CRASH COURSE ON ALGEBRA AND GEOMETRY

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1. Basic definitions

Here, I will state some definitions of various objects that you have hopefully seen before. If you are unfamiliar with something, please interrupt me.

Definition 1.1. An *abelian group* is a tuple $(G, \cdot, 1)$ of a set G, a multiplication

$$\cdot \colon G \times G \to G,$$

and an element $1 \in G$ satisfying the following properties:

(1) For any $g \in G$, $1 \cdot g = g \cdot 1 = g$;

(2) For any $g_1, g_2, g_3 \in G$, $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$;

(3) For any $g \in G$, there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = 1$;

(4) For any $g_1, g_2 \in G$, $g_1 \cdot g_2 = g_2 \cdot g_1$.

Some examples of abelian groups include $\mathbb{Z}^n$ (with addition), $\mathbb{Z}/n\mathbb{Z}$ (with addition), and $\mathbb{C} \setminus 0$ (with multiplication).

Definition 1.2. Let G be an abelian group and X be a set. Then an *action of* G *on* X is a map

$$\cdot \colon G \times X \to X$$

satisfying the following axioms:

(1) For any $x \in X$, $1 \cdot x = x$;

(2) For any $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$.

A simple example of a group action is if $X = G$ and then $g_1 \cdot g_2$ is simply the multiplication in G.

Definition 1.3. A *commutative ring* is a tuple $(R, +, \cdot, 0, 1)$ consisting of a set R, an addition $+\colon R \times R \to R$, a multiplication $\cdot \colon R \times R \to R$, and $0, 1 \in R$ satisfying the following properties:

(1) For any $r \in R$, $0 + r = r + 0 = r$;

(2) For any $r_1, r_2 \in R$, $r_1 + r_2 = r_2 + r_1$;

(3) For any $r \in R$, there exists $-r \in R$ such that $r + (-r) = (-r) + r = 0$;

(4) For any $r_1, r_2, r_3 \in R$, $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$;

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(5) For any $r \in R$, $1 \cdot r = r \cdot 1 = r$;

(6) For any $r_1, r_2 \in R$, $r_1 \cdot r_2 = r_2 \cdot r_1$;

(7) For any $r_1, r_2, r_3 \in R$, $r_1 \cdot (r_2 \cdot r_3) = (r_1 \cdot r_2) \cdot r_3$;

(8) For any $r_1, r_2, r_3 \in R$, $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$.

Some examples of commutative rings are $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{C}[x]$, and $\mathbb{Z}/n\mathbb{Z}$.

Definition 1.4. Let R be a commutative ring. An *ideal* $I \subset R$ is a subset satisfying the following properties:

(1) For any $a_1, a_2 \in I$, $a_1 + a_2 \in I$;

(2) For any $a \in I$, $-a \in I$;

(3) For any $a \in I$ and $r \in R$, $r \cdot a \in I$.

Some examples of ideals are $n\mathbb{Z} \subset \mathbb{Z}$ for any integer n, $R \subseteq R$ for any commutative ring R, $\{0\} \subset R$ for any commutative ring R, and

$$(a_1, \ldots, a_k) \coloneqq a_1 R + \cdots + a_k R \subseteq R$$

for any $a_1, \ldots, a_k \in R$.

For any ideal $I \subset R$ of a ring R, we can form the *quotient ring* R/I as follows. Define the equivalence relation $\sim$ on R by $r \sim s$ if $r - s \in I$. Then the ring R/I is the set of equivalence classes $[r]$ with the operations

$$[r] \cdot [s] = [r \cdot s] \qquad [r] + [s] = [r + s].$$

This is well-defined because if $r \sim r'$ and $s \sim s'$, then

$$\begin{aligned}(r+s) - (r'+s') &= (r-r') + (s-s') \in I \\ rs - r's' &= rs - rs' + rs' - r's' \\ &= r(s-s') + (r-r')s' \in I.\end{aligned}$$

Clearly addition and multiplication are commutative and associative, and finally it is easy to see that

$$\begin{aligned}[0] + [r] &= [0 + r] = [r] \\ [1] \cdot [r] &= [1 \cdot r] = [r] \\ [-r] + [r] &= [-r + r] = [0],\end{aligned}$$

so $[0]$ and $[1]$ are the additive and multiplicative units, respectively.

2. Some geometry

2.1. **Rings and varieties.** The most basic object we will study is the n-dimensional complex vector space $\mathbb{C}^n$. You will learn during the semester that this is a toric variety, and the combinatorial data associated to it is the cone

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n \geqslant 0\},$$

which is often called the first orthant. Polynomial functions on $\mathbb{C}^n$ are of course simply given by polynomials

$$\mathbb{C}[x_1, \ldots, x_n]$$

in n variables. In fact, all smooth (you will learn what this means later) toric varieties locally look like $\mathbb{C}^n$ for some n.

The second important object is the n-*dimensional torus*

$$(\mathbb{C}^\times)^n \coloneqq \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1, \dots, x_n \neq 0\}.$$

Because all of the coordinates are nonzero, we can take their inverse, so functions on $(\mathbb{C}^\times)^n$ are given by *Laurent polynomials*

$$\mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}] = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]/(x_1y_1 - 1, \dots, x_ny_n - 1).$$

Note that we can multiply elements of $(\mathbb{C}^\times)^n$ via the formula

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) \coloneqq (x_1y_1, \dots, x_ny_n).$$

The unit for this multiplication is clearly $(1, \dots, 1)$, and elements are invertible via

$$(x_1, \dots, x_n)^{-1} \coloneqq (x_1^{-1}, \dots, x_n^{-1}).$$

This gives the torus the structure of an *abelian group*. Now there is an *action* of $(\mathbb{C}^\times)^n$ on $\mathbb{C}^n$ given by the formula

$$(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1x_1, \dots, t_nx_n).$$

You will learn later that this is a key feature of all toric varieties, but this viewpoint will arise only a few times this semester.

In general, if $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$, then there is an algebraic variety[1]

$$V(f_1, \dots, f_k) \coloneqq \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid f_1(\mathbf{x}) = \cdots = f_k(\mathbf{x}) = 0\}$$

whose polynomial functions are given by the ring

$$\mathbb{C}[x_1, \dots, x_n]/(f_1, \cdots, f_k).$$

We will require that this ring is an *integral domain*, or in other words that if $r \cdot s = 0$, then either $r = 0$ or $s = 0$. This will geometrically correspond to the variety having only one component.

2.2. **More general varieties.** In general, *algebraic varieties* are constructed by gluing together things that look like $V(f_1, \dots, f_k)$. I will not explain how gluing works in general, but instead I will give a few examples. The most important example is called *projective space* and has many different representations. The first is as the quotient

$$(\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\})/\mathbb{C}^\times,$$

where $\mathbb{C}^\times$ acts by scaling:

$$t \cdot (x_0, \dots, x_n) = (tx_0, \dots, tx_n).$$

Therefore, any point in $\mathbb{P}^n$ can be described by *homogeneous coordinates* $[X_0, \dots, X_n]$, where for any nonzero $t \in \mathbb{C}$, the coordinates $[X_0, \dots, X_n]$ and $[tX_0, \dots, tX_n]$ describe the same point.

Because at least one of the coordinates must be nonzero, suppose that $X_0 \neq 0$. Then dividing by X_0, any point in $(X_0 \neq 0) \subset \mathbb{P}^n$ can be described by the coordinates

$$\left[1, \frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right].$$

[1] This is not strictly true, but will be OK in all the examples we consider.

If we set $x_i = \frac{X_i}{X_0}$, we see that we have a copy of $\mathbb{C}^n$. But now we can consider the chart where $X_1 \neq 0$, and here the coordinates are now

$$\left[\frac{X_0}{X_1}, 1, \frac{X_2}{X_1}, \dots, \frac{X_n}{X_1}\right].$$

If we write $y_i = \frac{X_i}{X_1}$, then we have another copy of $\mathbb{C}^n$. These two copies of $\mathbb{C}^n$ overlap where both $X_0, X_1 \neq 0$. In this region, to convert the x_i to the y_i, we can see that

$$\begin{aligned} y_0 &= \frac{X_0}{X_1} = \left(\frac{X_1}{X_0}\right)^{-1} = x_1^{-1} \\ y_2 &= \frac{X_2}{X_1} = \frac{X_2}{X_0} \cdot \frac{X_0}{X_1} = x_2 x_1^{-1} \\ &\vdots \\ y_n &= \frac{X_n}{X_1} = \frac{X_n}{X_0} \cdot \frac{X_0}{X_1} = x_n x_1^{-1}. \end{aligned}$$

The change of coordinates between the $X_i \neq 0$ and $X_j \neq 0$ charts is similar.

Instead of beginning with the description of $\mathbb{P}^n$ as a quotient (in fact all toric varieties can be described this way), we could in principle have started with $n+1$ copies of $\mathbb{C}^n$ with coordinates

$$x_1^0, \dots, x_n^0, x_0^1, \widehat{x}_1^1 \dots, x_n^1, \dots, x_0^k \dots, \widehat{x}_k^k, \dots, x_n^k \dots, x_0^n, \dots, x_{n-1}^n$$

(here the hat means that x_k^k is omitted) and then specified the transition maps

$$x_i^\ell = x_i^k (x_\ell^k)^{-1}$$

between the k-th and ℓ-th copies of $\mathbb{C}^n$ whenever $k \neq \ell$ (here, we set $x_k^k = 1$ for the purpose of this formula).

Later, you will learn that $\mathbb{P}^n$ is a toric variety associated to the fan obtained by considering all cones generated by subsets of up to n of the vectors

$$e_1, e_2, \dots, e_n, -e_1 - e_2 - \cdots - e_n,$$

where e_i is the vector with 1 in the i-th coordinate and 0 in the other coordinates.

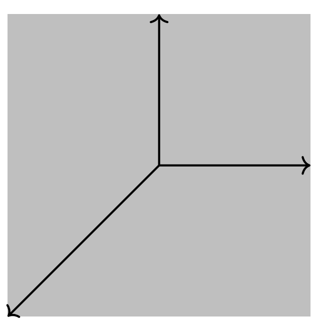

FIGURE 1. Fan of $\mathbb{P}^2$

3. Introduction to toric varieties

We will now turn to the subject of this seminar. A classical viewpoint on toric varieties can be seen in the sequence of inclusions

$$(\mathbb{C}^\times)^n \subset \mathbb{C}^n \subset \mathbb{P}^n.$$

Other examples include products of projective spaces and quotients of $\mathbb{C}^n$ by finite abelian groups. We can define a toric variety to be a variety X of dimension n with an action of $(\mathbb{C}^\times)^n$ such that there is an orbit isomorphic to $(\mathbb{C}^\times)^n$. This definition explains the original name of toric varieties as *torus embeddings*, but it completely obscures the relationship with combinatorics that we will emphasize this semester.

3.1. Definition of a toric variety.

Instead, we will construct a toric variety as follows. We will first consider a lattice $N = \mathbb{Z}^n$ and a *fan* Σ in N, which is a collection of strongly convex rational polyhedral cones in $N_\mathbb{R} = \mathbb{R}^n$.

Definition 3.1. A *strongly convex rational polyhedral cone* $\sigma \subset \mathbb{R}^n$ is a cone

$$\mathbb{R}_{\geqslant 0} \cdot v_1 + \cdots + \mathbb{R}_{\geqslant 0} \cdot v_k$$

such that

(1) $v_1, \ldots, v_k \in \mathbb{Z}^n$;

(2) If $0 \neq v \in \sigma$, then $-v \notin \sigma$.

We will call these simply cones. We then consider the lattice $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ with the pairing (in practice just the usual dot product)

$$\langle -, - \rangle : M \otimes N \to \mathbb{Z} \qquad \langle u, v \rangle = u(v).$$

Then we define the *dual cone* $\sigma^\vee \subseteq M_\mathbb{R} = \mathbb{R}^n$ by the formula

$$\sigma^\vee := \{u \in M_\mathbb{R} \mid \langle u, v \rangle \geqslant 0 \text{ for all } v \in \sigma\}.$$

Then we consider the semigroup $S_\sigma := \sigma^\vee \cap M$, and then we may consider the commutative ring

$$\mathbb{C}[S_\sigma] = \bigoplus_{u \in S_\sigma} \mathbb{C} \cdot x^u,$$

where $x^v \cdot x^{u'} = x^{u+u'}$ and $1 = x^0$. This determines an algebraic variety X_σ. Now if $\tau \subset \sigma$ is a face, $\sigma^\vee \subseteq \tau^\vee$, and therefore $S_\sigma \subseteq S_\tau$, so there is an inclusion $\mathbb{C}[S_\sigma] \subseteq \mathbb{C}[S_\tau]$. This defines a map $X_\tau \subseteq X_\sigma$. Thus, if τ is a face of both σ, σ', X_σ and $X_{\sigma'}$ are glued together along X_τ. From the fan Σ, this gluing process determines an algebraic variety, which is called X_Σ. For this seminar, we will take the perspective that a toric variety is something abstractly determined by the fan Σ, which will be the primary object of study.

3.2. **Some examples.** We will conclude with some examples.

Example 3.2. Consider the fan in $\mathbb{R}^2$ consisting of the cone

$$\sigma = \{(x,y) \mid x \geqslant 0, y \geqslant 0\}.$$

Then the dual cone is

$$\begin{aligned} \sigma^\vee &= \{(a,b) \mid ax + by \geqslant 0 \text{ for all } x, y \geqslant 0\} \\ &= \{(a,b) \mid a, b \geqslant 0\}. \end{aligned}$$

Therefore, $S_\sigma = \left\{(a,b) \in \mathbb{Z}^2 \mid a, b \geqslant 0\right\}$, and therefore,

$$\mathbb{C}[S_\sigma] = \bigoplus_{a,b \geqslant 0} \mathbb{C} \cdot x^a y^b = \mathbb{C}[x,y],$$

so $X_\sigma = \mathbb{C}^2$.

Example 3.3. Consider the fan in Figure 1 with σ_1 the first quadrant and σ_2, σ_3 numbered counterclockwise. Also write $v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1)$. Then we can compute

$$\begin{aligned} \sigma_1^\vee &= \{(a,b) \mid a, b \geqslant 0\} \\ \sigma_2^\vee &= \{(a,b) \mid b \geqslant 0, a + b \leqslant 0\} \\ \sigma_3^\vee &= \{(a,b) \mid a \geqslant 0, a + b \leqslant 0\}. \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{C}[\sigma_1] &= \mathbb{C}[x,y] \\ \mathbb{C}[\sigma_2] &= \mathbb{C}[x^{-1}, x^{-1}y] \\ \mathbb{C}[\sigma_3] &= \mathbb{C}[y^{-1}, xy^{-1}]. \end{aligned}$$

If we set $x = \frac{X_1}{X_0}$ and $y = \frac{X_2}{X_0}$, these glue to form $\mathbb{P}^2$ in the way described in Section 2.2.

Example 3.4. Consider the fan in $\mathbb{R}^n$ defined by the cone $\sigma = \{0\}$. Then $\sigma^\vee = M = \mathbb{R}^n$, so $S_\sigma = \mathbb{Z}^n$. Finally, we obtain $\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, so $X_\sigma = (\mathbb{C}^\times)^n$, as discussed in Section 2.1.