Talk - 9/20/23

Title: Expanding upon Affine Toric Varieties

First we start with a few definitions:

Def. Semigroup: A set S alongside a binary operation · which satisfies the associativity property. So for $(S, \cdot)$ such that

$$\forall a, b, c \in S, (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

An additive semigroup is when the binary operation is addition.

Def. Prime Ideal: An ideal P of a commutative ring R is prime if it satisfies

- If for $a, b \in R$ and $ab \in P$ then $a \in P$ or $b \in P$
- P is not the whole ring R

This definition expands upon our definition of prime numbers. "if *p* is a prime number and if *p* <u>divides</u> a product *ab* of two <u>integers</u>, then *p* divides *a* or *p* divides *b*"

Def. Maximal Ideal: A maximal ideal, J, of a Ring R is an Ideal if it is not equal to R such that there are no ideals which are a subset of R and contain an a subset J.

Lemma: if m is a max ideal then m is prime

Def. Spec(A): Assuming A is a ring then Spec(A) is the set of all prime ideals of A.

Def. Specm(A): A closed point of Spec(A) corresponding to the maximum ideal.

Affine Toric Variety

Now we can get into defining an affine toric variety. From Gordan's lemma we know that

$$S_\sigma = \sigma^\wedge \cap M$$

where N is a lattice and $M = Hom(N, Z)$ is the dual lattice that consists of the grid points in this space.

Then the group ring $C[S]$ "C adjoined S" is a commutative algebra. It is a complex vector space so we can have its basis be χ^u as u spans S. Therefore addition in S is determined by multiplication in the complex space for two points in $u', u \in S$ like so:

$$\chi^u \cdot \chi^{u'} = \chi^{u+u'}$$

Therefore generators of S uniquely correspond to χ^u for the $\mathbb{C}[S]$.

Now we let A be any finitely generated commutative C-algrebra and we pick the generators of A so we can write it as

$$C[X_1, X_2, ..., X_m]/I$$

where I is an Ideal. Then $Spec(A)$ is all prime ideals of this and $Specm(A)$ is the point corresponding the the maximum ideal.

We let $A = C[S]$ then we denote the complex Affine variety as Spec(A) then any homomorphism (structure preserving map of the form $f(x) \cdot f(y) = f(x \cdot y)$) from A to B determines a morphism from $Spec(A)$ to $Spec(B)$ (the equivalent for sets).

Then we choose a point $f \in A$ and we define

$$A_f = A[1/f]$$

so $Spec(A_f) > Spec(A[1/f])$ because we take out what doesn't invert. When $A = C[S]$ is constructed from a semigroup the points correspond to homomorphisms of semigroups from S to $\mathbb{C}$.

and when $S = S_\sigma$ is convex rational polyhedral cone (from last lecture, also a semigroup) then we let $A_\sigma = C[S_\sigma]$ and

$$U_\sigma = Spec(C[S_\sigma]) = Spec(A_\sigma)$$

where this is the corresponding **Affine Toric Variety.**

Examples

#1:

Let the cone σ to be defined by the points (0,1) and (n, -1) then $\sigma^\wedge$ is generated by (1, 0), (1,1) , …. So, if we label these with $v_0, ..., v_n$ then we can get the relations

$$v_0 + v_2 = 2v_1$$

which generalizes to

$$v_n + v_{n+2} = 2v_{n+1}$$

so it looks like

$$\mathbb{C}[S] = \mathbb{C}[x_1, ..., x_n] = \frac{\mathbb{C}[Y, Z]}{(x_j * x_{j+2} - x_{j+1}^2)_0^n}$$

where x_i corresponds with v_i and the denominator is the ideal at 0.

#2:

If we look at a group $S = \{2, 3, 4, 6, 7, ...\}$ then S is generated by the elements 2 and 3. Then we can write

$$\mathbb{C}[S] = \mathbb{C}[X^2, X^3] = \frac{\mathbb{C}[Y, Z]}{Z^2 - Y^3}$$

So $Spec(\mathbb{C}[S])$ is a rational curve with a cusp.

Lemmas/Exercises

Prove: If τ is a face of σ then the map $U_\tau \to U_\sigma$ embeds U_τ as a principle open subset of U_σ.

Prove: If τ is a subset of σ and the map $U_\tau \to U_\sigma$ embeds U_τ as a principle open subset of U_σ then τ must be a face of σ.