KUDLA PROGRAM FOR UNITARY SHIMURA VARIETIES

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Abstract. In this paper, we first review and summarize some recent progress in Kudla program on unitary Shimura varieties. We show how the local arithmetic Siegel-Weil formula implies the global arithmetic Siegel-Weil formula for non-singular coefficients on $\mathrm{U}(n,1)$. In particular, the arithmetic Siegel-Weil formula for non-singular coefficients on $U(1,1)$ is true.

1. Introduction

Kudla's seminal work [Kud97] started the Kudla program which studies the relation among (arithmetic) special cycles and their relations with values/derivatives of Eisenstein series and L-functions. Starting with low dimensional orthogonal Shimura varieties ([KRY99], [KRY04], and [KRY06]), a lot of progress has been made in last few years in both orthogonal and unitary cases. For example, The modularity of geometric theta series was proved by Borcherds ([?]), Wei Zhang ([?]), and Bruinier-Raum ([?]); The local arithmetic Siegel-Weil formula at unramified prime was conjectured by Kudla and Rapoport ([?]) and recently proved by Li and Zhang ([LZ19], see also [BY20] for special cases); The arithmetic Siegel-Weil formula at infinity was proved by Yifeng Liu ([Liu11]) in unitary case and by Bruinier-Yang in orthogonal case ([BY20]). Garcia and Sankaran proved both unitary and orthogonal cases in more general setting, including singular coefficients ([GS19]). Modularity of arithmetic theta series of arithmetic divisors was proved recently by Bruinier-Howard-Kudla-Rapoport-Yang in unitary case ([BHK$^+$17], [?]) and by Howard-Medapusi-Pera in orthogonal case ([?]). Li and Liu have recently proved the inner product formula ([?]), generalizing early result of Liu on $U(1,1)$ ([?]) These are just a few examples, here are some (incomplete) references for other work on Kudla program and in particular arithmetic Siegel-Weil formula ([HY12], [?], [?], [?], [?], [S$^+$13], [San14], [San17], [DY19], [?], [Shi18], [Shi20], [HSY20],[?], [?], [?]). In [?], Kulda outlined his program with orthogonal Shimura varieties as main example. In this paper, we will summarize some of the recent progress in unitary case and prove the arithmetic Siegel-Weil formula for $U(1,1)$ at all non-singular coefficients. We also show how to derive the arithmetic Siegel-Weil formula from local arithmetic Siegel-Weil formula and local Siegel-Weil (see Section 3) for non-singular coefficients.

In Section 2, we review basics on Eisenstein series on $\mathrm{U}(n,n)$, their Fourier coefficients and Siegel-Weil formula. The only new result is the local Siegel-Weil formula (Theorem 2.9), which describe the relation between the local Whittaker functions and local orbital integrals. Its analogue was proved in [KRY06] and [BY20] in orthogonal cases. In Section 3.1, we describe the geometric theta series of cycles and Kudla's geometric Siegel-Weil formula. In Section 3.2, we describe the modularity result of arithmetic theta series of arithmetic divisors proved in

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[BHK+17]. There are still a lot of unknown about arithmetic cycles of higher codimension, including the definition. In Section 3.3, we describe what is known and what are the basic questions to be resolved. It is interesting to notice that the highest codimension case (arithmetic 0-cyles) is easier to deal with and more is known. In Section 3.4, we describe the arithmetic Siegel-Weil formula and local arithmetic Siegel Weil formula in the non-singular case. We show that the arithmetic Siegel-Weil formula is local in nature in the non-singular case and follows from local arithmetic Siegel-Weil formula and local Siegel-Weil formula (see Theorems 3.7 and 3.10). The arguments at a finite and the infinite prime are the same. The local arithmetic Siegel-Weil formula is given in Theorem 3.6 at the infinite prime and Conjecture 3.8 at a finite prime. The conjecture is vague at 'bad' primes in general. In Section 4, we restrict ourselves to $U(1,1)$ Shimura curves, make Conjecture 3.8 precise and prove it, and thus prove the arithmetic Siegel-Weil formula for all primes.

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2. Eisenstein series

2.1. **Eisenstein series on** $\mathrm{U}(n,n)$**.** In this subsection, we review degenerate Eisenstein series on $G' = \mathrm{U}(n,n)$ and its Fourier expansion for convenience of the reader. Let F be an imaginary quadratic field with $\mathcal{O}_F$ its rings of integers. Let $\mathbb{A}$ (resp. $\mathbb{A}_F$) be the ring of Adeles of $\mathbb{Q}$ (resp. F). For an algebraic group G over $\mathbb{Q}$, we denote $G(\mathbb{Q})\backslash G(\mathbb{A})$ by $[G]$. Let Her_n be algebraic group over $\mathbb{Z}$ whose R-points consist of the Hermitian $n\times n$ matrices over entries in $\mathcal{O}_F \otimes R$, and let $G' = \mathrm{U}(n,n)$ be the algebraic group over $\mathbb{Z}$ such that

$$G'(R) = \{g \in \mathrm{GL}_{2n}(\mathcal{O}_F \otimes R) : g w {}^t\bar{g} = w\},$$

with $w = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$. Here I_n is the identity matrix of order n. Let $P' = N'M'$ be the standard Siegel parabolic subgroup of G with

$$N'(R) = \{n(b) = \left(\begin{smallmatrix} I_n & b \\ 0 & I_n \end{smallmatrix}\right) : b \in \mathrm{Her}_n(R)\},$$

and

$$M'(R) = \{m(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{smallmatrix}\right) : a \in \mathrm{GL}_n(\mathcal{O}_F \otimes R)\}.$$

Notice that M' is isomorphic to $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$. More generally, we set for $0 \le r \le n$

$$N'_r = \{n_r(b) = n(\left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right)) : b \in \mathrm{Her}_r(\mathbb{Q})\}$$

and

$$w_r = \begin{pmatrix} I_{n-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_r \\ 0 & 0 & I_{n-r} & 0 \\ 0 & I_r & 0 & 0 \end{pmatrix}.$$

One has $w_r = w$ and $n_r(b) = n(b)$ for $r = n$. We will simply write $N' = N'(\mathbb{Q})$, $N'_r = N'_r(\mathbb{Q})$, and $M' = M'(\mathbb{Q})$. Sometimes we also do the same for G' and P' if the context is clear.

Recall the Bruhat decomposition

$$G' = \bigcup_{r=0}^{n} P' w_r P'. \tag{2.1}$$

Lemma 2.1. *For $0 \le r \le n$, let Q_r be the standard paraboblic subgroup of GL_n consisting of the matrices $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ with D being square matrices of order r. Let*

$$M'_r = \{m(a) : a \in Q_r(F) \backslash \mathrm{GL}_n(F)\}.$$

Then

$$P'(\mathbb{Q}) \backslash G'(\mathbb{Q}) = \bigcup_{r=0}^{n} w_r N'_r M'_r.$$

Note that the term with $r = 0$ is simply $\{1\}$ and the term with $r = n$ is wN. The proof amounts to verify

$$P' \backslash P' w_r P' = w_r N'_r M'_r$$

via direct calculation and is left to the reader.

Let

$$\mathbb{H}_n^u = \{\tau \in M_n(\mathbb{C}) : \frac{\tau - {}^t\bar{\tau}}{2i} > 0\}$$

be the Hermitian symmetric domain for $G'(\mathbb{R})$. Every $\tau \in \mathbb{H}_n^u$ can be written uniquely as

$$\tau = u + iv, \quad u \in \mathrm{Her}_n(\mathbb{R}), \quad v \in \mathrm{Her}_n^+(\mathbb{R}),$$

with $u = \frac{1}{2}(\tau + {}^t\bar{\tau})$ and $v = \frac{1}{2i}(\tau - {}^t\bar{\tau})$. Here $\mathrm{Her}_n^+(\mathbb{R})$ is the set of positive definite Hermitian matrices of order n. We will simply write $v > 0$ for $v \in \mathrm{Her}_n^+(\mathbb{R})$. The group $G'(\mathbb{R})$ acts on $\mathbb{H}_n^u$ linear fractionally

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)(\tau) = (A\tau + B)(C\tau + D)^{-1}. \tag{2.2}$$

The stabilizer of i is a maximal subgroup of $G'(\mathbb{R})$ given by

$$K'_\infty = \{\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) : A^t\bar{A} + B^t\bar{B} = I, \quad A^t\bar{B} = B^t\bar{A}\} = U(2n) \cap \mathrm{U}(n,n) = \mathrm{U}(n) \times \mathrm{U}(n),$$

with identification $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \mapsto (A + iB, A - iB)$. Write $g_\tau = n(u)m(a)$ with $a^t\bar{a} = v$. We can choose and will a with $\det a > 0$ in this paper. Notice that $g_\tau(i) = \tau$. This implies that $G'(\mathbb{R})$ acts on $\mathbb{H}^u$ transitively, and

$$G'(\mathbb{R}) = P'(\mathbb{R}) K'_\infty = N'(\mathbb{R}) M'(\mathbb{R}) K'_\infty.$$

Let $\chi = \prod \chi_v$ be a unitary idele class character (Hecke character) of $F^\times \backslash \mathbb{A}_F^\times$, and extend it to a character of $P'(\mathbb{Q}) \backslash P'(\mathbb{A})$ via $\chi(n(b)m(a)) = \chi(\det a)$. Consider the induced representation $I(s, \chi) = \mathrm{Ind}_{P'(\mathbb{A})}^{G'(\mathbb{A})} \chi |\,|_{\mathbb{A}}^s$, consisting of smooth functions Φ on $G'(\mathbb{A})$ satisfying

$$\Phi(n(b)m(a)g, s) = \chi(\det a) |\det a|^{s + \rho_n} \Phi(g, s), \quad \rho_n = \frac{n}{2}.$$

An element $\Phi \in I(s, \chi)$ is often called a section. It is called factorizable if $\Phi = \prod_{p \le \infty} \Phi_p$ with $\Phi_p \in I(s, \chi_p)$ (the local induced representation). We say that Φ_∞ is of weight $m = (m_1, m_2)$ if for all $k = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \in K'_\infty$

$$\Phi_\infty(gk, s) = \Phi(g, s)(\det(A + iB))^{m_1} (\det(A - iB))^{m_2}. \tag{2.3}$$

We will write Φ_∞^m for the unique section of weight $m = (m_1, m_2)$ with $\Phi_\infty^m(1) = 1$. Notice that the restriction of Φ_∞^m on Sp_{2n} is of weight $m_1 - m_2$.

For a section $\Phi \in I(s, \chi)$, the associated Eisenstein series is defined to be

$$E(g, s, \Phi) = \sum_{\gamma \in P'(\mathbb{Q}) \backslash G'(\mathbb{Q})} \Phi(\gamma g, s) \tag{2.4}$$

which is absolutely convergent when $\Re(s)$ is big, has memomorphic analytic continuation to the whole complex s-plane, and is holomorphic on the unitary axis $\Re(s) = 0$. The Eisenstein series has the following Fourier expansion:

$$E(g, s, \Phi) = \sum_{T \in \mathrm{Her}_n(\mathbb{Q})} E_T(g, s, \Phi), \tag{2.5}$$

where $\psi = \prod \psi_p$ is the 'canonical' additive character of $\mathbb{A}/\mathbb{Q}$, and

$$E_T(g, s, \Phi) = \int_{[\mathrm{Her}_n]} E(n(b)g, s, \Phi)\psi(-\operatorname{tr}(bT))db \tag{2.6}$$

is the T-th Fourier coefficient of $E(g, s, \Phi)$.

For a Hermitian matrix T and an invertible matrix a of the same size, we write $T[a] = aTa^*$ and $a^* = {}^t\bar{a}$.

Theorem 2.2. *Let $0 \le r \le n$ and $T \in \mathrm{Her}_n(\mathbb{Q})$. Let*

$$A_r(T) = \{A \in Q_r(F) \backslash \mathrm{GL}_n(F) : T[{}^t\bar{a}^{-1}] = \left(\begin{smallmatrix} 0 & 0 \\ 0 & T_a \end{smallmatrix}\right) \text{ for some } a \in A\}$$

for some Hermitian matrix T_a of order r. Finally let, for $T' \in \mathrm{Her}_r(\mathbb{Q})$,

$$W_{T'}^{(r)}(g, s, \Phi) = \int_{\mathrm{Her}_r(\mathbb{A})} \Phi(w_r n_r(b) g, s)\psi(-\operatorname{tr}(bT')db,$$

the product of local 'Whittaker' functions of lever r. Then

$$E_T(g, s, \Phi) = \sum_{r \ge r(T)} \sum_{[a] \in A_r(T)} W_{T_a}^{(r)}(m(a)g, s, \Phi).$$

Here $r(T)$ is the rank of T. In particular, when T is non-singular and factorizable,

$$E_T(g, s, \Phi) = W_T(g, s, \Phi) = \prod_p W_{T,p}(g, s, \Phi_p)$$

is the product of local Whittaker functions.

Proof. Lemma 2.1 and substitution $b[a] \mapsto b$ imply

$$\begin{aligned} E_T(g, s, \Phi) &= \sum_{r=0}^{n} \int_{[\mathrm{Her}_n]} \sum_{c \in \mathrm{Her}_r, m(a) \in M_r} \Phi(w_r n_r(c) m(a) n(b) g, s)\psi(-\operatorname{tr}(bT))db \\ &= \sum_{r=0}^{n} \int_{[\mathrm{Her}_n]} \sum_{c \in \mathrm{Her}_r, m(a) \in M_r} \Phi(w_r n_r(c) n(b) m(a) g, s)\psi(-\operatorname{tr}(bT[{}^t\bar{a}^{-1}]))\, db. \end{aligned}$$

Write $b = \left(\begin{smallmatrix} b_1 & b_{12} \\ b_{12}^* & b_2 \end{smallmatrix}\right)$, and note

$$w_r n(\left(\begin{smallmatrix} b_1 & b_{12} \\ b_{12}^* & 0 \end{smallmatrix}\right)) w_r^{-1} = n(\left(\begin{smallmatrix} b_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)) m(\left(\begin{smallmatrix} 1 & -b_{12} \\ 0 & 1 \end{smallmatrix}\right)),$$

we see

$$\int_{[\mathrm{Her}_n]} \sum_{c\in \mathrm{Her}_r, m(a)\in M_r} \Phi(w_r n_r(c) n(b) m(a) g, s)\psi(-\operatorname{tr}(bT[{}^t\bar{a}^{-1}]))\, db$$

$$= \int_{\mathrm{Her}_r(\mathbb{A})} \int_{\substack{b_1 \in [\mathrm{Her}_{n-r}] \\ b_{12} \in [\mathrm{Res}_{F/\mathbb{Q}} M_{n-r,r}]}} \sum_{m(a)\in M_r} \Phi(w_r n_r(b_2) m(a) g, s)\psi(-\operatorname{tr}(bT[{}^t\bar{a}^{-1}])\, db$$

$$= \sum_{[a]\in A_r(T)} W^{(r)}_{T_a}(m(a)g, s, \Phi)$$

as claimed. □

Let

$$j_r : \mathrm{U}(r,r) \to \mathrm{U}(n,n), \quad \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mapsto \begin{pmatrix} I_{n-r} & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & I_{n-r} & 0 \\ 0 & C & 0 & D \end{pmatrix} \tag{2.7}$$

be a natural embedding. To distinguish the size r on induced representation, we put superscript in $I^{(r)}(s,\chi)$ to indicate that it is an induced representation on $\mathrm{U}(r,r)$. In particular, $I^{(n)}(s,\chi) = I(s,\chi)$. Then embedding map j_r induces a $\mathrm{U}(r,r)$-equivariant map for every $g \in G'(\mathbb{A})$:

$$j^*_{r,g} : I^{(n)}(s,\chi) \to I^{(r)}(s + \frac{n-r}{2}, \chi), \quad (j^*_{r,g}\Phi)(h,s) = \Phi(j_r(h)g, s). \tag{2.8}$$

The following proposition is clear.

Proposition 2.3. *Let the notation be as in Theorem 2.2. For $\Phi \in I(s,\chi)$ and $g \in G'(\mathbb{A})$, we have*

$$W^{(r)}_T(g, s, \Phi) = W_{T_a}(m(a), s + \frac{n-r}{2}, j^*_{r,g}\Phi)$$

is a (usual) Whittaker function for a section $j^*_{r,g}\Phi \in I^{(r)}(s + \frac{n-r}{2}, \chi)$.

Proposition 2.4. *Assume T is a positive semi-definite with rank $r(T) = n-1$. Choose $a \in \mathrm{GL}_n(F)$ such that $T[{}^t\bar{a}^{-1}] = \mathrm{diag}(0, T_a)$. Then*

$$E_T(g,s,\Phi) = W_{T_a}(1, s + \frac{1}{2}, \Phi^{(1)}_{m(a)g}) + W_{T_a}(1, s - \frac{1}{2}, \Phi^{(2)}_{m(a^{-1})g})$$

is the sum of T_a-th coefficients of two Eisenstein series on $\Phi^{(1)}_g = j^*_{n-1,g}\Phi \in I^{(n-1)}(s + \frac{1}{2}, \chi)$ and $\Phi^{(2)}_g \in I^{(n-1)}(s - \frac{1}{2}, \chi)$. Here

$$\Phi^{(2)}_g(h) = \int_{b_1\in\mathbb{A}} \int_{b_{12}\in\mathbb{A}_F^{n-1}} \Phi(w_n n(\begin{pmatrix} b_1 & b_{12} \\ b_{12}^* & 0 \end{pmatrix}) w_{n-1}^{-1} j_{n-1}(h) g, s) db_1\, db_{12}. \tag{2.9}$$

Proof. For $c \in \mathrm{Her}_r$, we will denote $n_-(c) = \left(\begin{smallmatrix} I_r & 0 \\ c & I_r \end{smallmatrix}\right)$ and $n(c) = \left(\begin{smallmatrix} I_r & c \\ 0 & I_r \end{smallmatrix}\right)$. Then $j_r(n(c)) = n(\left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right))$. In this proof, let $r = n-1$. It is easy to verify

$$\begin{aligned} & w_n n(\begin{pmatrix} b_1 & b_{12} \\ b_{12}^* & 0 \end{pmatrix}) w_r^{-1} j_r(n(c)) \\ &= w_n n_-(\left(\begin{smallmatrix} 0 & 0 \\ 0 & -c \end{smallmatrix}\right)) w_n^{-1} w_n n_-(\left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right)) n(\begin{pmatrix} b_1 & b_{12} \\ b_{12}^* & 0 \end{pmatrix}) n_-(\left(\begin{smallmatrix} 0 & 0 \\ 0 & -c \end{smallmatrix}\right)) w_r^{-1} \\ &= n(\left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right)) w_n m(\left(\begin{smallmatrix} 1 & -b_{12}c \\ 0 & 1 \end{smallmatrix}\right)) w_n^{-1} w_n n(\begin{pmatrix} b_1 - b_{12}cb_{12}^* & b_{12} \\ b_{12}^* & 0 \end{pmatrix}) w_r^{-1}. \end{aligned}$$

So

$$\begin{aligned}\Phi_g^{(2)}(n(c)h) &= \int_{\mathbb{A}}\int_{\mathbb{A}_F^r} \Phi(w_n n(\left(\begin{smallmatrix} b_1 - b_{12}cb_{12}^* & b_{12} \\ b_{12}^* & 0\end{smallmatrix}\right)) w_r^{-1} j_r(h)g, s) db_1\, db_{12} \\ &= \Phi_g^{(2)}(h).\end{aligned}$$

Similarly, for $a \in \mathrm{GL}_r(\mathbb{A}_F)$,

$$w_n n(\left(\begin{smallmatrix} b_1 & b_{12} \\ b_{12}^* & 0\end{smallmatrix}\right)) w_r^{-1} j_r(m(a)) = m((\begin{smallmatrix} 1 & 0 \\ 0 & a\end{smallmatrix})) w_n n(\left(\begin{smallmatrix} b_1 & b_{12}a \\ (b_{12}a)^* & 0\end{smallmatrix}\right)) w_r^{-1}.$$

This implies

$$\Phi_g^{(2)}(m(a)h) = |\det a|^{s+\frac{n}{2}} |\det a|^{-1} \Phi_g^{(2)}(h) = |\det a|^{s-\frac{1}{2}+\frac{n-1}{2}} \Phi_g^{(2)}(h).$$

So $\Phi_g^{(2)} \in I^{(n-1)}(s - \frac{1}{2}, \chi)$. Now a substitution $b \mapsto b[a]$ gives

$$\begin{aligned}W_T(g, s, \Phi) &= \int_{\mathrm{Her}_n(\mathbb{A})} \Phi(wn(b)g, s)\psi(-\operatorname{tr} bT) db \\ &= \int_{\mathrm{Her}_r(\mathbb{A})} \int_{b_1 \in \mathbb{A}} \int_{b_{12} \in \mathbb{A}_F^r} \Phi(w_n m(a) n(b) m(a^{-1})g, s)\psi(-\operatorname{tr}(b_2 T_a)) db_{12}\, db_1\, db_2 \\ &= \int_{\mathrm{Her}_r(\mathbb{A})} \Phi^{(2)}_{m(a^{-1})g}(wn(b_2))\psi(-\operatorname{tr}(b_2 T_a)) db_2 \\ &= W_{T_a}(1, s - \frac{1}{2}, \Phi^{(2)}_{m(a^{-1})g}).\end{aligned}$$

Since $T_a > 0$, it is easy to check that $A_r(T)$ consists of one element $[a]$. Now the proposition follows from Theorem 2.2 and Proposition 2.3.

□

2.2. **Weil representation, Rallis map, and Eisenstein series.** Let V_p be a (non-degenerate) Hermitian space over F_p of dimension m. Let χ be an idele class character of $\mathbb{A}_F^\times$ with $\chi|_{\mathbb{A}^\times} = \epsilon_{F/\mathbb{Q}}^m$, where $\epsilon_{F/\mathbb{Q}}$ is the quadratic idele character of $\mathbb{A}^\times$ associated to $F/\mathbb{Q}$ (i.e., the Dirichelet character associated to $F/\mathbb{Q}$). Let $K' = K^{(n)} = \prod_p K'_p$ be the maximal compact subgroup of $\mathrm{U}(n,n)(\mathbb{A})$ with

$$K'_p = \begin{cases} \mathrm{U}(n,n)(\mathbb{Z}_p) & \text{if } p < \infty, \\ \mathrm{U}(n) \times \mathrm{U}(n) & \text{if } p = \infty.\end{cases}$$

The reductive dual pair $(\mathrm{U}(V_p), \mathrm{U}(n,n)(\mathbb{Q}_p))$ gives a Weil representation $\omega_{\chi_p} = \omega_{V_p,\chi}$ of $\mathrm{U}(n,n)(\mathbb{Q}_p)$ on the Schwartz function space $S(V_p^n)$. In particular,

$$\begin{aligned}&\omega_{\chi_p}(n(b))\phi(x) = \psi_p(\operatorname{tr}(b(x,x)))\phi(x), \\ (2.10)\qquad &\omega_{\chi_p}(m(a))\phi(x) = \chi_p(\det a)|\det a|_{F_p}^{\frac{n}{2}}\phi(xa), \\ &\omega_{\chi_p}(w)\phi(x) = \gamma(V_p^n)\hat{\phi}(x) = \gamma_p(V_p^n)\int_{V_p^r} \phi(y)\psi(-\operatorname{tr}_{F_p/\mathbb{Q}_p}\operatorname{tr}(x,y)))dy,\end{aligned}$$

where $\gamma(V_p^n)$ is some local Weil index (a root of unity). On the other hand, $\mathrm{U}(V_p)$ acts on $S(V_p^n)$ linearly. From the above formula, one can see that there is an $\mathrm{U}(n,n)$-intertwining operator–Rallis map

$$(2.11)\qquad \lambda_p : S(V_p^n) \to I(s_n, \chi_p), \quad \lambda_p(\phi)(g) = \omega_{\chi_p}(g)\phi(0),$$

where $s_n = \frac{m-n}{2}$. Let $\Phi = \Phi_\phi \in I(s, \chi_p)$ be the associated standard section (i.e., $\Phi|_{K_p}$ is independent of s) such that $\Phi(s_r) = \lambda_p(\phi)(g)$. We will simply denote $\lambda_p(\phi)$

for this Φ_ϕ. We first record some general results on local Weil representation and Rallis map, which are of independent interest and are in more general situation.

Let F/F_0 be a quadratic etale extension of a local field F_0 of residue character p. Let V be a (non-degenerate) Hermitian space over F of dimension m and let χ be a character of $F^\times$ with $\chi|_{F_0^\times} = \epsilon_{F/F_0}^m$, where ϵ_{F/F_0} is the quadratic character of $F_0^\times$ associated to F/F_0. When $p < \infty$, let ψ be an unramified character of F_0 and $\psi_F(x) = \psi(\mathrm{tr}_{F/F_0}(x))$. When $p = \infty$, $F_0 = \mathbb{R}$ and $F = \mathbb{C}$, let $\psi(x) = e(x) = e^{2\pi i x}$. Let $\omega_\chi = \omega_{V,\chi,\psi}$ be the associated Weil representation of $\mathrm{U}(n,n)$ on $S(V^n)$ and λ the associated Rallis map.

Lemma 2.5. *Assume $p < \infty$ and let $\mathcal{H} = \partial^{-1} \oplus \mathcal{O}_F$ be the hyperbolic plane with Hermitian form $(y,z) = y_1\bar{z}_2 + y_2\bar{z}_1$, where $\partial = \partial_{F/F_0}$ is the relative different of F/F_0. Then the following is true.*

(1) $K' = \mathrm{U}(n,n)(\mathcal{O}_{F_0})$ acts on $\mathrm{Char}(\mathcal{H}^r)$ trivially via the Weil representation ω_1, where 1 stands for the trivial character of $F^\times$.

(2) Let $V_t = V \oplus (\mathcal{H} \otimes_{\mathcal{O}_F} F)^t$, and $\phi^{(t)} = \phi \otimes \phi_t$, with $\phi_t = \mathrm{Char}((\mathcal{H}^t)^n)$. Let Φ and $\Phi^{(t)}$ be the standard sections in $I(s,\chi)$ associated to ϕ and $\phi^{(t)}$ respectively. Then

$$\Phi(g, s_n + t) = \Phi^{(t)}(g, s_n + t) = \omega_\chi(g)\phi^{(t)}(0).$$

In particular,

$$W_T(g, s_n + t, \phi) = W_T(g, s_n + t, \phi^{(t)}).$$

Proof. (sketch) (1) $K' = \mathrm{U}(n,n)(\mathcal{O}_{F_0})$ is generated by Weyl group elements w_j, $n(b)$, and $m(a)$ with $b \in \mathrm{Her}_n(\mathcal{O}_{F_0})$, $a \in \mathrm{GL}_n(\mathcal{O}_F)$. Since $\mathcal{H}$ is unimodular, it is easy to check for $\phi_1 = \mathrm{Char}(\mathcal{H}^n)$

$$\omega_1(w_j)\phi_1 = \phi_1.$$

It is clear that

$$\begin{aligned}\omega_1(n(b))\phi_1(x) &= \psi(\mathrm{tr}\, b(x,x))\phi_1(x) = \phi_1(x),\\ \omega_1(m(a))\phi_1(x) &= |\det a|_F \phi_1(xa) = \phi_1(x).\end{aligned}$$

This proves (1). For (2), notice that every $g \in \mathrm{U}(n,n)$ can be written as $g = n(b)m(a)k$ with $k \in K'$. In this case,

$$\begin{aligned}\Phi^{(t)}(g, s_n + t) &= \omega_\chi \phi^{(t)}(0)\\ &= \omega_\chi(g)\phi(0)\omega_1(n(b)m(a)k)\phi_r(0)\\ &= \omega_\chi(g)\phi(0)|\det a|_F^t\\ &= \Phi(g, s_n + t)\end{aligned}$$

as claimed. □

The above lemma is an analogue of [Kud97, Appendix]. Similarly, Kudla and Millson show the following result,

Lemma 2.6. *(1) Assume that V_∞ is of signature (p,q) with $p + q = m$. Let $\chi(z) = (\frac{z}{|z|})^{\kappa(\chi)}$ for an integer $\kappa(\chi)$ (with $\chi(-1) = (-1)^{\kappa(\chi)}$). Given an orthogonal decomposition*

$$V_\infty = V^+ \oplus V^-, \quad x = x^+ + x^-,$$

where $V^{\pm}$ are positive (negative) definite subspaces of V_∞. Then the Gaussian majorant

$$\phi_\infty(x) = e^{-\pi \operatorname{tr}[(x^+, x^+) - (x^-, x^-)]}$$

is a $K' = \mathrm{U}(n) \times \mathrm{U}(n)$-eigenfunction, under the Weil representation ω_χ. The associated standard section $\Phi_\infty^\ell \in I(s, \chi)$ is of weight $\ell = (\frac{p-q+\kappa(\chi)}{2}, \frac{-p+q+\kappa(\chi)}{2})$. In particular, the Gaussian ϕ_∞ on a positive definite V_∞ gives the standard section Φ_∞^ℓ on $\mathrm{U}(n,n)(\mathbb{R})$ of weight $\ell = (\frac{\kappa(\chi)+m}{2}, \frac{\kappa(\chi)-m}{2})$.

(2) Let $\mathcal{H} = \mathbb{C}^2$ be the hyperbolic Hermitian plane with Hermitian form $(y, z) = y_1\bar{z}_1 - y_2\bar{z}_2$, and let $\phi_t = \phi_\infty^{\otimes t} \in S(\mathcal{H}^t)$ with ϕ_∞ being a Gaussian majorant defined above associated to $\mathcal{H}$, and let $\phi^{(t)} = \phi \otimes \phi_t$ for any $\phi \in S(V^r)$. Let Φ and $\Phi^{(t)} \in I(s, \chi)$ be the standard sections associated to ϕ and $\phi^{(t)}$, then

$$\Phi(g, s_n + t) = \Phi^{(t)}(g, s_n + t),$$

and in particular

$$W_T(g, s_n + t, \phi) = W_T(g, s_n + t, \phi^{(t)}).$$

Lemma 2.7. *Assume that F/F_0 is unramified and that $L \subset V$ is an $\mathcal{O}_F$-unimodular lattice. Then $\Phi = \lambda(\mathrm{Char}(L^n)) \in I(s, \chi)$ is spherical, i.e., $\Phi_p|_{K_p'} = 1$. Here χ is an unramified character of $F^\times$ with $\chi|_{F_0^\times} = \epsilon_{F/F_0}^m$.*

Now let V be a global (non-degenerate) Hermitian space over F of dimension m, and let

$$\lambda = \otimes \lambda_p : S(V^n(\mathbb{A})) \to I(s_n, \chi)$$

be the $\mathrm{U}(n,n)(\mathbb{A})$-equivariant map. For a Schwartz function $\phi = \otimes\phi_p \in S(V^n(\mathbb{A}))$, there are two ways to produce modular forms on $[\mathrm{U}(n,n)]$: one is the Eisenstein series $E(g, s, \phi) = E(g, s, \lambda(\phi))$. The other is the so-called theta integral. Recall the theta kernel

$$\theta(g, h, \phi) = \sum_{x \in V^r} \omega_\chi(g)\phi(h^{-1}x),$$

which is an automorphic form on $[\mathrm{U}(V) \times \mathrm{U}(n,n)]$. Averaging over $[\mathrm{U}(V)]$, the theta integral, if convergent,

$$I(g, \phi) = \frac{1}{\mathrm{Vol}([\mathrm{U}(V)])} \int_{[\mathrm{U}(V)]} \theta(g, h, \phi) dh \tag{2.12}$$

is an automorphic form on $\mathrm{U}(n,n)$. The following is a special case of the well-known Siegel-Weil formula, as developed by Siegel, Weil, Kudla-Rallis, Ichino, and Gan-Qiu-Takeda.

Theorem 2.8. *(Siegel-Weil formula) Let the notation be as above, and assume that $r = 0$ or $m - r > n$ (Weil convergency condition), where r is the Witt split index of V. Then the theta integral $I(g, \phi)$ is absolutely convergent, the Eisenstein series is holomorphic at $s = s_n$, and*

$$I(g, \phi) = \kappa E(g, s_n, \phi).$$

Here $\kappa = 2$ or 1 depending on whether $s_n = 0$ or not.

Now we briefly describe its local analogue—local Siegel-Weil formula, which will be used in Section 3 in proving arithmetic Siegel-Weil formula. We refer to [BY20, Section 2] and [KRY06, Section 5.3.1] for detail. Let $F_0 = \mathbb{R}$ or a p-adic local field, and let F be a quadratic extension of F_0. Let ψ be an non-trivial additive character.

Let V be a non-degenerate Hermitian space over F of dimension n with $H = \mathrm{U}(V)$, and let $G' = \mathrm{U}(n,n)$. For a $n \times n$ Hermitian matrix T, let

$$\Omega(T) = \{x \in V^n : T(x) := (x,x) = T\}.$$

Then for a non-singular T, one has

$$H \cong \Omega(T) : h \mapsto hx.$$

Through the moment map

$$T : V^n \to \mathrm{Her}_n : x \mapsto T(x) = (x,x)$$

and the above identification $H \cong \Omega(T)$, one can show that there is a unique Haar measure dh associated to prefixed Haar measures on V^n and Her_n such that for any $\phi \in S(V^n)$

$$\int_{V^n} \phi(v)dv = \int_{\mathrm{Her}_n} O_T(\phi)dT, \tag{2.13}$$

where

$$O_T(\phi) = \int_{h\in H} \phi(h^{-1}x)dh \tag{2.14}$$

is the orbit integral of ϕ over $\Omega(T)$. Here x is any element of $\Omega(T)$.

Theorem 2.9. *(local Siegel-Weil formula) Let dh be the Haar measure on H associated to the self-dual Haar measures on V^n and Her_n (with respect to ψ), as determined by (2.13). Then for any $\phi \in S(V^n)$*

$$W_T(1,0,\phi) = \gamma(V^n)O_T(\phi) \tag{2.15}$$

for all non-singular $T \in \mathrm{Her}_n$. Moreover,

(1) *When F_0 is a p-adic local field and $L \subset V$ is a lattice. Let $K_L = \{h \in H : hL = L\}$ and $\phi_L = \mathrm{Char}(L^n)$. Let T be a Gram matrix of L, then*

$$W_T(1,0,\phi_L) = \gamma(V^n)\,\mathrm{Vol}(K_L, dh).$$

(2) *When $F_0 = \mathbb{R}$, $F = \mathbb{C}$, and V positive definite, let $\phi(x) = e^{-2\pi\,\mathrm{tr}(x,x)} \in S(V^n)$. In this case, $H = \mathrm{U}(n)$. Then for any positive definite Hermitian matrix T of order n,*

$$W_T(\tau,0,\phi) = \gamma(V^n)\,\mathrm{Vol}(\mathrm{U}(n),dh)q^T, \quad q^T = e^{2\pi i\,\mathrm{tr}(T\tau)}.$$

Proof. (sketch) (2.15) follows formally from

$$\begin{aligned} W_T(1,0,\phi) &= \int_{\mathrm{Her}_n} \gamma(V^n) \int_{V^n} \phi(x)\psi(\mathrm{tr}(b(x,x)))dx\psi(-\mathrm{tr}(bT))db \\ &= \int_{\mathrm{Her}_n} \gamma(V^n) \int_{\mathrm{Her}_n} O_{T'}(\phi)\psi(\mathrm{tr}\, b(T'-T))dT'db \\ &= \gamma(V^n)O_T(\phi). \end{aligned}$$

(1) follows directly from (2.15) and the fact that $(x,x) = T, x \in L^n$ implies that x_i form a basis of L.

For (2), write $\tau = u + iv$ and $v = a^t\bar{a}$ with $\det a > 0$. Then

$$\begin{aligned} W_T(\tau, 0, \phi) &= (\det v)^{-n/2} W_T(n(u)m(a), 0, \phi) \\ &= \psi(\mathrm{tr}(Tu)) W_T(1, 0, \phi_a) \\ &= \gamma(V^n)\psi(\mathrm{tr}(Tu)) O_T(\phi_a) \\ &= \gamma(V^n) \operatorname{Vol}(\mathrm{U}(n), dh) q^T \end{aligned}$$

as claimed. Here $\phi_a(x) = \phi(xa)$. □

2.3. **Incoherent Hermitian spaces and Incoherent Eisenstein Series.** The case $m = n$ is more interesting as observed by Kudla ([Kud97]). Locally let $R(V_p)$ be the image of λ_p in $I(0, \chi_p)$, which is always irreducible. When p is split in F, there is only one non-degenerate Hermitian space of dimension n (denoted by V_p^+) and $I(0, \chi_p) = R(V_p^+)$. When p is finite and non-split in F, there are exactly two non-degenerate Hermitian spaces $V_p^\pm$ of dimension n, depending on

$$\epsilon(V_p) = \epsilon_{F_p/\mathbb{Q}_p}((-1)^{n(n-1)/2} \det V_p), \tag{2.16}$$

and

$$I(0, \chi_p) = R(V_p^+) \oplus R(V_p^-).$$

When $p = \infty$, one has

$$I(0, \chi_\infty) = \oplus_{q=0}^n R(V_\infty^{p,q}),$$

where $V_\infty^{p,q}$ is the Hermitian space over $\mathbb{C}$ of signature (p, q). Putting them together, one has

$$I(0, \chi) = \oplus_V R(V) \oplus (\oplus_{\mathcal{C} \text{ incoherent}} R(\mathcal{C})). \tag{2.17}$$

Here the first sum is over all global Hermitian spaces over F of dimension n, $R(V) = \otimes R(V_p)$, and the second sum is over all **incoherent** Hermitian spaces $\mathcal{C} = \otimes C_p$ over $\mathbb{A}_F$ of dimension n, in the sense that it cannot come from a global Hermitian space, i.e., $\epsilon(\mathcal{C}) = \prod_p \epsilon(\mathcal{C}_p) = -1$. The Siegel-Weil formula deals with those sections in $R(V)$. The 'incoherent' Eisenstein series from the sections in $R(\mathcal{C})$, as Kudla called it, turns out to be more interesting. Let $\phi = \otimes\phi_p \in S(\mathcal{C}^n)$, and let $E(g, s, \phi)$ be the associated Eisenstein series for $\lambda(\phi)$. Then Kudla observed that the central value vanishes automatically: $E(g, 0, \phi) = 0$. A natural question is what does the central derivative $E'(g, 0, \phi)$ stand for?

We will assume that $\mathcal{C}_\infty$ is positive definite. So there is a global Hermitian space V of signature $(n-1, 1)$ with $V(\mathbb{A}_f) \cong \mathcal{C}_{\mathbb{A}_f}$, and an associated Shimura variety M (see Section 3 for detail). Kudla had a conjectured formula to reinterpret the central derivative $E'(g, 0, \phi)$ via arithmetic cycles on the integral model of X—the (mostly conjectured) arithmetic Siegel-Weil formula ([Kud97], [**?**]). Before we start to review the Shimura varieties and special divisors in next section, we recall the following fact for $r(T) = n$. When $r(T) = n$, T is non-singular, Theorem 2.2 asserts

$$E_T(g, s, \phi) = \prod_{p \le \infty} W_{T,p}(g_p, s, \phi_p).$$

Let

$$\mathrm{Diff}(\mathcal{C}, T) = \{p \le \infty : \det \mathcal{C}_p / \det T \notin \mathrm{N}_{F/\mathbb{Q}} \mathbb{A}_F^\times\}. \tag{2.18}$$

Proposition 2.10. *The following are true.*

(1) $|\mathrm{Diff}(\mathcal{C}, T)| \ge 1$ *is odd.*

(2) *If* $p \in \mathrm{Diff}(\mathcal{C}, T)$*, then for every* $\phi \in S(\mathcal{C}^n)$*, the local Whittaker function vanishes at the center* $s = 0$*:* $W_{T,p}(g, 0, \phi) = 0$*. The converse is true for* $p < \infty$*.*

(3) *We have*

$$\mathrm{ord}_{s=0} E_T(g, s, \phi) \geq |\,\mathrm{Diff}(\mathcal{C}, T)|.$$

(4) $E'_T(g, 0, \phi) = 0$ *unless* $\mathrm{Diff}(\mathcal{C}, T) = \{p\}$ *consists of a single prime* p*. In such a case*

$$E'_T(g, 0, \phi) = \frac{W'_{T,p}(g_p, 0, \phi_p)}{W_{T,p}(g_p, 0, \tilde{\phi}_p)} E_T(g, 0, \tilde{\phi}).$$

Here $\tilde{V}$ *is a neighboring global Hermitian space of* $\mathcal{C}$ *in the sense* $\tilde{V}_q \cong \mathcal{C}_q$ *for* $q \neq p$ *and* $\tilde{V}_p$ *and* $\mathcal{C}_p$ *give two different Hermitian spaces over* F_p *of the same dimension,* $\tilde{\phi}_p \in S(\tilde{V}_p^r)$ *with* $W_{T,p}(g_p, 0, \tilde{\phi}_p) \neq 0$*, and finally* $\tilde{\phi} = \otimes \tilde{\phi}_q$ *with* $\tilde{\phi}_q = \phi_q$ *for* $q \neq p$*.*

3. Unitary Shimura varieties and arithmetic cycles

In this section, we will review the Shimura varieties of unitary type $(n-1, 1)$ and its arithmetic cycles. We will mainly follow [BHK+17] but with Kudla green functions ([Kud97], [?]) and Garcia-Sankaran green currents ([GS19]). Then we will discuss Kudla's program on modularity and arithmetic Siegel-Weil formula and some recent progress.

Let W_0 and W be Hermitian spaces over $F = \mathbb{Q}(\sqrt{d})$ of signature $(1, 0)$ and $(n-1, 1)$ respectively, and let $V = \mathrm{Hom}_F(W_0, W)$ with Hermitian form given by

$$(f_1(x_1), f_2(x_2)) = (x_1, x_2)(f_1, f_2), \quad x_i \in W_0,\ f_i \in V.$$

Let $G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$ be the subgroup of pairs for which the similitude factors are equal. We denote by $\nu : G \to \mathbb{G}_m$ the common similitude character, and note that $\nu(G(\mathbb{R})) \subset \mathbb{R}_{>0}$.

Let $\mathcal{D}(W_0) = \{y_0\}$ be a one-point set, and define

$$\mathcal{D}(W) = \{\text{negative definite } \mathbb{C}\text{-lines } y \subset W_{\mathbb{R}}\}, \tag{3.1}$$

so that $H(\mathbb{R})$ acts on the connected hermitian domain

$$\mathcal{D} = \mathcal{D}(W_0) \times \mathcal{D}(W).$$

Notice that we can identify $\mathcal{D}$ with $\mathcal{D}(V)$, which consists of all negative $\mathbb{C}$-lines in $V_{\mathbb{R}}$: fix a F-basis $\{v_0\}$ of W_0 and $a = (v_0, v_0) > 0$, then

$$(V, (\ ,\ )) \cong (W, a^{-1}(\ ,\ )),$$

which gives $\mathcal{D} \cong \mathcal{D}(W) \cong \mathcal{D}(W_0) \times \mathcal{D}(W)$. Notice also that G acts on V via $(gf)(x) = g(f(g^{-1}x))$, which gives the following exact sequence

$$1 \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m \to G \to H := \mathrm{U}(V) \to 1. \tag{3.2}$$

So H also acts on $\mathcal{D}(V)$, which is compatible with the action of H on $\mathcal{D}$. Let K be a compact open subgroup of $G(\mathbb{A}_f)$, then the orbifold quotient

$$M_K(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K$$

is the space of complex points of a smooth F-stack of dimension $n-1$, denoted $M = M_K$. The reason to use (G, V) instead of simply $\mathrm{GU}(W)$ will be apparent when define integral model and integral special cycles later.

Let

$$\omega = \{v \in V_{\mathbb{R}} :\ (v,v) < 0\}$$

be the tautological line bundle over $\mathcal{D}$, with hermitian metric $\|v\|^2 = -(v,v)$. The metrized line bundle descends to a line bundle over M_K, denoted by $\hat{\omega} = (\omega, \| \|)$—the metrized line bundle of modular forms of weight 1.

3.1. **Kudla's geometric Siegel-Weil formula.** Let $1 \le i \le m$. Given $x = (x_1, \cdots, x_m) \in V^m$ such that the subspace $V(x)$ generated by x_i is negative definite of dimension $r = r(x)$, and $h \in G(\mathbb{A}_f)$, let $\mathcal{D}_x$ be the negative lines in $V_{\mathbb{R}}$ perpendicular to $V(x)$, G_x be the stabilizer of x in G, and $K_{x,h} = G_x(\mathbb{A}_f) \cap hKh^{-1}$. Then the cycle $Z(x,h) \subset M$ of codimension r is given by

$$Z(x,h) = G_x(\mathbb{Q})\backslash\mathcal{D}_x \times G_x(\mathbb{A}_f)/K_{x,h} \to M. \tag{3.3}$$

For $T \in \mathrm{Her}_m(\mathbb{Q})$ positive definite of rank r and $\phi \in S(V^m_{\mathbb{A}_f})$, if there is $x \in V^m$ with $T(x) = ((x_i, x_j)) = T$, Kudla defined the weighted cycle

$$Z^{\mathrm{Naive}}(T,\phi) = \sum_{h \in G_x(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K} \phi(h^{-1}x) Z(x,h) \in \mathrm{CH}^r(M).$$

Let $Z(T,\phi) = Z^{\mathrm{Naive}}(T,\phi)\cdot(\omega^{-1})^{m-r} \in \mathrm{CH}^m(M)$ and let ($\tau \in \mathbb{H}^u_m$)

$$\theta^{geo}_m(\tau,\phi) = \sum_{T \in \mathrm{Her}_m(\mathbb{Q}), T \ge 0} Z(T,\phi) q^T \in \mathbb{C}[[q]] \otimes \mathrm{CH}^m(M), \quad q^T = e(\mathrm{tr}(T\tau)) \tag{3.4}$$

be the generating series of geometric special cycles of codimension m. Kudla conjectured that it is a unitary modular form of weight n valued in $\mathrm{CH}^m_{\mathbb{C}}(M)$, Chow group with coefficients in $\mathbb{C}$([**?**]). In particular, $Z(T,\phi)$ generates a finite subspace of $\mathrm{CH}^m_{\mathbb{C}}(M)$. Its analogue in orthogonal case was proved by Wei Zhang (formally modular) [**?**] and Bruinier-Raum (formal modular is modular) [**?**]. Kudla's geometric Siegel-Weil formula ([**?**]) claims that

$$\theta^{geo}_m(\tau,\phi)\cdot\omega^{n-m} = C \cdot E^{(m)}(\tau, s_m, \phi) \tag{3.5}$$

for some non-zero explicit constant C. These theta functions have the following compatibility property

$$\theta^{geo}_{m_1}(\tau_1,\phi_1)\cdot\theta^{geo}_{m_2}(\tau_2,\phi_2) = \theta^{geo}_{m_1+m_2}(\mathrm{diag}(\tau_1,\tau_2), \phi_1 \otimes \phi_2). \tag{3.6}$$

Assume $n = 2r$ is even. For any cuspital automorphic representation π of $\mathrm{U}(r,r)$, Li and Liu proved in [**?**] that the π-part of θ^{geo}_r is cohomologically trivial and its height pairing with itself is equal to the the central derivative of the 'doubling' L-series $L(s,\pi)$ under some technical conditions.

3.2. **Arithmetic theta series of arithmetic divisors.** We are mostly follow [BHK+17]. Assume that $\mathfrak{a}_0$ is an $\mathcal{O}_F$-self dual lattice of W_0, $\mathfrak{a}$ is an $\mathcal{O}_F$-self dual lattice of W, and let $L = \mathrm{Hom}_{\mathcal{O}_F}(\mathfrak{a}_0, \mathfrak{a}) \subset V$ be an $\mathcal{O}_F$-self dual lattice of V. Let

$$K = \{(h_0, h_1) \in G(\mathbb{A}_f) :\ h_0\mathfrak{a}_0 = \mathfrak{a}_0, h_1\mathfrak{a} = \mathfrak{a}\}. \tag{3.7}$$

Then the Shimura variety $M = M_K$ has a regular integral model $\mathcal{M} = \mathcal{M}^{\mathrm{Kra}}$ over $\mathcal{O}_F$. Let $\mathcal{M}_{1,0}$ be the moduli stack over $\mathcal{O}_F$ of CM elliptic curves with $\mathcal{O}_F$-action. Let $\mathcal{M}^{Kra}_{n-1,1}$ be the moduli stack over $\mathcal{O}_F$, which assigns to an $\mathcal{O}_F$-scheme S the groupoid of quadruples $(A, \iota, \psi, \mathcal{F}_A)$ in which

(1) $A \to S$ is an abelian scheme of relative dimension n,
(2) $\iota : \mathcal{O}_F \to \mathrm{End}(A)$ is an action of $\mathcal{O}_F$,

(3) $\lambda : A \to A^\vee$ is a principal polarization satisfying $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_F$,
(4) $\mathcal{F}_A \subset \mathrm{Lie}(A)$ is an $\mathcal{O}_F$-stable $\mathcal{O}_S$-module, local direct summand of rank $n-1$ satisfying *Krämer's condition*: $\mathcal{O}_F$ acts on $\mathcal{F}_A$ via the structure map $\mathcal{O}_F \to \mathcal{O}_S$, and acts on the line bundle $\mathrm{Lie}(A)/\mathcal{F}_A$ via the complex conjugate of the structure map.

Over F, the item $\mathcal{F}_A$ and the Kramer condition is automatic. Let $\mathcal{M} = \mathcal{M}^{\mathrm{Kra}}$ be the moduli stack which classifies, for any $\mathcal{O}_F$-scheme S, pairs

$$(A_0, A) \in \mathcal{M}_{(1,0)}(S) \times \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}(S) \tag{3.8}$$

for which there exists, at every geometric point $s \to S$, an isomorphism of hermitian $\mathcal{O}_{F,\ell}$-modules

$$\mathrm{Hom}_{\mathcal{O}_F}(T_\ell A_{0,s}, T_\ell A_s) \cong \mathrm{Hom}_{\mathcal{O}_F}(\mathfrak{a}_0, \mathfrak{a}) \otimes \mathbb{Z}_\ell \tag{3.9}$$

for every prime $\ell \neq p$. Then $\mathcal{M}/\mathcal{O}_F$ is the canonical model for M. Notice that $\mathrm{Hom}_{\mathcal{O}_F}(A_0, A)$ has a positive definite Hermitian form

$$(f_1, f_2) = \lambda_{A_0}^{-1} \circ f_2^\vee \circ \lambda_A \circ f_1 \in \mathrm{End}_{\mathcal{O}_F}(A_0) = \mathcal{O}_F. \tag{3.10}$$

For a positive integer $m > 0$, let $\mathcal{Z}(m)$ be the moduli stack of tuples (A_0, A, x) with $(A_0, A) \in \mathcal{M}$ and

$$x \in \mathrm{Hom}_{\mathcal{O}_F}(A_0, A), \quad (x, x) = m. \tag{3.11}$$

It is easy to check that $\mathcal{Z}(m)(\mathbb{C}) = Z(m) = Z(m, \mathrm{Char}(\omega))$.

Let (A_0, A) be the universal object over $\mathcal{M}$, let $\mathcal{F}_A \subset \mathrm{Lie}(A)$ be the universal subsheaf of Krämer's moduli problem. Define the *line bundle of weight one modular forms* ω on $\mathcal{M}$ by

$$\boldsymbol{\omega}^{-1} = \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A)/\mathcal{F}_A. \tag{3.12}$$

It was shown [BHK+17, Section 2] that $\boldsymbol{\omega}_{\mathbb{C}} \cong \omega$. Via this identification and the metric on ω, we have a metrized line bundle $\hat{\boldsymbol{\omega}}$ over $\mathcal{O}_F$.

There is a canoncal toriodal compactification $\mathcal{M}^*$ of $\mathcal{M}$. Then divisors $\mathcal{Z}(m)$ also extends to $\mathcal{Z}^*(m)$ in $\mathcal{M}^*$, and so does $\hat{\boldsymbol{\omega}}$ (which we still denote by the same symbol). The boundary $\mathcal{B} = \mathcal{M}^* - \mathcal{M}$ are indexed by 'cusps' Φ ([BHK+17, Definition 3.1.1]): $\mathcal{B} = \sum \mathcal{B}_\Phi$. Let

$$\mathcal{B}(m) = \sum b_\Phi(m) \mathcal{B}_\Phi, \quad b_\Phi(m) = \frac{m}{n-2} \#\{x \in L_0 : (x, x) = m\}.$$

where $(L_0, (\cdot, \cdot))$ is a hermitian $\mathcal{O}_F$-module of signature $(n-2, 0)$, which depends on Φ. Finally, let $\mathcal{Z}^{tot}(m) = \mathcal{Z}^*(m) + \mathcal{B}(m)$ for $m > 0$ and

$$\mathcal{Z}^{tot}(0) = \boldsymbol{\omega}^{-1} + \mathrm{Exc} \in \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{M}^*).$$

Here Exc is the exceptional devisor of $\mathcal{M}$ (see for example [BHK+17]).

Theorem 3.1. *Let $\epsilon_{F/\mathbb{Q}} : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ be the Dirichlet character determined by $F/\mathbb{Q}$. The formal generating series*

$$\sum_{m \geq 0} \mathcal{Z}^{tot}(m) \cdot q^m \in \mathrm{CH}^1(\mathcal{M}^*)[[q]]$$

is modular of weight n, level $\Gamma_0(D)$, and character $\epsilon^n_{F/\mathbb{Q}}$ in the following sense: for every $\mathbb{Q}$-linear functional $\alpha : \mathrm{CH}^1(\mathcal{M}^*) \to \mathbb{C}$, the series

$$\sum_{m \geq 0} \alpha(\mathcal{Z}^{tot}(m)) \cdot q^m \in \mathbb{C}[[q]]$$

is the q-expansion of a classical modular form of the indicated weight, level, and character.

Bruinier constructed 'automorphic' Green functions $Gr^B(m)$ for $Z(m)$ in his thesis using regularized theta lifting ([**?**], see also [**?**]), the behavior of $Gr^B(m)$ was studied in [**?**] and its divisor in $\mathcal{M}^*$ is $\mathcal{Z}^{tot}(m)(\mathbb{C})$ (which is the reason for the definition of $\mathcal{Z}^{tot}(m)$). Let $\widehat{\mathcal{Z}}^{tot}_B(m) = (\mathcal{Z}^{tot}(m), Gr^B(m))$ and

$$\widehat{\mathcal{Z}}^{tot}(0) = \widehat{\boldsymbol{\omega}}^{-1} + (\mathrm{Exc}, -\log(D)).$$

The main result of [BHK+17, Theorem B] is the following theorem.

Theorem 3.2. *The arithmetic theta series of arithmetic divisors*

$$\theta^{ar}_B(\tau) = \sum_{m\geq 0} \widehat{\mathcal{Z}}^{tot}_B(m) q^m \tag{3.13}$$

is a modular form of weight n, level $\Gamma_0(D)$, and character $\epsilon^n_{F/\mathbb{Q}}$ with value in $\widehat{\mathrm{CH}}^1(\mathcal{M}^*)$.

The modularity has important applications, we refer to [**?**] and [**?**] for some of the applications.

There is another systematic way to construct Green functions associated to $Z(m)$, given by Kudla [Kud97], which is more relevant to this paper and is given as follows.

For any $z \in \mathcal{D}$, we have the orthogonal decomposition

$$V_{\mathbb{R}} = z \oplus z^{\perp}, \quad x = x_z + x_{z^{\perp}},$$

Following Kudla ([Kud97]), we define $R(x, z) = -(x_z, x_z)$, and

$$\xi_0(x, z) = \Gamma(1, 2\pi R(x, z)), \tag{3.14}$$

with partial Gamma function for $a > 0$ and $\Re(s) > 0$

$$\Gamma(s, a) = \int_a^{\infty} e^{-t} t^s \frac{dt}{t}.$$

It is easy to check that $\xi(x, z)$ is smooth in $\mathcal{D} - \mathcal{D}_x$ and has log singularity along $\mathcal{D}_x$. Actually, Kudla showed that it is a Green function for the divisor $\mathcal{D}_x$:

$$dd^c \xi_0(x, z) + \delta_{\mathcal{D}_x} = [\phi_{KM,0}(x, z)], \tag{3.15}$$

where $\phi_{KM,0}$ is the (smooth) Kudla-Millson Schwartz $(1, 1)$-form which is Poincaré dual for the cycle $\mathcal{D}_x$, see [Kud97, Proposition 11.1].

Finally, let for $m \in \mathbb{Z}$, $v > 0$, and $(z, h) \in M$

$$\Xi(m, v, z, h) = \sum_{\substack{x \in V \\ (x,x)=m}} \varphi(h^{-1}x) \cdot \xi_0(x\sqrt{v}, z). \tag{3.16}$$

It is smooth on M^* for $m \leq 0$ and is a Green function for $\mathcal{Z}^{tot}(m)(\mathbb{C})$. Let

$$\widehat{\mathcal{Z}}^{tot}_K(m, v) = \begin{cases} (\mathcal{Z}^{tot}(m), \Xi(m, v)) & \text{if } m > 0, \\ (0, \Xi(m, v)) & \text{if } m < 0, \\ \widehat{\omega}^{-1} + (\mathrm{Exc}, -\log(|Dv|) + \Xi(0, v)) & \text{if } m = 0. \end{cases}$$

By [**?**, Theorem 1.3], Kudla's arithmetic theta function

$$\theta^{ar}_K(\tau) = \sum_{m\in\mathbb{Z}} \widehat{\mathcal{Z}}^{tot}_K(m, v) q^m \tag{3.17}$$

is a (non-holomorphic) modular form of weight n, level $\Gamma_0(D)$, and character $\epsilon_{F/\mathbb{Q}}^n$ with value in $\widehat{\mathrm{CH}}^1(\mathcal{M}^*)$.

3.3. **Generating functions of higher codimension arithmetic cycles.** This section is mostly speculative. For a positive definite Hermitian matrix $T \in \mathrm{Her}_m(\mathbb{Z})$, let $\mathcal{Z}^{\mathrm{Naive}}(T)$ be the moduli stack over $\mathcal{O}_F$, representing $(A_0, A, f = (f_1, \cdots, f_m))$ with $(A_0, A) \in \mathcal{M}(S)$ and $f_i \in \mathrm{Hom}_{\mathcal{O}_F}(A_0, A)$ such that

$$T(f) = ((f_i, f_j)) = T.$$

It is a Deligne-Mumford stack and its generic fiber $\mathcal{Z}^{\mathrm{Naive}}(T)/F = Z(T, \mathrm{Char}(L^m))$, which will be denoted simply by $Z(T)$. However, $\mathcal{Z}^{\mathrm{Naive}}(T)$ could have lower codimension at some prime p and could thus be not equi-dimensional. To make it an element in $\mathrm{CH}^m(\mathcal{M})$, we view it as a quotient of filtration of certain K-group, and define $\mathcal{O}_{\mathcal{Z}(T)}$ as an element there as 'derived tensor' of $\mathcal{O}_{\mathcal{Z}(t_i)}$ over $\mathcal{Z}^{Naive}(T)$:

$$\mathcal{O}_{\mathcal{Z}(T)} = (\mathcal{O}_{\mathcal{Z}(t_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_m)})_{\mathcal{Z}^{Naive}(T)}.$$

Here t_i is the i-th diagonal element of T, $\otimes^{\mathbb{L}}$ stands for the derived tensor product and we take the part of the derived intersection that is supported on $\mathcal{Z}^{Naive}(T)$ ([**?**]). Thus $\mathcal{Z}(T)$ is an element in $\mathrm{CH}^m(\mathcal{M})$. For positive semi-definite T of rank $r < m$, we modify it by $(\boldsymbol{\omega}^{-1})^{m-r}$ to get an element $\mathcal{Z}(T)$ in $\mathrm{CH}^m(\mathcal{M})$.

As for Kudla Green currents, the star product

$$\xi_0^m(x, z) = \xi_0(x_1) * \xi_0(x_2) * \cdots * \xi_0(x_m)$$

is a Green current for $\mathcal{D}_x = \{z \in \mathcal{D} : z \perp x_i\}$. The Kudla Green current for $Z^{Naive}(T, \phi)$ is then given by, for a positive Hermitian matrix $v = a^t \bar{a}$,

$$\Xi^{Naive}(T, v, \phi)(z, h) = \sum_{\substack{x \in V^r \\ (x,x)=T}} \varphi(h^{-1}x) \cdot \xi_0^m(xa, z). \tag{3.18}$$

This gives $\hat{Z}^{Naive}(T, v, \phi) = (Z^{Naive}(T), \Xi^{Naive}(T, v, \phi)) \in \mathrm{CH}^r(M)$ with r being rank of T. Define

$$\hat{Z}(T, v, \phi) = (Z(T), \Xi(T, v, \phi)) = \hat{Z}^{Naive}(T, v, \phi) \otimes ((\hat{\boldsymbol{\omega}})^{-1})^{m-r} \in \mathrm{CH}^m(M).$$

We drop ϕ when it is $\mathrm{Char}(L^m)$. Recently, Garcia and Sankaran ([GS19]) discovered a more conceptional way to define a Green current $\Xi_{GS}(T, v)$ for $Z(T)$, see [GS19], which is equivalent to Kudla's Green current in arithmetic Chow group. We will identify both as $\Xi(T, v)$. Actually, $\Xi(T, v)$ is well-defined for all Hermitian T and is smooth when T not positive semi-definite. A **basic** question is to understand its boundary behavior, and show that it is a Green current on M^* for

$$Z^{tot}(T) = Z(T) + Z_B(T)$$

for some boundary cycle $Z_B(T)$. Assume that $Z_B(T)$ has a canonical integral model $\mathcal{Z}_B(T)$, and thus an integral cycle $\mathcal{Z}^{tot}(T) = \mathcal{Z}(T) + \mathcal{Z}_B(T)$, and arithmetic cycle $\widehat{\mathcal{Z}}^{tot}(T, v) = (\mathcal{Z}^{tot}(T), \Xi(T, v)) \in \widehat{CH}^m(\mathcal{M}^*)$. The 0-th term $\mathcal{Z}^{mod}(0, v)$ might need more modification, and we skip it here.

Conjecture 3.3. *The generating series*

$$\theta_m^{ar}(\tau) = \sum_T \widehat{\mathcal{Z}}^{tot}(T, v) q^T$$

is a unitary modular form for $\mathrm{U}(m, m)$ *of weight* n.

Another basic question is to construct Bruinier type Green currents $Gr^B(T)$ to make a similar generating series

$$\theta_m^{ar,B}(\tau) = \sum_{T \geq 0} \widehat{\mathcal{Z}}^B(T) q^T, \tag{3.19}$$

a holomorphic unitary modular form of weight n. Here

$$\widehat{\mathcal{Z}}^B(T) = (\mathcal{Z}^{tot}, Gr^B(T)).$$

We certainly expect the compatibility properties

$$\theta_{m_1}^{ar}(\tau_1) \cdot \theta_{m_2}^{ar}(\tau_2) = \theta_{m_1+m_2}^{ar}(\mathrm{diag}(\tau_1, \tau_2)).$$

3.4. **Arithmetic Siegel-Weil formula.** Although the previous section is very speculative, the case $m = n$ can be made more precise. In this case, we have the arithmetic degree map

$$\widehat{\deg} : \widehat{CH}^n(\mathcal{M}^*) \to \mathbb{C}.$$

Kudla's arithmetic Siegel-Weil formula is, roughly speaking, the following conjecture.

Conjecture 3.4. *Up to some modification at 'ramified' primes, one has*

$$\widehat{\mathrm{Deg}} \theta_n^{ar}(\tau) = C E'(\tau, 0, \lambda(\phi)\Phi_\infty^\ell).$$

Here $C \neq 0$ is some explicit constant, $\phi = \mathrm{Char}(\omega^n)$, and $\Phi_\infty^\ell \in I(s, \chi_\infty)$ is the standard section of weight $\ell = (\frac{n+\kappa(\chi_\infty)}{2}, \frac{-n+\kappa(\chi_\infty)}{2})$ (associated to the the Gaussian in $S(\mathcal{C}_\infty^n)$). In another word, one has for every Hermitian $n \times n$ matrix T,

$$\widehat{\mathrm{Deg}} \widehat{\mathcal{Z}}^{tot}(T, v) q^T = C E_T'(\tau, 0, \lambda(\phi)\Phi_\infty^\ell). \tag{3.20}$$

This conjecture is only known for orthogonal type of signature $(0, 2)$ and $(1, 2)$ completely, see [KRY99], [KRY06], [**?**] and [DY19]. Much more has been known for non-singular T, which essentially follows a local arithmetic Siegel-Weil formula and a usual Siegel-Weil formula.

Lemma 3.5. *Assume that T is non-singular Hermitian of order n, and let $\mathcal{C}$ be the incoherent Hermitian space over $\mathbb{A}_F$ such that $\mathcal{C}_{\mathbb{A}_f} \cong V_{\mathbb{A}_f}$ and $\mathcal{C}_\infty$ is positive definite as in Section 2.3. Then the following holds.*

1. *When $|\mathrm{Diff}(\mathcal{C}, T)| > 1$, then $\widehat{\mathcal{Z}}^{tot}(T, v) = 0$.*
2. *When $\mathrm{Diff}(\mathcal{C}, T) = \{\infty\}$, T is not positive definite, one has*
$$\widehat{\mathcal{Z}}^{tot}(T, v) = (0, \Xi(T, v)).$$
3. *When $\mathrm{Diff}(\mathcal{C}, T) = \{p\}$ with $p < \infty$, p is non-split in F, and*
$$\widehat{\mathcal{Z}}^{tot}(T, v) = (\mathcal{Z}(T), 0).$$
is supported at p.

Proof. (sketch) First notice that $Z(T) = 0$. This is because if there is $\vec{x} = (x_1, \cdots, x_n) \in V^n$ with $(x, x) = T$, then x_i form a basis of V and $D_{\vec{x}} = \emptyset$. By [KR14, Lemma 2.7], $\mathcal{Z}(T) = \emptyset$ if T is not positive definite. This proves (2). If $\mathcal{Z}^{Naive}(T)(\overline{\mathbb{F}}_p)$ is not empty with an element $(A_0, A, \cdots, \vec{x})$, then $\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_F}(A_0, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ represents T. Away from p and ∞, $\mathbb{V}_q \cong V_q$ by definition of $\mathcal{M}$. By (2) $\mathbb{V}_\infty$ is positive definite. Since $\mathcal{C}$ is incoherent, $\mathbb{V}_p$ and $V_p = \mathcal{C}_p$ have 'opposite' determinant, and so $\mathrm{Diff}(\mathcal{C}, T) = \{p\}$. This proves (1). Finally, when $\mathrm{Diff}(\mathcal{C}, T) = \{p\}$, $\Xi(T, v) = 0$, no boundary is added to $\mathcal{Z}(T)$, and we have (3). $\square$

By definition of arithmetic degree and the above lemma, we have for a non-singular Hermitian matrix T:
(3.21)

$$\widehat{\deg}\hat{\mathcal{Z}}(T,v) = \begin{cases} \frac{1}{2}\int_{X_K}\Xi(T,v)dx & \text{if } \mathrm{Diff}(\mathcal{C},T)=\{\infty\}, \\ \chi(\mathcal{Z}^{Naive}(T), \mathcal{Z}(t_1)\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\mathcal{Z}(t_n))\log \mathrm{N}(\mathfrak{p}) & \text{if } \mathrm{Diff}(\mathcal{C},T)=\{p\}, \\ 0 & \text{otherwise.}\end{cases}$$

Here $\mathfrak{p}$ is the unique prime ideal of F above p, and $t_1, \cdots, t_n$ are diagonal elements of T.

3.5. **Local Arithmetic Siegel-Weil formula and 'proof' of arithmetic Siegel-Weil formula for non-singular T.** In this subsection, we describe the local arithmetic Siegel-Weil formulas and use it to derive the arithmetic Seigel formula (3.20).

Local Arithmetic Siegel-Weil formula and 'proof' of arithmetic Siegel- Weil formula for non-singular T.

We first assume $\mathrm{Diff}(\mathcal{C},T)=\{\infty\}$, so T is non-singular but not positive definite. Given $x \in V^n$ with $(x,x)=T$, define

$$\mathrm{ht}_\infty(x) = \frac{1}{2}\int_{\mathcal{D}} \xi_0^n(x,z) \tag{3.22}$$

to be the local height at x. Yifeng Liu ([Liu11]) proved the following local arithmetic Siegel-Weil formula and arithmetic Siegel-Weil formula at infinity. Garcia and Sankaran ([GS19]) gave a different proof with more generality (including singular terms).

Theorem 3.6. *(Local arithmetic Siegel-Weil at infinity) (Liu) Let the notation be as above. Then*

$$ht_\infty(x) = (-1)^n B_\infty \cdot W'_{T,\infty}(1,0,\Phi^\ell_\infty)e^{-2\pi \operatorname{tr} T} = \frac{W'_{T,\infty}(\tau,0,\Phi^\ell_\infty)}{W_{\tilde{T},\infty}(\tau,0,\Phi^\ell_\infty)}$$

for any positive definite Hermitian form $\tilde{T}$ with $\operatorname{tr}\tilde{T} = \operatorname{tr} T$ (if exist). Here

$$\Gamma_n(s) = (2\pi)^{\frac{n(n-1)}{2}}\prod_{j=0}^{n-1}\Gamma(s-j).$$

and

$$B_\infty^{-1} = \gamma(V^n)\frac{(2\pi)^{n^2}}{\Gamma_n(n)} = \gamma(V^n)\,\mathrm{Vol}(\mathrm{U}(n),dh),$$

where the Haar measure in the unitary ball is given in Theorem 2.9. In particular, the local height $ht_\infty(x)$ depends only on $(x,x)=T$, not on x itself.

Proof. When T has signature $(n-1,1)$, this follows from [Liu11, Theorem 4.17, Proposition 4.5]. The factor $(-1)^n$ comes from

$$\gamma(V^n) = (-1)^n\gamma(\mathcal{C}_\infty^n).$$

When T has signature (p,q) with $q \geq 2$, the left hand side is zero as there is no $x \in V^n$ with $(x,x)=T$ (as V has signature $(n-1,1)$), and $W_{T,\infty}(\tau,0,\Phi^\ell)=0$ by [Liu11, Proposition 4.5]. The last identify for B_∞ comes from Theorem 2.9. ☐

For the arithmetic Siegel-Weil formula, we can do a little more general without using lattice L. Let $G_0 = \mathrm{GU}(W_0)$ and $H = \mathrm{U}(V)$, and identify

$$G \cong G_0 \times H, \quad (g_0, g) \mapsto (g_0, g_0^{-1} g), \tag{3.23}$$

and assume $K = K_0 \times K_u$ with $K_0 \cong \hat{\mathcal{O}}_F^\times$ being the maximal compact subgroup of $G_0(\mathbb{A}_f) = \mathbb{A}_{F,f}^\times$, and $K_u \subset H(\mathbb{A}_f)$ a compact open subgroup.

Theorem 3.7. *(Arithmetic Siegel-Weil at infinity) (Liu) For any $\phi \in S(V(\mathbb{A}_f)^n)^K$ (K-invariant) and nonsingular T which is not positive definite, we have*

$$\widehat{\deg} \widehat{\mathcal{Z}}(T, v, \phi) q^T = C \cdot E_T'(\tau, 0, \phi).$$

Here $C = \frac{(-1)^n h_F}{w_F \operatorname{Vol}(K_u, dh) \operatorname{Vol}(\mathrm{U}(n), dh)}$ with the Haar measure given in Theorem 2.9, and $w_F = |\mathcal{O}_F^\times|$ being the number of roots of unity in $\mathcal{O}_F^\times$.

Proof. Here we sketch a short proof using local Siegel-Weil formula and local arithmetic Siegel-Weil formula, following [BY20, Section 7]. Assume that there is $x \in V^n$ with $(x, x) = T$ (otherwise both sides are zero), fix such an x, then for any $y \in V^n$ with $(y, y) = T$, there is a unique $h \in H$ with $hx = y$ as $x_1, \cdots, x_n$ form a basis of V. Notice also that G_0-factor in $G_0 \times H$ acts on V and $\mathcal{D}$ trivially. So we have By definition

$$\begin{aligned}
\widehat{\deg} \widehat{\mathcal{Z}}(T, v, \phi) &= \frac{1}{2} \int_{(G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}_f)/K_0) \times H(\mathbb{Q}) \backslash \mathcal{D} \times H(\mathbb{A}_f)/K_u} \frac{1}{|G_0(\mathbb{Q}) \cap K|} \\
&\quad \cdot \sum_{h_0 \in H(\mathbb{Q})} \phi((h_0 h)^{-1} x) \xi_0^n (h_0^{-1} x a, z) dh \\
&= \frac{h_F}{2 w_F} \int_{\mathcal{D} \times H(\mathbb{A}_f)/K_u} \phi(h^{-1} x) \xi_0^n (x a, z) dh \\
&= \frac{h_F}{w_F \operatorname{Vol}(K_u, dh)} \mathrm{ht}_\infty(xa) \int_{H(\mathbb{A}_f)} \phi(h^{-1} x) dh.
\end{aligned}$$

The factor w_F in the denominator comes from the fact that $\mathcal{M}$ is a stack with a point $\mathbf{x} = (A_0, A, ..)$ counting with multiplicity $\frac{1}{\mathrm{Aut}(\mathbf{x})} = 1/w_F$. By local Siegel-Weil formula (Theorem 2.9) and local arithmetic Siegel-Weil formula (Theorem 3.6), we have

$$\begin{aligned}
\widehat{\deg} \widehat{\mathcal{Z}}(T, v, \phi) &= \frac{(-1)^n h_F}{\operatorname{Vol}(K_u, dh) \prod_{p \le \infty} \gamma(V_p^n) \operatorname{Vol}(\mathrm{U}(n), dh)} \\
&\quad \cdot W'_{aTa^*, \infty}(1, 0, \Phi^\ell) e^{-2\pi \operatorname{tr}(Tv)} W_{T,f}(1, 0, \phi) \\
&= \frac{(-1)^n h_F}{\operatorname{Vol}(K_u, dh) \operatorname{Vol}(\mathrm{U}(n), dh)} E_T'(\tau, 0, \phi) q^{-T}
\end{aligned}$$

as claimed. Here we use the fact

$$W_{aTa^*, \infty}(1, s, \Phi) e^{2\pi i \operatorname{tr}(Tu)} = W_T(\tau, s, \Phi) |\det v|^s$$

for $\tau = u + iv, v = aa^*$, and recall that for $g_\tau = n(u) m(a)$ $(\det a > 0)$

$$W_T(\tau, s, \Phi) = |\det v|^{-\frac{n}{2}} W_T(g_\tau, s, \Phi).$$

□

The arithmetic Siegel-Weil formula (3.20) for the case $\mathrm{Diff}(\mathcal{C}, T) = \{p\}$ for a (nonsplit) finite prime p can be dealt similarly although we need some more notation and need some modification at ramified primes.

First we introduce the Rapoport-Zink space. Let $\breve{F}_p$ be the completion of the maximal unramified extension of F_p with ring of integers $\mathcal{O}_{\breve{F}_p}$. We also denote $W = W(\bar{\mathbb{F}}_p)$. Let's fix a point $(\underline{E}^\circ, \underline{A}^\circ)$ in the supersingular locus of $\mathcal{M}(\bar{\mathbb{F}}_p)$, which induces a tuple

$$(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}}, \mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}, \mathrm{Id}_{\mathbb{X}}, \mathcal{F}_{\mathbb{X}}),$$

where $\mathbb{Y} := E^\circ([p^\infty])$ and $\mathbb{X} := A^\circ([p^\infty])$. This will serve as a base point of the Rapoport-Zink space. Given the base point $(\underline{E}^\circ, \underline{A}^\circ)$, we can define another Hermitian space

$$V^{(p)} = \mathrm{Hom}^0_{\mathcal{O}_F}(E^\circ, A^\circ) := \mathrm{Hom}_{\mathcal{O}_F}(E^\circ, A^\circ) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with Hermitian form

$$(x, y)_{V^{(p)}} = \lambda_E^\vee \circ y^\vee \circ \lambda_A \circ x \in \mathrm{End}^0_{\mathcal{O}_F}(E) \simeq F.$$

Notice that

$$V^{(p)} \otimes \mathbb{A}_f^p = \mathrm{Hom}_{F \otimes \mathbb{A}_f^p}(V(E^\circ)^p, V(A^\circ)^p) \simeq V \otimes \mathbb{A}_f^p \tag{3.24}$$

and

$$V^{(p)} \otimes \mathbb{Q}_p \cong \mathrm{Hom}_{\mathcal{O}_{F_p}}(\mathbb{Y}, \mathbb{X}) \otimes \mathbb{Q}_p \not\simeq V \otimes \mathbb{Q}_p. \tag{3.25}$$

Here $V(E^\circ)$(*resp.* $V(A^\circ)$) is the rational Tate module of E°(*resp.* A°). So $V^{(p)}$ is the neighboring global Hermitian space of the incoherent space $\mathcal{C}$ at p while our original V is the neighboring global Hermitian space of $\mathcal{C}$ at ∞.

For $\mathbf{x} \in V^{(p)}$, it induces

$$\mathbf{x}_p \in := V_p^{(p)} \cong \mathbb{V} = \mathrm{Hom}_{\mathcal{O}_{F_p}}(\mathbb{Y}, \mathbb{X}) \otimes \mathbb{Q}_p$$

and

$$\mathbf{x}^p = (\mathbf{x}_q)_{q \neq p, \infty} \in \mathrm{Hom}_{F \otimes \mathbb{A}_f^p}(V(E^\circ)^p, V(A^\circ)^p).$$

Let $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}_p}}$ be the category of $\mathcal{O}_{\breve{F}_p}$-schemes S such that $\pi \cdot \mathcal{O}_S$ is a locally nilpotent ideal sheaf. We define the RZ-space $\mathcal{N}^{\mathrm{Kra}}_{(n-1,1)}$ over $\mathcal{O}_{\breve{F}_p}$ to be the following moduli functor (see [KR11], [RTW14] and [Krä03]): for $S \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}_p}}$, $\mathcal{N}^{\mathrm{Kra}}_{(n-1,1)}(S)$ is the groupoid of isomorphism classes of tuples $(X, \iota, \lambda, \rho, \mathcal{F}_X)$ given as follows:

(1) X is a p-divisible group over S of dimension n and relative height $2n$;
(2) $\iota : \mathcal{O}_{F_p} \to \mathrm{End}(X)$ is an $\mathcal{O}_F$-action on X satisfying Kottwitz condition:

$$\mathrm{char}(\iota(\pi)|\mathrm{Lie}X) = (T - \pi)^{n-1}(T + \pi);$$

(3) $\lambda : X \to X^\vee$ is a principal-polarization whose associated Rosati involution induces on $\mathcal{O}_{F_p}$ the nontrivial automorphism over $\mathbb{Z}_p$;
(4) $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, k} \bar{S}$ is a $\mathcal{O}_{F_p}$-linear quasi-isogeny of height 0 such that λ and $\rho^*(\lambda_{\mathbb{X}})$ differ locally on $\bar{S}$ by a factor in $\mathbb{Z}_p^\times$;
(5) (the Krämer condition) $\mathcal{F}_X$ is an $\mathcal{O}_{F_p} \otimes \mathcal{O}_S$-submodule of $\mathrm{Lie}(X)$ with $\mathcal{O}_S$-rank $n-1$ and is a direct $\mathcal{O}_S$-summand of $\mathrm{Lie}(X)$. $\mathcal{O}_{F_p}$ acts on $\mathcal{F}_X$ via the structure morphism $\mathcal{O}_{F_p} \to \mathcal{O}_S$ and acts on $\mathrm{Lie}(X)/\mathcal{F}_X$ via the Galois conjugate of the structure morphism.

When p is inert, the Krämer condition follows from (2) and is not needed. In fact, in this case $\mathcal{F}_X \subset \mathrm{Lie}(X)$ is canonically determined by X as long as X satisfies Condition 2.

Let $\mathcal{N}_{(1,0)}$ be the similar Rapoport-Zink space with the framing object $\mathbb{X}$ replaced by $\mathbb{Y}$. Then $\mathcal{N}_{(1,0)} \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ and has a universal p-divisible group $\mathcal{Y}$ over it. For every $S \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}_p}}$, we have $\mathcal{N}_{(1,0)}(S) = \{Y = \mathcal{Y}_S\}$.

The RZ-space we will consider is

$$\mathcal{N} = \mathcal{N}_{(1,0)} \times_{\mathrm{Spf}\, \mathcal{O}_{\breve{F}}} \mathcal{N}^{\mathrm{Kra}}_{(n-1,1)}.$$

Adding $\mathcal{N}_{(1,0)}$ enables us to define cycles naturally.

For a $0 \neq \mathbf{x} \in \mathbb{V}$, let $\mathcal{Z}(\mathbf{x})$ be the closed (formal) subscheme of $\mathcal{N}$ such that $(Y, \iota_Y, \lambda_Y, \rho_Y, X, \iota_X, \lambda_X, \rho_X, \mathcal{F}_X) \in \mathcal{N}(S)$ if $\rho_X^{-1} \circ \mathbf{x} \circ \rho_Y$ lifts to an $\mathcal{O}_F$ homomorphism $x : Y \to X$. According to [How15], $\mathcal{Z}(\mathbf{x})$ is a divisor. For a full rank lattice $M \subset \mathbb{V}$ with a basis $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, let

$$\mathrm{ht}_p(\mathbf{x}) = \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(\mathbf{x}_n)})$$

be the local height at p. It is known that $\mathrm{ht}_p(\mathbf{x})$ is independent of the choice of the basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of M by [How19] and is often denoted by $\mathrm{Int}_p(M)$. Let T be a gram matrix of M. When p is inert, Kudla and Rapoport proposed a conjectural formula for this number ([**?**]), which was recently proved by Chao Li and Wei Zhang ([LZ19]).

In order to take the ramified primes into consideration as well, We propose the following vague conjectural formula for the convenience of proving (3.20) in such a case.

Conjecture 3.8. *(Local arithmetic Siegel-Weil formula at finite prime) Let p be a non-split prime and $\mathfrak{p}$ be the prime ideal of F above p. Then there exist $n-1$ sections $\phi_p^i \in S(\mathbb{V}^n)$ and the associated sections $\Phi_p^i(s) \in I(s, \chi)$ ($1 \le i \le n-1$), and rational functions (in p^s) $c_p^i(s)$ with $c_p^i(0) = 0$ such that*

$$\mathrm{ht}_p(\mathbf{x}) \log \mathrm{N}(\mathfrak{p}) = \mathrm{Int}_p(M) \cdot \log \mathrm{N}(\mathfrak{p}) = \frac{W'_{T,p}(1, 0, \Phi_p^*)}{W_{S_p,p}(1, 0, \Phi_p^0)} \tag{3.26}$$

where Φ_p^0 is the standard section associated to $\mathrm{Char}(L_p^n)$ and

$$\Phi_p^* = \Phi_p^0 + \sum_{i=1}^{n-1} c_p^i(s) \Phi_p^i.$$

When p is inert we can simply take all the $\Phi_p^i(s)$ to be zero.

To make the conjecture precise, we need to determine Φ_p^i and $c_p^i(s)$. The following observation gives some hints. Let $(t_{ij}) = T$, and $\mathrm{v}(T) = \min\{\mathrm{val}_\pi(t_{ij})\}$. According to [Shi18, Theorem 1.2], if $\mathrm{v}(T) < 0$, then

$$\mathrm{ht}_p(\mathbf{x}) = 0.$$

Let $(t_{ij}) = T$. As a result, we want $W'_{T,p}(1, 0, \Phi_p^*) = 0$ when $\min\{\mathrm{v}_\pi(t_{ij})\} \le -1$. $W'_{T,p}(1, 0, \Phi_p^0)$ is zero if $\min\{\mathrm{v}_\pi(t_{ij})\} \le -2$, but in general nonzero if $\min\{\mathrm{v}_\pi(t_{ij})\} = -1$. So the correction terms should be nontrivial when $\min\{\mathrm{v}_\pi(t_{ij})\} = -1$. Here for a Hermitian lattice L we define $\chi(L) = \chi(\mathrm{disc}(L))$, where χ is the quadratic character associated with the quadratic extension F/F_0.

By [Jac62], there are $n-1$ equivalent classes of Hermtian lattices.

$$\mathcal{H}_\epsilon^{n,i} := \mathcal{H}^i \oplus I_{n-2i,\epsilon} \quad \text{for } 1 \le i \le \frac{n}{2}, \quad \epsilon = \pm 1 \tag{3.27}$$

where $\mathcal{H}$ is the hyperbolic plane, and $I_{n-2i,\epsilon}$ is the unimodular Hermitian lattice of rank $n-2i$ with $\chi(I_{n-2i,\epsilon}) = \chi(\mathcal{H}_\epsilon^{n,i}) = \epsilon$. When $n = 2r$ is even, we take $I_{0,\epsilon} = 0$ and $\mathcal{H}_1^{n,r} = \mathcal{H}^r$. Then we set $\Phi_p^i(s)$ be the standard section associated with $\mathrm{Char}(\mathcal{H}_{\chi(M)}^{n,i})$. Setting $W'_{T,p}(1, 0, \Phi_p^*) = 0$ for T the Gram matrix of $\mathcal{H}_{\chi(M)}^{n,i}$ for all possible i gives $n-1$ equations, which uniquely determines $c_p^i(s)$ by a direct calculation. For more detail, see [?]. We prove the case $n = 3$ in [?].

Remark 3.9. *The local arithmetic Siegel-Weil formula—Kudla-Rapoport conjecture—is usually stated in terms of local density polynomials. The local Whittaker function differs from the local density polynomial by a simple factor. Indeed, let L be an integral $\mathcal{O}_{F_p}$-lattice of rank m with a gram matrix S, and let $H_n = \mathrm{Her}_n(\mathbb{Z}_p)$, and*

$$T \in H_n^\vee = \{T = (t_{ij}) \in \mathrm{Her}_n(\mathbb{Q}_p) : \mathrm{ord}_p(t_{ii}) \ge 0, \ \textit{and} \ \mathrm{ord}_{\mathfrak{p}}(t_{ij}\partial_F) \ge 0\}$$

be the dual of $\mathrm{Her}_n(\mathbb{Z}_p)$ with respect to the form $\psi(\mathrm{tr}(XY))$. Let

$$\alpha(L, T) = \int_{\mathrm{Her}_n(\mathbb{Q}_p)} \int_{L^n} \psi(\mathrm{tr}(b((x, x) - T)))dxdb$$

be the local density defined in [HSY20] and [Shi20] with $\mathrm{Vol}(L, dx) = (\mathrm{Vol}(H_n, db) = 1$. With this definition, we have

$$\begin{aligned} \alpha(L, T) &= |d|_p^{-\frac{n(n-1)}{2}} p^{ln(n-2m)} |\{X \in M_{m,n}(\mathcal{O}_{F_p}/p^l) : S[X] - T \in p^l H_n\}| \\ &= p^{ln(n-2m)} |\{X \in M_{m,n}(\mathcal{O}_{F_p}/p^l) : S[X] - T \in p^l H_n^\vee\}| \end{aligned} \tag{3.28}$$

for sufficiently large l, where d is the discriminant of F. Notice that $\alpha(L, T)$ only depends on the Gram matrix S of L. The classical local density

$$\alpha^{cl}(S, T) = \lim_{l \to \infty} p^{ln(n-2m)} |\{X \in M_{m,n}(\mathcal{O}_{F_p}/p^l) : S[X] - T \in p^l H_n\}|$$

differs from the $\alpha(L, T)$ by a factor of $|d|_p^{\frac{n(n-1)}{2}}$. They are the same in unramified cases. Simple calculation gives the following relation between local Whittaker functions and local densities ($s_m = \frac{m-n}{2}$):

$$\begin{aligned} W_{T,p}(1, s_m, \mathrm{Char}(L^n)) &= \gamma(L^n) |\det L|_p^n |d|_p^{\frac{n(2m+n-1)}{4}} \alpha(L, T) \\ &= \gamma(L^n) |\det L|_p^n |d|_p^{\frac{n(2m-n+1)}{4}} \alpha^{cl}(S, T). \end{aligned} \tag{3.29}$$

The local density polynomial $\alpha(L, T, X)$ is determined by

$$\alpha(L, T, p^{-2r}) = \alpha(L_r, T)$$

with $L_r = L \oplus \mathcal{H}^r$, where $\mathcal{H}$ is the hyperbolic Hermitian plane given in Lemma 2.5. So

$$W_{T,p}(1, s_m + r, \mathrm{Char}(L^n)) = \gamma(L^n) |\det L|_p^n |d|_p^{\frac{n(2m+n-1)}{4}} \alpha(L, T, p^{-2r}). \tag{3.30}$$

In case of the conjecture, $m = n$ and $s_m = 0$.

Globally, instead of the Eisenstein series in Conjecture 3.4, we need a modified one: $E(\tau, s, \Phi^*)$ for $\Phi^* = \otimes \Phi_p^*$ where Φ_p^* is the one given in Conjecture 3.8 when p is non-split in F, the standard section Φ_p associated of $\mathrm{Char}(L_p^n)$ when p is split, and is Φ_∞^ℓ (the one associated to Gaussian ϕ_∞ in $S(\mathcal{C}_\infty^n)$).

Theorem 3.10. *Assume that Conjecture 3.8 is true, and assume* $\mathrm{Diff}(\mathcal{C}, T) = \{p\}$. *Then we have*

$$\widehat{\deg} \mathcal{Z}(T) q^T = C \cdot \mathcal{E}'_T(\tau, 0, \Phi^*),$$

where $C = \frac{(-1)^n h_F}{w_F \operatorname{Vol}(K_u, dh) \operatorname{Vol}(\mathrm{U}(n), dh)}$ *is the same constant as in Theorem 3.7 with*

$$K_u = K_L = \{h \in \mathrm{U}(V)(\mathbb{A}_f) : hL = L\}.$$

Proof. The case $p = \infty$ is Theorem 3.7 as

$$W_{T,q}(1, 0, \Phi_q^*) = W_{T,q}(1, 0, \Phi_q)$$

for any prime q. Assume that p is finite and nonsplit in F. We again identify G with $G_0 \times H$ as before with $G_0 = \mathrm{GU}(W_0)$ and $H = \mathrm{U}(V)$. Under this identification, our compact open $K = K_0 \times K_u$ with $K_0 = \hat{\mathcal{O}}_F^\times$. Let $H^{(p)} = U(V^{(p)})$ and $G^{(p)} = G_0 \times H^{(p)}$ be the analogue of G. Notice that with this identification, G_0 acts on $V^{(p)}$ and $\mathcal{N}$ trivially. To simplify the notation in the proof, we write $V^{(p)} = \tilde{V}$, $H^{(p)} = \tilde{H}$ and so on.

By [KR14, Lemma 2.21], $\mathcal{Z}(T)$ is supported in the supersingular locus $\mathcal{M}^{ss}$ of $\mathcal{M}$. Let $\widehat{\mathcal{M}}^{ss}$ be the completion of the formal completion of $\mathcal{M} \times_{\operatorname{Spec} \mathcal{O}_F} \operatorname{Spec} \mathcal{O}_{\breve{F}_p}$ along its supersingular locus. By the p-adic uniformization theorem (([RZ96, Theorem 6.30]),[KR14, Theorem 5.5])

$$\begin{aligned} \widehat{\mathcal{M}}^{ss} &\cong \tilde{G}(\mathbb{Q}) \backslash (\mathcal{N}' \times \tilde{G}(\mathbb{A}_f^p)/K^p) \\ &\cong (G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}_f)/K_0) \times (\tilde{H}(\mathbb{Q}) \backslash \mathcal{N} \times H(\mathbb{A}_f^p)/K_u^p). \end{aligned}$$

Here $\mathcal{N}' = (G_0(\mathbb{Q}_p)/K_{0,p}) \times \mathcal{N}$. Under this identification we have ([KR14, Proposition 6.3] and [San17, Proposition 4.4])

$$\widehat{\mathcal{Z}(T)} \cong (G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}_f)/K_0) \times \bigsqcup_{h \in \tilde{H}(\mathbb{Q}) \backslash \tilde{H}(A_f^p)/K_u^p} \bigsqcup_{\substack{\mathbf{x} \in \Omega^{(p)}(T) \\ h^{-1}\mathbf{x}^p \in L^n \otimes \hat{\mathbb{Z}}^{(p)}}} \mathcal{Z}(\mathbf{x}_p),$$

where $\widehat{\mathcal{Z}(T)}$ is the closure of $\mathcal{Z}(T)$ in $\widehat{\mathcal{M}}^{ss}$, $\mathcal{Z}(\mathbf{x}_p)$ is the special cycle in $\mathcal{N}$ associated to $\mathbf{x}_p \in \tilde{V}_p^n = \mathbb{V}^n$, and

$$\Omega^{(p)}(T) = \{\mathbf{x} \in \tilde{V}^n : (\mathbf{x}, \mathbf{x}) = T\}.$$

Fix one $\mathbf{x} \in \Omega^{(p)}(T)$, then

$$\tilde{H}(\mathbb{Q}) \cong \Omega^{(p)}(T), \quad h \mapsto h\mathbf{x}.$$

So we have by Conjecture 3.8 and Theorem 2.9

$$\begin{aligned} \widehat{\deg}(\hat{\mathcal{Z}}(T)) &= \frac{h_F}{w_F} \sum_{h \in \tilde{H}(\mathbb{Q}) \backslash H(A_f^p)/K_u^p} \sum_{h_0 \in \tilde{H}(\mathbb{Q})} \phi_f^p(h^{-1} h_0^{-1} \mathbf{x}^p) \operatorname{ht}_p(\mathbf{x}_p) \log(\mathrm{N}(\mathfrak{p})) \\ &= \frac{h_F}{w_F} \frac{W'_{T,p}(1, 0, \Phi_p^*)}{W_{S_p,p,}(1, 0, \Phi_p)} \int_{H(\mathbb{A}_f^p)/K_u^p} \phi_f^p(h^{-1}\mathbf{x}^p) dh \\ &= \frac{h_F}{w_F \operatorname{Vol}(K_u^p, dh)} \frac{W'_{T,p}(1, 0, \Phi_p^*)}{W_{S_p,p}(1, 0, \Phi_p)} \gamma(V(\mathbb{A}_f^p)^n)^{-1} W_{T, \mathbb{A}_f^p}(1, 0, (\Phi^*)_f^p) \\ &= \frac{h_F}{w_F \operatorname{Vol}(K_u^p, dh)} C_1 E'_T(\tau, 0, \Phi^*), \end{aligned}$$

with

$$C_1^{-1} = \gamma(V(\mathbb{A}_f^p)^n) W_{S_p,p,}(1, 0, \Phi_p) W_{T,\infty}(\tau, 0, \Phi^\ell).$$

Here S_p is a Gram matrix of L_p. Recall that T is positive definite, Theorem 2.9 implies

$$C_1^{-1} = \prod_{p\le\infty} \gamma(\mathcal{C}_p^n) \operatorname{Vol}(K_{L,p}, dh) \operatorname{Vol}(\mathrm{U}(n), dh) q^T = (-1)^n \operatorname{Vol}(K_{L,p}, dh) \operatorname{Vol}(\mathrm{U}(n), dh) q^T.$$

Put this back into the formula above, we prove the theorem. □

In fact we can make C more explicit. Let δ_d denote the number of distinct prime factors of d and define

$$L(2s, n, \epsilon_{F/\mathbb{Q}}) = \prod_{i=1}^n L(2s+i, \epsilon_{F/\mathbb{Q}}^i), \text{ where } L(2s+i, \epsilon_{F/\mathbb{Q}}^2) := \zeta(2s+i).$$

Then we have the following expressions of C.

Proposition 3.11.

$$C = \begin{cases} \frac{\Gamma_n(n) h_F |d|^{\frac{n(n+1)}{4}}}{(2\pi)^{n^2} 2^{\delta_d} w_F} L(0, n, \epsilon_{F/\mathbb{Q}}), & \text{if } n \text{ is odd,} \\ \frac{-\Gamma_n(n) h_F |d|^{\frac{n(n-1)}{4}}}{(2\pi)^{n^2} 2^{\delta_d} w_F} L(0, n, \epsilon_{F/\mathbb{Q}}) \prod_{p|d}(p^{\frac{n}{2}} + \epsilon(V_p)), & \text{if } n \text{ is even.} \end{cases}$$

Here $\epsilon(V_p)$ is defined as in (2.16).

Proof. To make C explicit, we need to compute $\operatorname{Vol}(K_u, dh)$ and $\operatorname{Vol}(\mathrm{U}(n), dh)$. According to Theorem 2.9,

$$\operatorname{Vol}(K_u, dh) = \prod_{p<\infty} \gamma_p(V^n)^{-1} W_T(1, 0, \operatorname{Char}(L_p^n))$$

where T is the Gram matrix of L. Recall from Remark 3.9 that ($m = n$ in this case)

$$\begin{aligned} W_{T,p}(1, 0, \operatorname{Char}(L^n)) &= \gamma(L^n) |\det L|_p^n |d|_p^{\frac{n(2m+n-1)}{4}} \alpha(L, T) \\ &= \gamma(L^n) |\det L|_p^n |d|_p^{\frac{n(2m-n+1)}{4}} \alpha^{cl}(S, T). \end{aligned} \tag{3.31}$$

When n is odd, Theorem 7.3 of [GY00] implies that

$$\alpha_p^{cl}(L_p, L_p) = \begin{cases} \prod_{i=1}^n (1 - \epsilon_{F_p/\mathbb{Q}_p}(p)^i p^{-i}), & \text{if } p \nmid d, \\ 2 \prod_{i=1}^{\frac{n-1}{2}} (1 - p^{-2i}), & \text{if } p|d. \end{cases}$$

Then

$$\begin{aligned} \operatorname{Vol}(K_u, dh) &= \prod_{p<\infty} |N(\det L)|_p^{\frac{n}{2}} |d|_p^{\frac{n(n+1)}{4}} \alpha_p^{cl}(L_p, T) \\ &= |d|^{-\frac{n(n+1)}{4})} L(0, n, \epsilon_{F/\mathbb{Q}})^{-1}. \end{aligned}$$

Also, according to Theorem 3.7,

$$\operatorname{Vol}(\mathrm{U}(n), dh) = \frac{(2\pi)^{n^2}}{\Gamma_n(n)} = (2\pi)^{\frac{n(n+1)}{2}} \prod_{i=1}^{n-1} (i!)^{-1}.$$

A combination of the above shows that

$$C = \frac{(-1)^n h_F}{w_F \mathrm{Vol}(K_u, dh)\mathrm{Vol}(\mathrm{U}(n), dh)} = \frac{\Gamma_n(n) h_F |d|^{\frac{n(n+1)}{4}}}{(2\pi)^{n^2} 2^{\delta_d} w_F} L(0, n, \epsilon_{F/\mathbb{Q}}).$$

When n is even, [GY00, Theorem 7.3] implies that

$$\alpha_p^{cl}(L_p, L_p) = \begin{cases} \prod_{i=1}^n (1 - \epsilon_{F_p/\mathbb{Q}_p}(p)^i p^{-i}), & \text{if } p \nmid d\ , \\ 2(1 - \epsilon(V_p) p^{-\frac{n}{2}}) \prod_{i=1}^{\frac{n}{2}-1}(1 - p^{-2i}), & \text{if } p | d\ . \end{cases}$$

Then

$$\mathrm{Vol}(K_u, dh) = \prod_{p<\infty} |N(\det L)|_p^{\frac{n}{2}} |d|_p^{\frac{n(n+1)}{4}} \alpha_p^{cl}(L_p, T) = |d|^{-\frac{n(n+1)}{4}} 2^{\delta_d} L(0, n, \epsilon_{F/\mathbb{Q}})^{-1} \prod_{p|d} \frac{p^{\frac{n}{2}}}{p^{\frac{n}{2}} + \epsilon(V_p)}.$$

Plugging this to the formula of C, one proves the proposition. □

To deal with singular coefficients of (3.20) is the same as to prove the following conjecture at their non-singular coefficients.

Conjecture 3.12. *For $0 \le m \le n-1$, there is a 'normalized' Eisenstein series $\mathbb{E}(\tau, s, \Phi^*)$ on $\mathrm{U}(m,m)$ and a constant $C \neq 0$ with*

$$\phi_m^{ar}(\tau) \cdot \widehat{\boldsymbol{\omega}}^{n-m} = C \cdot \mathbb{E}'(\tau, \frac{n-m}{2}, \Phi^*).$$

Here Φ_p^* is some modification of the standard section associated of $\mathrm{Char}(L_p^m)$ for finite p (modification only happens at 'bad' primes), and $\Phi_\infty = \Phi_\infty^\ell$. The case $m = 0$ amounts to give an explicit formula for the arithmetic volume $\widehat{\omega}^n$.

Here the normalization for the Eisenstein series is important as its value at $\frac{n-m}{2}$ is non-zero, see [KRY04], [KRY06], [DY19], and [?].

4. The arithmetic Siegel-Weil formula on Shimura curves of type $U(1,1)$

In this section, we restrict to the case $n = 2$ and make Conjecture 3.8 a theorem with precise modification and thus prove Theorem 3.10 unconditionally in this case. Actually, we will relax the condition a little. Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of conductor $D = D(B)$, and let $\mathcal{O}$ be an Eichler order of square-free index N with $(N, D) = 1$. We also assume $2 \nmid dND$. Locally

$$\mathcal{O}_p = \begin{cases} \mathcal{O}_{B_p} & \text{if } p|D, \\ L_0(p) & \text{if } p|N, \\ M_2(\mathbb{Z}_p) & \text{if } p \nmid ND. \end{cases} \tag{4.1}$$

Here $\mathcal{O}_{B_p}$ is the maximal order of the division algebra B_p, and

$$L_0(p) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}_p) :\ p|c \}.$$

We assume and fix an embedding $i : \mathcal{O}_F \hookrightarrow \mathcal{O}$, which extend to an embeding $F \hookrightarrow B$. Via this embedding, we will simply view $\mathcal{O}_F$ as a subring of $\mathcal{O}$. Choose an $\delta \in \mathcal{O}$

with $\delta^2 = \Delta \in \mathbb{Z}$ primes to dND such that $x\delta = \delta\bar{x}$. Recall that $d = d(F)$ is the discriminant of F. Then $B = F + F\delta$. Let $\mathrm{tr}_F(x + y\delta) = x$, and let $z \mapsto z^\iota$ be the main involution of B.

Let $L = \mathcal{O}$ with $\mathcal{O}_F$-Hermitian form

$$(z_1, z_2) = \mathrm{tr}_F(z_1 z_2^\iota),$$

and let $V = L \otimes_{\mathbb{Z}} \mathbb{Q} = B$ be the Hermitian space of signature $(1,1)$. We mention that every Hermitian space over F of signature $(1,1)$ can be obtained this way.

Let B act on the left F-vector space V via right multiplication, which gives a ring embedding (with respect to the basis $\{1, \delta\}$)

$$\alpha : B \hookrightarrow \mathrm{End}_F(V) = M_2(F), \quad (x + y\delta)h = (x, y)\alpha(h) \begin{pmatrix} 1 \\ \delta \end{pmatrix}. \tag{4.2}$$

or explicitly,

$$\alpha(h_1 + h_2\delta) = \begin{pmatrix} h_1 & h_2 \\ \bar{h}_2\delta^2 & \bar{h}_1 \end{pmatrix}. \tag{4.3}$$

Recall that

$$\mathrm{GU}(V) = \{h \in \mathrm{GL}_2(F) : h \,\mathrm{diag}(1, -\delta^2)\,{}^t\bar{h} = \nu(h)\,\mathrm{diag}(1, -\delta^2)\}$$

with similitude $\nu(h) \in Q^\times$. The following lemma can be verified via calculation and is left to the reader.

Lemma 4.1. *The map (4.2) induces an embeddding $\alpha : B^\times \to GU(V)$ with similitude $\nu(\alpha(h)) = \det h = hh^\iota$ being the reduced norm. Moreover, we have*

$$\alpha(B^1) = SU(V), \text{ where } B^1 \text{ consists of norm } 1 \text{ elements in } B^\times,$$
$$\mathrm{U}(V) = \alpha(B^1) \ltimes U(1),$$
$$GU(V) = \alpha(B^\times) \ltimes U(1).$$

Here $U(1) = F^1$ is embedded into $\mathrm{U}(V)$ via $\epsilon \mapsto \mathrm{diag}(1, \epsilon)$.

Let $\mathfrak{a}_0$ be a fractional ideal of F with Hermitian form $(x, y) = \frac{x\bar{y}}{\mathrm{N}(\mathfrak{a}_0)}$, and $W_0 = \mathfrak{a}_0 \otimes_{\mathbb{Z}} \mathbb{Q} = F$ with the associated Hermitian form. Let $W = B$ with Hermitian form

$$(z_1, z_2)_W = \frac{(z_1, z_2)}{\mathrm{N}(\mathfrak{a}_0)} = \frac{\mathrm{tr}_F(z_1 z_2^\iota)}{\mathrm{N}(\mathfrak{a}_0)},$$

and let $\mathfrak{a} = \mathfrak{a}_0\mathcal{O} \subset W$ be an integral $\mathcal{O}_F$ lattice of W. Then it is easy to check that

$$L \cong \mathrm{Hom}_{\mathcal{O}_F}(\mathfrak{a}_0, \mathfrak{a})$$

as Hermitian $\mathcal{O}_F$ lattices, and $V = \mathrm{Hom}_F(W_0, W)$. Recall that

$$G = \{(g_0, g) \in \mathrm{GU}(W_0) \times \mathrm{GU}(W) : \nu(g_0) = \nu(g)\} \cong G_0 \times H, \quad (g_0, g) \mapsto (g_0, g_0^{-1}g)$$

with $G_0 = \mathrm{GU}(W_0) = \mathrm{res}_{F/\mathbb{Q}}\mathbb{G}_m$ and $H = \mathrm{U}(V)$. Recall also

$$K = \{(g_0, g_1) \in H(\mathbb{A}_f) : g_0\mathfrak{a}_0 = \mathfrak{a}_0, g_1\mathfrak{a} = \mathfrak{a}\} \cong K_0 \times K_L$$

as in Section 3. The associated Shimura curve M over F has the property

$$M(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K.$$

Since $\mathfrak{a}$ is not $\mathcal{O}_F$-unimodular, the integral model $\mathcal{M}$ has a slightly different moduli interpretation as in Section 3, which we now describe. Write $D = D_1D_2$ and $N = N_1N_2$ with

$$N_1|d, \quad (N_2, d) = 1, \quad D_1|d, \quad (D_2, d) = 1, \quad d|ND, \quad d \equiv x^2 \mod 4N_2. \tag{4.4}$$

$d \equiv x^2 \mod 4N_2$ is forced by the condition $\mathcal{O}_F \subset \mathcal{O}$, which also implies that the prime divisors of D are inert in $\mathcal{O}_F$. It is easy that the above assumptions about $d,\ N,\ D$ implies that

$$D_2N_2L^\vee \subset L \subset L^\vee \text{ and } [L^\vee : L] = (D_2N_2)^2.$$

Here $L^\vee$ is the dual of L with respect to the Hermitian form. Let $\mathcal{M}^{\mathrm{Kra}}_{(1,1)} = \mathcal{M}^{\mathrm{Kra},D_2N_2}_{(1,1)}$ be the moduli stack over $\mathcal{O}_F$, which assigns to an $\mathcal{O}_F$-scheme S the groupoid of quadruples $(A, \iota, \lambda, \mathcal{F}_A)$ in which

1. $A \to S$ is an abelian scheme of relative dimension 2,
2. $\iota : \mathcal{O}_F \to \mathrm{End}(A)$ is an action of $\mathcal{O}_F$,
3. $\lambda : A \to A^\vee$ is a polarization satisfying $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_F$ such that
$$\ker(\lambda) \subset A[D_2N_2], \quad \text{and} \quad |\ker(\lambda)| = (D_2N_2)^2,$$
4. $\mathcal{F}_A \subset \mathrm{Lie}(A)$ is an $\mathcal{O}_F$-stable local direct summand of $\mathrm{Lie}(A)$ of rank 1 satisfying *Krämer's condition*: $\mathcal{O}_F$ acts on $\mathcal{F}_A$ via the structure map $\mathcal{O}_F \to \mathcal{O}_S$, and acts on the line bundle $\mathrm{Lie}(A)/\mathcal{F}_A$ via the complex conjugate of the structure map.

Notice that only condition (3) changes.

Now we define $\mathcal{M} = \mathcal{M}_{\mathcal{O}}$ to be the substack of $\mathcal{M}_{(1,0)} \times_{\mathrm{Spec}\,\mathcal{O}_F} \mathcal{M}^{\mathrm{Kra},D_2N_2}_{(1,1)}$ that associates to an $\mathcal{O}_F$-scheme S the groupoid of pairs

$$(A_0, A) \in \mathcal{M}_{(1,0)}(S) \times_{\mathrm{Spec}\,\mathcal{O}_F} \mathcal{M}^{\mathrm{Kra}}_{\mathcal{O},(1,1)}(S) \tag{4.5}$$

such that

$$\mathrm{Hom}_{\mathcal{O}_F}(T_\ell(A_{0,s}), T_\ell(A_s)) \simeq L \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \tag{4.6}$$

as a Hermitian lattice over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ at any geometric point $s : \mathrm{Spec}\, k \to S$ for every $\ell \neq \mathrm{char}(k(s))$. As an $\mathcal{O}_F$-stack, $\mathcal{M}_{\mathcal{O}}$ is regular and flat. Its complex uniformization is given by the following proposition.

Proposition 4.2.

$$\mathcal{M}_{\mathcal{O}}(\mathbb{C}) \cong G(\mathbb{Q})\backslash \mathbb{D} \times G(\mathbb{A}_f)/K.$$

The special cycles $\mathcal{Z}(T)$ can be defined exactly the same as in Section 3. Local Rapoport-Zink space $\mathcal{N} = \mathcal{N}_{1,0} \times \mathcal{N}^{\mathrm{Kra}}_{1,1}$ can be modified similarly. For example, The RZ-space $\mathcal{N}^{\mathrm{Kra}}_{(1,1)}$ over $\mathcal{O}_{\breve{F}_p}$ is the moldui formal stack which represents the following (see [RTW14] and [Krä03]): for $S \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}_p}}$, $\mathcal{N}^{\mathrm{Kra}}_{(1,1)}(S)$ is the groupoid of isomorphism classes of tuples $(X, \iota, \lambda, \rho, \mathcal{F}_X)$ given as follows:

1. X is a p-divisible group over S of dimension 2 and relative height 4;
2. $\iota : \mathcal{O}_{F_p} \to \mathrm{End}(X)$ is an $\mathcal{O}_{F_p}$-action on X satisfying Kottwitz condition:
$$\mathrm{char}(\iota(\pi)|\mathrm{Lie}X) = (T - \pi)(T + \pi) = T^2 - \pi_0;$$
3. $\lambda : X \to X^\vee$ is a quasi-polarization whose associated Rosati involution induces on $\mathcal{O}_F$ the nontrivial automorphism over F_0, and
$$\ker(\lambda) \subset X[p], \quad |\ker(\lambda)| = p^{2\mathrm{val}_p(D_2N_2)};$$
4. $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, k} \bar{S}$ is a $\mathcal{O}_{F_p}$-linear quasi-isogeny of height 0 such that λ and $\rho^*(\lambda_{\mathbb{X}})$ differ locally on $\bar{S}$ by a factor in $\mathcal{O}^\times_{\mathbb{Z}_p}$;

(5) $\mathcal{F}_X$ is an $\mathcal{O}_{F_p} \otimes \mathcal{O}_S$-submodule of $\mathrm{Lie}(X)$ with $\mathcal{O}_S$-rank 1 and is a direct $\mathcal{O}_S$-summand of $\mathrm{Lie}(X)$;
(6) $\mathcal{O}_{F_p}$ acts on $\mathcal{F}_X$ via the structure morphism $\mathcal{O}_{F_p} \to \mathcal{O}_S$ and acts on $\mathrm{Lie}(X)/\mathcal{F}_X$ via the Galois conjugate of the structure morphism.

Only condition (3) is slightly different.

Fix a basic element $(\mathbb{Y}, \mathbb{X}) \in \mathcal{N}(\bar{\mathbb{F}}_p)$ and $\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X}) \otimes \mathbb{Q}$ as before. For a full rank lattice $M = M_{\mathbf{x}_1,\mathbf{x}_2} \subset \mathbb{V}$ with a basis $\{\mathbf{x}_1, \mathbf{x}_2\}$, the local height (intersection number) is given by

$$\mathrm{ht}_p(\mathbf{x}) = \mathrm{Int}_p(M) = \mathcal{Z}(\mathbf{x}_1) \cdot \mathcal{Z}(\mathbf{x}_2) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(\mathbf{x}_2)}).$$

Now we define the modification section $\Phi_p* = \Phi_p + c_p(s)\Phi_p^c$ and $\Phi^* = \otimes \Phi_p^*$ in Conjecture 3.8 and Theorem 3.10 precisely as follows. Here Φ_p is the standard section in $I(s, \chi_p)$ associated to $\mathrm{Char}(L_p^2)$ for $p < \infty$ and $\Phi_\infty = \Phi_\infty^\ell$, $\ell = (\frac{2+\kappa(\chi_\infty)}{2}, \frac{-2+\kappa(\chi_\infty)}{2})$. Let $\Phi_p^c \in I(s, \chi_p)$ be the standard section associated to $\mathrm{Char}(\mathcal{H}_p^2)$, where $\mathcal{H}_p = \partial_{F_p}^{-1} \oplus \mathcal{O}_{F_p}$ is the hyperbolic plane with Hermitian form $(x, y) = x_1\bar{y}_2 + x_2\bar{y}_1$. Let

$$c_p(s) = \frac{p^s - p^{-s}}{1 - p^2} \begin{cases} 1 & \text{if } p|D \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\Phi_p^* = \Phi_p$ for $p \nmid D$. Notice that we need to make a little modification for p inert in F and $p|D$.

Theorem 4.3. *Let the notation be as above. Then Conjecture 3.8 holds in this case, and Theorem 3.10 holds conditionally:*

$$\widehat{\deg}(\hat{\mathcal{Z}}(T, v))q^T = C \cdot E'_T(\tau, 0, \Phi^*)$$

with

$$C = \frac{h_F^2}{24 \cdot 2^{\delta_d} \cdot w_F^2} \prod_{p|D}(p-1) \prod_{p|N}(p+1).$$

Proof. We verify the formula case by case and may assume $\mathrm{Diff}(\mathcal{C}, T) = \{p\}$ with $p < \infty$ non-split in F (the case $p = \infty$ is Theorem 3.7). In particular, T is positive definite,

Case 1: When $p \nmid dND$, i.e., everything is unramified at p, this is a special case of Li and Zhang's main theorem in [LZ19].

Case 2: When $p|D$ and $p \nmid d$, this is the case proved by Sankaran in [San17]. Briefly, Remark 3.9 aserts

$$W_{T,p}(1, r, \mathrm{Char}(L_p^n)) = \gamma(V_p^n)|\det L|_p^n \alpha(L, T, p^{-2r}). \tag{4.7}$$

(in [San17, Proposition 4.11], $\alpha(S_r, T)$ should be $\alpha(S_{2r}, T)$). Write $\alpha'(L, T) = -\frac{d\alpha(L,T,X)}{dX}|_{X=1}$ as in [San17], [Shi20], and [HSY20]. In our case, $n = m = 2$, we have

$$\frac{W'_{T,p}(1, 0, \mathrm{Char}(L_p^2))}{W_{S_p,p}(1, 0, \mathrm{Char}(L_p^2))} = 2\log p \frac{\alpha'(L_p, T)}{\alpha(L_p, S_p)},$$

and

$$\frac{W_{T,p}(1, 0, \mathrm{Char}(\mathcal{H}^2))}{W_{S_p,p}(1, 0, \mathrm{Char}(L_p^2))} = p^2 \frac{\alpha(\mathcal{H}, T)}{\alpha(L_p, S_p)}.$$

Here S_p is a Gram matrix of L_p. By [San17, Corollaries 2.17, 3.6], we have for $(x,x)=T$

$$\mathrm{ht}_p(x) = \frac{\alpha'(L,T)}{\alpha(L,S)} - \frac{p^2}{p^2-1}\frac{\alpha(\mathcal{H},T)}{\alpha(L,S)}.$$

So ($\mathfrak{p} = p\mathcal{O}_F$ in this case)

$$\mathrm{ht}_p(x)\log(\mathrm{N}(\mathfrak{p})) = \frac{W'_{T,p}(1,0,\mathrm{Char}(L_p^2))}{W_{S_p,p}(1,0,\mathrm{Char}(L_p^2))} + c'_p(0)\frac{W_{T,p}(1,0,\mathrm{Char}(\mathcal{H}^2))}{W_{S_p,p}(1,0,\mathrm{Char}(L_p^2))}$$

as claimed.

Case 3: When $p|d$ and $p|D$, this is basically [HSY20, Theorem 1.3] together with the above argument in Case 2. We leave the detail to the reader.

Case 4: When $p|d$ and $p|N$, this is basically [Shi20, Theorem 7.1].

The case $p\nmid d$ and $p|N$ does not exist as $\mathcal{O}_F \subset \mathcal{O}$.

Now we compute C explicitly. In the current situation, according to Theorem 7.3 of [GY00], we have

$$\alpha_p^{cl}(L_p,L_p) = \begin{cases} (1-p^{-1})(1-p^{-2}), & \text{if } p \text{ splits in } F \text{ and } p\nmid N,\\ p(1-p^{-1})^2, & \text{if } p \text{ splits in } F \text{ and } p|N,\\ (1+p^{-1})(1-p^{-2}), & \text{if } p \text{ is inert and } p\nmid ND,\\ p(1+p^{-1})^2, & \text{if } p \text{ is inert and } p|D,\\ 2\frac{(p+1)}{p}, & \text{if } p|d \text{ and } p|D,\\ 2\frac{(p-1)}{p}, & \text{if } p|d \text{ and } p|N. \end{cases}$$

The rest of the calculation for C is the same as that in Proposition 3.11. □

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