Local Properties of Toric Varieties

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1 Notation and Definition review

To begin, lets review some definitions and notation.

Definition: a semigroup is a set whose elements are associative on a group operation.

$$\forall a, b, c \in S : a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

Essentially, we can think of a semi-group in exactly the same way we think about a group, except a semi-group does not necessarily contain the identity or inverse elements like a group would. For now, we can just think of the semi-group as a set

We also want to remember a couple of things in regards to notation:

$$\sigma = \text{cone}$$
$$\sigma^\vee = \text{dual}$$
$$U_\sigma = \text{ affine variety of } \sigma$$
$$\text{N and M are lattices}$$
$$\sigma^\perp = \text{elements perpendicular to } \sigma$$

2 Local Properties of Toric Varieties

Local in this sense means up close on the chosen point.

For any cone σ in a lattice N, the corresponding affine variety U_σ has a distinguished point, which we denote by x_σ. This point in U_σ is given by a map of semigroups:

$$S_\sigma = \sigma^\vee \cap M \to \{1, 0\} \subset \mathbb{C}^* \cup \{0\} = \mathbb{C}$$

defined by the rule

$$\text{u} \mapsto \begin{cases} 1 \text{ if u} \in \sigma^\perp \\ 0 \text{ otherwise} \end{cases}$$

Lets look at an example of this:

*See Appendix[1]

This function is well-defined, since $\sigma^\perp$ is a face of $\sigma^\vee$, which implies that the sum of two elements in $\sigma^\vee$ cannot be in $\sigma^\perp$ unless both are in $\sigma^\perp$. In other words:

$$u_1, u_2 \in \sigma^\vee\text{: if } u_1 \notin \sigma^\perp\text{, then there exists } v \in \sigma \text{ such that } u_1 \cdot v > 0, u_2 \geq 0 \text{ so } (u_1 + u_2) \cdot v > 0$$

3 Nonsingularity

Restating the difference between M and N Remember, N and M are lattices created by the spans of vector spaces. For our purposes, we can think of $N = \mathbb{Z}^n$. M is a little different; M is defined as some homomorphism that takes the dot product of N and $\mathbb{Z}$, but essentially is also $\mathbb{Z}^n$. ($M = hom(N, \mathbb{Z}) = \mathbb{Z}^n$).
Definition: U_σ is non-singular if and only if σ is generated by a basis of N.

Exercise 1
We want to show that if σ spans $N_\mathbb{R}$, then x_σ is the unique fixed point of the action of the torus T_N on U_σ. So lets say σ spans $N_\mathbb{R}$, or in other words, σ spans $\mathbb{Z}^n$. Then we know from the previous example that $\sigma^\perp = \{0\}$. Now, take a ring $A = A_\sigma$ and let m be the maximal ideal of A corresponding to the point x_σ (the maximal ideal is generated by x_σ, the vectors in $\sigma^\perp$), so m is generated by all χ^u for nonzero u in S_σ. The square m^2 is generated by all χ^u for those u that are sums of two elements of $S_\sigma \backslash \{0\}$.
Definition The cotangent space m/m^2 has a basis of images of elements χ^u for those u in $S_\sigma \backslash \{0\}$ that are not the sums of two vectors.
An example of this is in Appendix[1,5]
One characterization of nonsingularity is that the cotangent space m/m^2 is n-dimentional, since $dim(U_\sigma) = dim(T_N) = n$. This inplies in particular that $\sigma^\vee$ cannot have more than n edges (and that the minimal generators along these edges must generate S_σ).
$\implies$ the minimal generators for S_σ must be a basis for M, and the dual is generated by a basis for N.
Hence U_σ is isomorphic to $\mathbb{C}^n$

Definition : for $u \in N, k \in \mathbb{Z} > 0$; A structure is saturated if $u \in N$ and $k \cdot u \in \sigma \cap N$ then $u \in \sigma \cap N$

A general cone has smaller dimension k. Let $N_\sigma = span(\sigma) \cap N$ ($N_\sigma = \sigma \cap N + (-\sigma \cap N)$) be the sublattice of N generated by $\sigma \cap N$. Since σ is saturated, N_σ is also saturated, so the quotient group $N(\sigma) = N/N_\sigma$ is also a lattice.

You can write this by splitting up the lattice: The dimension of $N(\sigma) = \mathbb{Z}^{n-k}$
then $N = N_\sigma \oplus R^{n-k}$, $\sigma = \sigma' \oplus 0$. ($\sigma'$ is a cone in N)
and $N \cong N_\sigma \oplus N(\sigma)$.
We do this for M, as well.
$(M = M' \oplus M'') \to N_\sigma \oplus N'' \to S_\sigma = ((\sigma')^\vee \cap M') \oplus M''$
So then

$$U_\sigma \cong U_{\sigma'} \times T_{N''} \cong U_{\sigma'} \times (\mathbb{C}^*)^{n-k}$$

This leads us to propose that an affine toric variety is nonsingular if and only if the cone is generated by part of a basis for lattice N, in which case

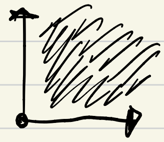
$$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}, k = dim(\sigma)$$

We therefore call a cone nonsingular if it is generated by part of a basis for a lattice, and we call a fan nonsingular if all of its cones are nonsingular, i.e., if the cooressponding toirc variety is nonsingular.

4 References

[Ful93] William Fulton. Introduction to Toric Varieties. (AM-131), Volume 131. Princeton University Press, Princeton, 1993.

1

σ = 

$\sigma^\vee$ = 

$\mathbb{C}[S_\sigma] = \mathbb{C}[x,y] \to \mathbb{C}$

$x=0$, $y=0$ $\to$ $A_\sigma = \mathbb{C}[S_\sigma]$

$x_\sigma = (0,0)$

kernel $\longrightarrow \mathbb{C}[S_\sigma] \to \mathbb{C}$

$\uparrow$

Maximal Ideal = point in U_σ

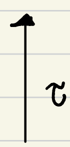
2

$$\boxed{\begin{array}{ll} U_\sigma & u_\sigma \cong f^{-1}(p) \\ \downarrow\uparrow & \downarrow \\ T_{N(\sigma)} = (\mathbb{C}^*)^k \to & p, \text{ any point} \end{array}}$$

torus stuff intuitively

Did not use

3



faces

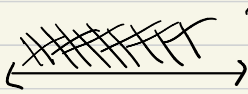
$(x_\tau = (1,0))$

$\uparrow$

$U_\tau = \mathbb{C}^* \times \mathbb{C}$

$\mathbb{C}\{S_\tau\} = \mathbb{C}[x, x^{-1}, y]$

$x \mapsto 1$, $y \mapsto 0$, $\to \mathbb{C}$

 $\tau^\vee$

$\tau^\perp = \longleftrightarrow$

Did not use

$$(u_1, u_2) \mapsto \begin{cases} 1 & (u_2 = 0) \\ 0 & \text{otherwise} \end{cases}$$

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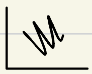
4  σ $\mathbb{C}[S_\sigma] = \mathbb{C}[x,y] \to \mathbb{C}$ $m/m^2 = (x,y)$

 $\sigma^\vee$

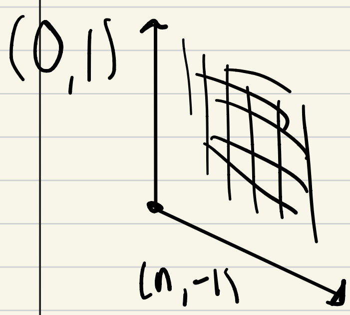
$x \to 0$, $y \to 0$ $A = \mathbb{C}[S_\sigma]$ $m = (x,y)$ $m/m^2 = (x,y)$ $\longleftarrow$ 2D vector space

$x_0 = (0,0)$ $m^2 = (x^2, xy, y^2)$

5. Example of singular

Not generated by a basis for $\mathbb{Z}^2$

x coordinate is always a multiple of n

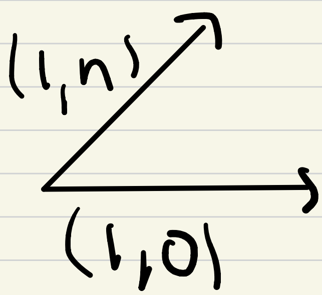
$\sigma =$ 

$n \geq 2$

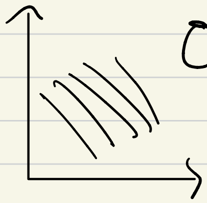
$S_\sigma = \langle (1,0), (1,1), \dots (1,n) \rangle$

$m/m^2 = \text{span}(x, xy, \dots, xy^n)$

dim $= n+1$; singular if $n \geq 2$

$\sigma^\vee =$ 

6.

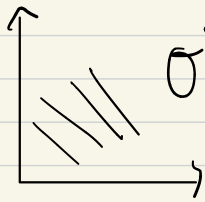


σ

Span
$\mathfrak{m}/\mathfrak{m}^2 = (x,y)$

dim $= 2$

nonsingular



$\sigma^\vee$

$$S_\sigma = (\mathbb{Z}_{\geq 0})^2$$

$$u \mapsto \begin{cases} 1 & u = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

did not use