AN OVERVIEW OF BATYREV'S MIRROR CONSTRUCTION

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1. Historical overview

Let Q be a quintic threefold in $\mathbb{P}^4$, that is a smooth hypersurface given by the vanishing locus of a homogeneous polynomial of degree 5 in 5 variables. The most famous example is the *Fermat quintic*, which is given by the equation

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0.$$

In 1991, four physicists stunned the mathematical community by predicting the number of rational curves (images of maps from $\mathbb{P}^1$) of low degrees on a general Q (for special Q, the number of curves will be different, or even infinite):

Table 1. Predicted number of rational curves of degree d

Degree	Predicted number of curves
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625

In fact, their prediction goes all the way to degree 10 (and is correct up to degree 9 and almost correct in degree 10), but as you can see, the number of digits increases rapidly. Besides the fact that physicists can do things at a much lower level of rigor, the main tool that they used was something called *mirror symmetry*. This allowed them to translate the problem into a much easier problem (computing integrals) on a different algebraic variety, called the *mirror quintic*.

Remark 1.1. The reason the physicists' predicted number of curves differs slightly from the actual number of curves is because they are really computing a **virtual** count of curves, called a *Gromov-Witten invariant*. These invariants are still the subject of intense study today.

At the time, it was actually very difficult for physicists to produce pairs of mirror varieties. In fact, there were many examples in which a mirror was expected to exist, but was unknown. This was rectified in 1994 by Batyrev in [Bat93]. Our goal today is to explain this construction. There are other mathematical predictions made by mirror symmetry, but they are far beyond the scope of this lecture:

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- Enumerative mirror symmetry, proven in [Giv98; LLY97].
- Homological mirror symmetry, conjectured in [Kon94].

2. Mirror symmetry

Mirror symmetry arises from a physical theory called string theory, which for reasons of mathematical consistency requires the universe to be 10-dimensional. Of course, the observed universe is 4-dimensional, so the physicists need a way to make the other 6 dimensions invisible. The strategy is something called *compactification*, which makes the 6 extra dimensions a very small compact manifold of size $\hbar$. By various physical considerations, there are several requirements on this 6-dimensional manifold:

(1) The manifold must have a complex structure. This means that there is a notion of holomorphic functions on it;

(2) The manifold must be simply connected (up to torsion). This means that any loop on it can be contracted to a point (again, up to torsion);

(3) The manifold must have trivial canonical divisor, or $K_X = 0$. Recall from Jennifer's second talk that the canonical divisor is the divisor of zeroes and poles of a meromorphic differential form.

Such a manifold is called a *Calabi-Yau threefold*.

For any choice of a Calabi-Yau threefold X, the physicists produce a *supercomformal field theory (SCFT)*, which has twisted (much simpler, and the ones actually studied by mathematicians) versions called the *A-model* and the *B-model*. They are controlled by parameters whose numbers are the dimensions of various vector spaces associated to X.

2.1. Hodge numbers.

The primary reference for this part is [GH94]. Let X be any smooth projective complex algebraic variety of dimension n (actually, this can be done in more generality, but I am not a differential geometer). Then X is a complex manifold, so locally, there are holomorphic coordinates $z_1, \ldots, z_n$. Then, there are real coordinates given by $z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n$. Think of $z = x + iy$ and $\bar{z} = x - iy$.

In these coordinates, differential forms on X (with values in $\mathbb{C}$) can locally be written as

$$\omega = \sum_{i_1,\ldots,i_p,j_1,\ldots,j_q} f_{i_1,\ldots,i_p,j_1,\ldots,j_q}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

We say that the form $f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ has type (p, q) because it has p holomorphic parts and q antiholomorphic parts.

Next, define the operator $\bar{\partial}$ by the formula

$$\bar{\partial}(\omega) = \sum_{k,i_1,\ldots,i_p,j_1,\ldots,j_q} \frac{\partial}{\partial \bar{z}} f_{i_1,\ldots,i_p,j_1,\ldots,j_q}\, d\bar{z}_k \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

in local coordinates. By the commutativity of mixed partials, we have $\bar{\partial}^2 = 0$, or in other words,

$$\bar{\partial}(\bar{\partial}(\omega)) = 0$$

for all differential forms ω. Finally, let $\Omega^{p,q}(X)$ denote the space of all differential forms of type (p,q) and define

$$Z^{p,q}(X) \coloneqq \left\{\omega \in \Omega^{p,q}(X) \mid \bar{\partial}(\omega) = 0\right\}.$$

Finally, we define the space

$$H^{p,q}(X) \coloneqq Z^{p,q}(X)/(\bar{\partial}\Omega^{p,q-1}(X))$$

and its dimension $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$, called a *Hodge number*. It is convenient to arrange the Hodge numbers of X into a diamond as below:

$$\begin{array}{ccccccccccc}
 & & & & h^{0,0} & & & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & & & \\
 & \iddots & & & \vdots & & & \ddots & & & \\
h^{m,0} & & \cdots & & \underset{\text{Serre}}{\curvearrowleft} & & \cdots & & h^{0,m} & & \updownarrow \text{ Hodge} \\
 & \ddots & & & \vdots & & & \iddots & & & \\
 & & h^{m,m-2} & & h^{m-1,m-1} & & h^{m-2,m} & & & & \\
 & & & h^{m,m-1} & & h^{m-1,m} & & & & & \\
 & & & & h^{m,m} & & & & & & \\
 & & & & \underset{\text{conjugation}}{\leftrightarrow} & & & & & &
\end{array}$$

2.2. **Hodge numbers and mirror symmetry.** Returning to mirror symmetry, it turns out that

- The A-model associated to a Calabi-Yau threefold X is controlled by *Kähler parameters*, which live in $H^{1,1}(X)$.
- The B-model associated to X is controlled by *complex parameters*, which live in $H^{2,1}(X)$.

Then the idea of mirror symmetry is that for a Calabi-Yau threefold X, there should be a mirror Calabi-Yau threefold $X^\vee$ such that the corresponding SCFTs are equivalent in a way that exchanges the A- and B-models (hence exchanges Kähler and complex parameters). In particular, we must have

$$h^{1,1}(X) = h^{2,1}(X^\vee)$$

and vice versa.

3. Reflexive polytopes and mirror symmetry

3.1. **Reflexive polytopes.** Recall the following definition from Jennifer's second lecture:

Definition 3.1. A lattice polytope P is *reflexive* if its facet presentation is

$$P = \{v \in M \mid \langle v, u_F \rangle \geqslant -1 \text{ for all facets } F\}.$$

Here, u_F is the first lattice point on the inward pointing normal ray to F.

Recall also that if P is reflexive, then the origin is the unique interior lattice point of P and the dual polytope (*c.f.* Peng's first talk)

$$P^\circ = \{u \in N \mid \langle v, u \rangle \geqslant -1 \text{ for all } v \in P\}.$$

is a reflexive lattice polytope. We can attempt to classify lattice polytopes, but this gets much harder every time the dimension increases by 1. In two dimensions, reflexive lattice polygons were classified last week:

Theorem 3.2. *There are* 16 *equivalence classes (up to* $GL_2(\mathbb{Z})$*) of reflexive polygons.*

In three dimensions, there are 4319 classes of reflexive polytopes, and in four dimensions, there are 473800776 equivalence classes of reflexive polytopes. The latter fact was discovered only in 2002.

3.2. **The Batyrev construction.** The construction in [Bat93] is actually quite simple. Let P be an $n+1$-dimensional reflexive polytope. This defines an $n+1$-dimensional Fano toric variety X_P, as explained by Jennifer last week. Recall from Peng's second talk that a divisor

$$\sum a_i D_i$$

on X_P defines a polytope, which is the set

$$\left\{v \in \mathbb{Z}^{n+1} \mid \langle v, u_i \rangle \geqslant -a_i \text{ for all } i\right\},$$

where u_i are the rays of the fan of X_P. In particular, P corresponds to the divisor

$$-\sum D_i = -K_{X_P},$$

as explained in Jennifer's talk.

As Jake explained in his second talk, the global sections of (the line bundle corresponding to) $-K_X$ are

$$\left\{x^v \mid v \in P \cap \mathbb{Z}^{n+1}\right\}.$$

In coordinates, if $v = (v_1, \ldots, v_{n+1})$, then $x^v := x_1^{v_1} \cdots x_{n+1}^{v_{n+1}}$. If we choose a general section

$$f := \sum_{v \in P \cap \mathbb{Z}^{n+1}} c_v x^v,$$

then the locus $V := (f = 0) \subset X_P$ defines a hypersurface in X_P, or a subvariety of dimension n. By general considerations in algebraic geometry (the adjunction formula), $K_{V_f} = 0$. In addition, V inherits the property of being simply connected up to torsion from X_P (the Lefschetz hyperplane theorem), so it is a Calabi-Yau variety. In general, X_P may be singular, so V may also be singular (the best we can hope for is that the singularities of V are only at the singularities of X_P, which is satisfied for a general V by Bertini's theorem).

If we consider P° instead, we obtain a Fano toric variety X_{P°, a global section

$$f^\circ \coloneqq \sum_{\nu^\circ \in P^\circ \cap \mathbb{Z}^{n+1}} c_{\nu^\circ} \chi^{\nu^\circ},$$

and a Calabi-Yau hypersurface

$$V^\circ \coloneqq (f^\circ = 0) \subset X_{P^\circ}.$$

Theorem 3.3 ([Bat93])**.** *The Hodge numbers (suitably defined) of* V *and* $V\circ$ *are related by*

$$h^{1,1}(V) = h^{n-1,1}(V^\circ) \qquad h^{n-1,1}(V) = h^{1,1}(V^\circ).$$

Of course, the physicists are looking for smooth objects, but we have only defined a possibly singular Calabi-Yau variety. Recall that we may assume that the singularities of V are only at the singularities of X_P, so we only need to consider the singularities of X_P. Recall the final result from Jennifer's first talk:

Theorem 3.4. *For any toric variety* X*, there is a refinement* $\widetilde{\Sigma}$ *of its fan* Σ *such that* $X_{\widetilde{\Sigma}} \to X$ *is a resolution of singularities.*

Unfortunately, if $f\colon \widetilde{X} \to X$ is a resolution of singularities, in general it is impossible to choose $\widetilde{X}$ such that $f^*K_X = K_{\widetilde{X}}$. Fortunately for the physicists, this is not a problem when $n = 3$.

Proposition 3.5. *Let* V *be a singular Calabi-Yau variety of dimension* 3*. Then there exists a resolution* $\widetilde{V} \to V$ *where* $\widetilde{V}$ *is a smooth Calabi-Yau variety (in other words, a Calabi-Yau threefold).*

This allows us to choose smooth models for V and V° defined above. By Theorem 3.3, they satisfy

$$h^{1,1}(V) = h^{2,1}(V^\circ) \qquad h^{2,1}(V) = h^{1,1}(V^\circ).$$

Warning 3.6. For any given V constructed from a 4-dimensional polytope, there may be more than one smooth model. However, in string theory, there are really **global** Kähler and complex *moduli spaces* of dimensions $h^{1,1}(V)$ and $h^{2,1}(V)$, respectively. The Kähler and complex parameters of different smooth models of V correspond looking at different open sets of the same global Kähler and complex moduli spaces.

Remark 3.7. A more modern perspective in mathematics is to not worry about trying to resolve the singularities of V by enlarging the class of geometric objects such that V becomes smooth. This requires the fan to contain extra data, see [BCS05].

4. The quintic threefold, revisited

Our goal is to construct the quintic threefold V with $h^{1,1}(V) = 1$ and $h^{2,1}(V) = 101$ and its mirror V°.

Recall that the fan of $\mathbb{P}^4$ has rays $e_1, e_2, e_3, e_4, -e_1 - e_2 - e_3 - e_4$. This implies that the polytope corresponding to $-K_{\mathbb{P}^4} = D_1 + D_2 + D_3 + D_4$ has the form

$$P = \left\{(\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4 \mid \nu_1, \nu_2, \nu_3, \nu_4 \geqslant -1, \nu_1 + \nu_2 + \nu_3 + \nu_4 \leqslant 1\right\}.$$

Its vertices are the points

$$(-1, -1, -1, -1), (4, -1, -1, -1), (-1, 4, -1, -1), (-1, -1, 4, -1), (-1, -1, -1, 4).$$

Given $\nu \in P \cap \mathbb{Z}^4$, we obtain the monomial $x_1^{\nu_1} \cdots x_4^{\nu_4}$. We want to homogeneize this, and to do this we replace it with the monomial

$$x_0^{-\nu_1-\nu_2-\nu_3-\nu_4+1} x_1^{\nu_1+1} x_2^{\nu_2+1} x_3^{\nu_3+1} x_4^{\nu_4+1}$$

as in [CLS11, Section 5.4]. For example, $x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1}$ is replaced by x_0^5 and $x_1^4 x_2^{-1} x_3^{-1} x_4^{-1}$ is replaced by x_1^5. In particular, all of the resulting monomials have nonnegative degrees in all of the variables and total degree 5 by construction. Therefore, V is the vanishing locus of a homogeneous quintic polynomial and is thus a quintic threefold.

There are $\binom{5+5-1}{5-1} = 126$ homogeneous polynomials of degree 5 in 5 variables. Note that V is not changed by scaling all c_ν by the same factor simultaneously. Then $\mathbb{P}^4$ has a 24-dimensional automorphism group, so in fact, we have 101 degrees of freedom, which recovers $h^{2,1}(V) = 101$.

The dual polytope P° has vertices given by the points $e_1, e_2, e_3, e_4, -e_1 - e_2 - e_3 - e_4$. The corresponding fan has rays which pass through the vertices of P. Note that

$$P^\circ \cap \mathbb{Z}^4 = \{0, e_1, e_2, e_3, e_4, -e_1 - e_2 - e_3 - e_4\},$$

so any f° has the form

$$f^\circ = c_5 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 + c_0 x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1}.$$

The homogeneization procedure is given by

$$x^\nu \to x_0^{-\nu_1-\nu_2-\nu_3-\nu_4+1} \prod_{i=1^4} x_i^{4\nu_i+1-\sum_{j\neq i} \nu_j},$$

so f° becomes

$$f^\circ = c_5 x_0 x_1 x_2 x_3 x_4 + c_0 x_0^5 + c_1 x_1^5 + c_2 x_2^5 + c_3 x_3^5 + c_4 x_4^5.$$

This makes sense because X_{P° is P^4/G, where $G = \mu_5^3$ acts by

$$(\zeta_1, \zeta_2, \zeta_3) \cdot [x_0 : x_1 : x_2 : x_3 : x_4] = [x_0 : \zeta_1 x_1 : \zeta_2 x_2 : \zeta_3 x_3 : \zeta_1^{-1} \zeta_2^{-1} \zeta_3^{-1} x_4].$$

Now again V° is unchanged by a global scalar and X_{P° has automorphism group of dimension 4, so we in fact have only 1 degree of freedom. This gives $h^{2,1}(V^\circ) = 1$, as desired.

References

[Bat93] Victor V. Batyrev. *Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties*. 1993. arXiv: alg-geom/9310003 [alg-geom].

[BCS05] Lev A. Borisov, Linda Chen, and Gregory G. Smith. "The orbifold Chow ring of toric Deligne-Mumford stacks". In: *J. Amer. Math. Soc.* 18.1 (2005), pp. 193–215. ISSN: 0894-0347,1088-6834. DOI: 10.1090/S0894-0347-04-00471-0. URL: https://doi.org/10.1090/S0894-0347-04-00471-0.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841. ISBN: 978-0-8218-4819-7. DOI: 10.1090/gsm/124. URL: https://doi.org/10.1090/gsm/124.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. Reprint of the 1978 original. John Wiley & Sons, Inc., New York, 1994, pp. xiv+813. ISBN: 0-471-05059-8. DOI: 10.1002/9781118032527. URL: https://doi.org/10.1002/9781118032527.

[Giv98] Alexander Givental. "A mirror theorem for toric complete intersections". In: *Topological field theory, primitive forms and related topics (Kyoto, 1996)*. Vol. 160. Progr. Math. Birkhäuser Boston, Boston, MA, 1998, pp. 141–175. ISBN: 0-8176-3975-6.

[Kon94] Maxim Kontsevich. *Homological Algebra of Mirror Symmetry*. 1994. arXiv: alg-geom/9411018 [alg-geom].

[LLY97] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. "Mirror principle. I". In: *Asian J. Math.* 1.4 (1997), pp. 729–763. ISSN: 1093-6106,1945-0036. DOI: 10.4310/AJM.1997.v1.n4.a5. URL: https://doi.org/10.4310/AJM.1997.v1.n4.a5.