REVIEW LECTURE

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1. Linear algebra review

1.1. Linear algebra over $\mathbb{R}$.

Definition 1.1. A $\mathbb{R}$-*vector space* V is a set V with the operations of addition and scalar multiplication satisfying the following axioms:

(1) Addition is commutative and associative;

(2) There exists $0 \in V$ such that $0 + v = 0$ for all $v \in V$;

(3) For any $v \in V$, there exists $-v \in V$ such that $(-v) + v = 0$;

(4) Scalar multiplication distributes over addition.

Definition 1.2. If $\{v_i\}_{i \in I}$ is a set of vectors in V, then the *span* $\operatorname{span}(v_i)_{i \in I}$ is the set of all linear combinations of the v_i (note that this must be a finite sum).

Definition 1.3. We say $v_1, \ldots, v_n \in V$ are *linear independent* if there does not exist $a_1, \ldots, a_n$, not all zero, such that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

A set $\{v_i\}_{i \in I}$ is *linearly independent* if any finite subset of the v_i is linearly independent.

Definition 1.4. Suppose $\{v_i\}_{i \in I}$ are linearly independent and $\operatorname{span}(v_i)_{i \in I} = V$. Then $\{v_i\}_{i \in I}$ is called a *basis* of V.

Proposition 1.5. *Every vector space* V *has a basis.*

We will denote the standard basis vectors of $\mathbb{R}^n$ by e_i.

Remark 1.6. This is equivalent to the Axiom of Choice.

It turns out that all bases have the same size, so if $\{v_i\}$ is a basis for V, then $|\{v_i\}|$ is called the *dimension* of V.

Definition 1.7. Let V be an $\mathbb{R}$-vector space. Then the *dual space* $V^\vee$ is defined to be the space $V^\vee := \operatorname{Hom}(V, \mathbb{R})$ of linear maps from V to $\mathbb{R}$.

Proposition 1.8. *If* V *is finite-dimensional, then* $\dim V = \dim V^\vee$.

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Proof. Let $n = \dim V$ and choose a basis $v_1, \ldots, v_n$ of V. Then define v_i^* by
$$v_i^*(v_j) = \delta_{ij}.$$
Every linear map $V \to \mathbb{R}$ is uniquely determined by its values on the v_i, so the v_i^* span $V^\vee$. The v_i^* are clearly linearly independent, so they form a basis of $V^\vee$. □

1.2. Linear algebra over $\mathbb{Z}$.

Definition 1.9. A *lattice* is an abelian group N with an isomorphism $N \cong \mathbb{Z}^n$ for some nonnegative integer n.

Spans and linear independence are defined exactly as they are over $\mathbb{R}$, so we can talk about bases for N (in fact the isomorphism $N \cong \mathbb{Z}^n$ is the same thing as a choice of basis for N).

Proposition 1.10. *A matrix $M \in M_n(\mathbb{Z})$ is invertible over $\mathbb{Z}$ (as in M^{-1} has integer entries) if and only if* $\det M = \pm 1$.

Proof. If A is the cofactor matrix of M, then $MA = \det(M)I_n$, and therefore $M^{-1} = \frac{1}{\det(M)}A$. If $\det M = \pm 1$, then $M^{-1} = \pm A$ has integer entries, so M is invertible over $\mathbb{Z}$. If M is invertible over $\mathbb{Z}$, then its determinant has a multiplicative inverse in $\mathbb{Z}$, so it must be ± 1. □

Definition 1.11. If N is a lattice, we define the *dual lattice* M to be the abelian group $\mathrm{Hom}(N, \mathbb{Z})$ of maps of abelian groups (linear maps over $\mathbb{Z}$) from N to $\mathbb{Z}$.

Proposition 1.12. *For any lattice N, $M \cong N$.*

The proof of this fact is exactly the same as over $\mathbb{R}$.

2. A word about projective spaces

2.1. **Toric description.** Consider the vectors $v_1 = e_1, \ldots, v_n = e_n, v_0 =:= -\sum_{i=1}^n e_i$. Then there is a cone σ_i generated by $v_j, j \neq i$. Then there is a fan Δ consisting of the σ_i and all of their faces. Denote the standard basis for $N = \mathbb{Z}^n$ by $e_1, \ldots, e_n$ and the basis for M by $e_1^*, \ldots, e_n^*$.

Definition 2.1. We will define *projective space* by $\mathbb{P}^n = X_\Delta$.

We will now explore some properties of $\mathbb{P}^n$. First note that for each i, σ_i is generated by a basis for N because when $i \neq 0$,
$$-e_i = v_0 + \sum_{j \neq i} v_i.$$
This implies that $U_{\sigma_i} \cong \mathbb{C}^n$. We will now put coordinates on the U_{σ_i}.

Note that $\sigma_0^\vee = \sigma_0$, while for $i = 1, \ldots, n$,
$$\sigma_i^\vee = \left\langle -e_i^*, e_j^* - e_i^* \mid j \neq i \right\rangle$$

because when $j \neq 0$, if $\tau_{ij} = \sigma_i \cap \sigma_j$ is the face of σ_i given by forgetting ν_j, the inward-pointing normal $\underline{p}_{ij} = (p_1, \dots, p_n)$ satisfies the equations

$$\begin{aligned} p_k &= 0 \qquad k \neq i, j \\ p_i + p_j &= 0 \end{aligned}$$

and must have $p_i < 0$ (because any point in σ_i has non-positive i-th coordinate).

Therefore,

$$\mathbb{C}[S_{\sigma_i}] = \mathbb{C}\left[\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{1}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right].$$

Then note that $\tau_{ij} \coloneqq \sigma_i \cap \sigma_j$ is generated by all of the vectors except ν_i, ν_j. Then we can compute

$$\tau_{ij}^\vee = \left\langle -e_i^*, e_j^* - e_i^* \mid j \neq i, e_i^* - e_j^* \right\rangle.$$

This is symmetric in i and j because $e_k^* - e_i^* + (e_i^* - e_j^*) = e_k^* - e_j^*$. Then

$$\mathbb{C}[S_{\tau_{ij}}] = \mathbb{C}\left[\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{1}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right],$$

and this has an automorphism given by multiplication by $\frac{x_i}{x_j}$ (this is the gluing map). If $i \neq 0, j = 0$, then we obtain

$$\mathbb{C}[S_{\tau_{ij}}] = \mathbb{C}[x_1, \dots, x_{i-1}, x_i, x_i^{-1}, x_{i+1}, \dots, x_n],$$

and there is an automorphism given by multiplication by x_i^{-1} (the gluing map).

If we set $x_i = \frac{X_i}{X_0}$, we obtain the uniform description

$$\begin{aligned} \mathbb{C}[S_{\sigma_i}] &= \mathbb{C}\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right], \\ \mathbb{C}[S_{\tau_{ij}}] &= \mathbb{C}\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}, \frac{X_i}{X_j}\right]. \end{aligned}$$

Then $\mathbb{C}[S_{\tau_{ij}}]$ has an automorphism given by multiplication by $\frac{X_i}{X_j}$, which recovers the description given in my first talk via gluing.

2.2. **A word on the quotient construction.** Recall that in my first talk, my first definition of $\mathbb{P}^n$ was as $(\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^\times$. I then used this to obtain the description of $\mathbb{P}^n$ via gluing that I just gave. We can in fact go in the opposite direction to recover the quotient construction from the toric description. You will see some of this story later in the semester, and one of you could give a talk about the quotient description of toric varieties if you want.

Consider the map $\mathbb{Z}^{n+1} \to \mathbb{Z}^n$ given by $e_i \mapsto \nu_i$. This has a kernel generated by $(1, \dots, 1)$, so there is an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^{n+1} \to \mathbb{Z}^n \to 0.$$

Then we can consider the map $\mathbb{C}^\times \to (\mathbb{C}^\times)^{n+1}$ given by $t \mapsto (t, \dots, t)$, and so we see a copy of $\mathbb{C}^\times$ scaling $\mathbb{C}^{\times n+1}$ as in the first talk. Then the polynomials that satisfy $f(tx) = tf(x)$ are simply the monomials $x_0, \dots, x_n$, and the locus where they all vanish is the origin, so we remove the origin and take the quotient (in general, there is a theory of stability that tells us how to do it).