsilon_2=-1$. *Finally,*

$$\alpha(\mathcal{H},T)=\frac{1+q}{q^2}\left[-2+(1+q)q^{\frac{n+1}{2}}\right]\neq 0.$$

Proof of Theorem 1.3. First notice that the Gram matrix of L can be chosen to be of the form $S=\mathrm{diag}(v,1)$ with $v\in\mathcal{O}_{F_0}^{\times}$ and that the Gram matrix of $\mathcal{H}$ is S'. Let $T=T(\mathbf{x}_1,\mathbf{x}_2)$. As L and $L_{\mathbf{x}_1,\mathbf{x}_2}$ represent two different Hermitian spaces of dimension 2, we have to have $\epsilon_1\epsilon_2=-1$ when T can be diagonalized, and $\epsilon_2=-1$ when T is anti-diagonal. The case that L is isotropic is proved in [Shi20]. Now we assume L is anisotropic, i.e, $\epsilon_2=-1$. There are two cases.

Case 1: We first assume that $T\approx\mathrm{diag}(u_1(-\pi_0)^{\alpha},u_2(-\pi_0)^{\beta})$ with $\alpha\leq\beta$. In this case $\epsilon_1=1$. When $\alpha<0$, both sides of the identity are automatically zero. So we assume $\alpha\geq 0$. Theorem 6.1 implies that

$$\begin{aligned}&\alpha(L,T,X)\\=&-q(1-X)\sum_{e=0}^{\alpha}(qX)^e-q^{\alpha+1}X^{\beta+1}(1-X)\sum_{e=0}^{\alpha}(q^{-1}X)^e-(q+1)(X^{\alpha+\beta+2}-1).\end{aligned}$$

So

$$\begin{aligned}-\alpha'(L,T)=&\frac{\partial}{\partial X}\alpha(L,T,X)|_{X=1}\\=&q\sum_{e=0}^{\alpha}q^e+q^{\alpha+1}\sum_{e=0}^{\alpha}q^{-e}-(q+1)(\alpha+\beta+2)\\=&2\sum_{e=1}^{\alpha+1}q^e-(q+1)(\alpha+\beta+2).\end{aligned}$$

Theorem 6.1(1) also implies

$$\alpha(L,S)=2(1+q), \tag{6.1}$$

and

$$\alpha(\mathcal{H},T)=2(1+q^{-1})(q^{\alpha}-q^{-1}).$$

Hence the right hand side of the formula in Theorem 1.3 is

$$\begin{aligned}&\frac{2}{\alpha(L,S)}[\alpha'(L,T)-\frac{q^2}{q^2-1}\alpha(\mathcal{H},T)]\\ =&\frac{1}{q+1}[-2\sum_{e=1}^{\alpha+1}q^e+(q+1)(\alpha+\beta+2)+\frac{2}{q-1}-\frac{2q^{\alpha+1}}{q-1}]\\ =&\alpha+\beta+2-2\sum_{e=0}^{\alpha}q^e=\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2})\end{aligned}$$

as claimed by Theorem 1.1.

Case 2: Now we treat the anti-diagonal case

$$T\approx\begin{pmatrix}0&\pi^n\\(-\pi)^n&0\end{pmatrix},\ n\text{ odd}.$$

The case $n<-1$ is trivial as both sides are clearly zero.

When $n\geq -1$, and let $r=\frac{n+1}{2}$. we have by Theorem 6.1

$$\begin{aligned}\alpha'(L,T)=&-\frac{\partial}{\partial X}\alpha(L,T,X)|_{X=1}\\ =&1-(q+1)\sum_{e=0}^{r}q^e+(q+1)(n+2).\end{aligned}$$

Combining this with Theorem 6.1 (2), we see that the right hand side of Theorem 1.3 is

$$\begin{aligned}&\frac{2}{\alpha(L,S)}[\alpha'(L,T)-\frac{q^2}{q^2-1}\alpha(\mathcal{H},T)]\\ =&\frac{1}{q+1}-\sum_{e=0}^{r}q^e+2r+1-\frac{q^r}{q-1}+\frac{2}{(q-1)(q+1)}\\ =&\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2})\end{aligned}$$

as claimed by Theorem 1.3.

□

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