Divisors of toric varieties

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Abstract

In this talk, we'll cover the divisors of varieties of different types, that is, *Weil* divisors and *Cartier* divisors, and their consequences on characterizing simplicial cones and a surjectivity of their constituents' inner products. This talk has been typed up following section 3.3 of *Introduction to Toric Varieties* authored by William Edgar Fulton in 1993 [Ful93]: Divisors, with results informed by those presented by graduate student Avi Zeff from *Intersection Theory* last published under Fulton in 1998 [Ful98], in Columbia's Spring 2021 seminar on intersection theory.

1 Types of divisors

Definition 1.1. A *Weil divisor* on a variety X is a finite formal sum $\sum_i a_i V_i$ of irreducible (if it cannot be written as the union of nonempty algebraic varieties), closed subvarieties of codimension one in X. The Weil divisors form a group $Z_{n-1}(X)$.

Definition 1.2. A *Cartier divisor* D consists of data about a covering of X by affine (if for any two distinct points in the set, the line passing through these points lie in the set itself) open sets U_α, that is, an open set that is isomorphic to $\operatorname{Spec} R$ for some commutative ring R and the nonzero rational functions f_α called *local equations*, such that ratios f_α/f_β are always supported (never zero) over $U_\alpha \cap U_\beta$, and *regular* (everywhere-defined, polynomial on the subvarieties). That is, it consists of the following data:

1. an open cover $\{U_\alpha\}_\alpha$ of X
2. for each index α, an associated rational function f_α on open U_α, defined up to multiplication by a *unit* (as in it does not admit any zeroes or poles) such that for any α, β we have f_α/f_β a unit on $U_\alpha \cap U_\beta$.

Remark 1.3. As such, like the Weil divisors, the Cartier divisors form an abelian group with $(\{U_\alpha, f_\alpha\}) + (\{U_\alpha, g_\alpha\}) = (\{U_\alpha, f_\alpha g_\alpha\})$ as the open covers are either the same or just refine to $\{U_\alpha \cap V_\beta\}$. So, we call this abeilan group $div(X)$ and for any Cartier divisor D the order $ord_V(D) = ord_V(f_\alpha)$ for α such that $U_\alpha \cap V$ is nonempty.

Definition 1.4. A *vector bundle* is a collection of vector spaces that varies in a geometric way over a given base space V: over each element $x \in V$ there is a vector space V_x, called the *fiber* over x, and as x varies in X, the fibers vary along in a geometric way. A *line bundle* is a vector bundle of dimension 1, that is, a vector bundle whose typical fiber is a 1-dimensional vector space (a line).

Definition 1.5. The *ideal sheaf* $\mathscr{O}(-D)$ of D is the *subsheaf* (since this is invertible, it is called a *line bundle*) of that of rational functions generated by f_α on U_α, that is, the inverse sheaf $\mathscr{O}(D)$ is the subsheaf of that of rational functions generated by $1/f_\alpha$ on U_α. Indeed, the sheaf's *transition functions* on U_α to U_β, that is, the quotient f_α/f_β (cf. projective functionals), so a Cartier divisor D determines a Weil divisor $[D] = \sum_{\text{codim}(V,X)=1} \text{ord}_V(D) \cdot V$ where the *order of vanishing* of an equation for D in the local ring along V the subvariety represents $\text{ord}_V(D)$. As such, the *associated Weil divisor* of a Cartier divisor D is $[D] = \sum_V ord_V(D) \cdot [V]$ for V a codimension 1 subvariety of X.

Definition 1.6. The *Picard group* $\text{Pic}(X)$ of a *ringed space* (a family of rings parametrized by open subsets of a topological space together with ring homomorphisms) X is the group of isomorphism classes of invertible sheaves (or line bundles) on X, with the group operation being tensor product.

2 Divisors on toric varieties

We now inspect divisors on a toric variety $X(\Delta) = X$ that map to themselves on torus $T = T_N$ (latticized). The irreducible, closed subvarieties of codimension one in X that are stable on the torus T corresponding to edges / rays of the fan on which X is constructed. Thus, edges $\tau_1, \dots, \tau_d$ and v_i the first lattice point met along τ_i gives us the divisors:

Definition 2.1. The *divisors* of a toric variety X on a fan Δ with edges $\tau_1, \dots, \tau_d$ are the *orbit closures* $D_i = V(\tau_i)$ and the *T-Weil divisors* are all sums $\sum_i a_i D_i$ for integers a_i, and so the Cartier divisors that are equivalent under the torus T are *T-Cartier divisors* – we say that two Cartier divisors D and D' are *linearly equivalent* if $D - D' = div(f)$ for some f.

Remark 2.2. In the way we defined a Cartier divisor, we attain a line bundle on X. That is, given a divisor $D = (\{U_\alpha, f_\alpha\})$, construct line bundle $L = \mathcal{O}(D)$ to be trivialized on each U_α with transition functions f_α/f_β. Then, the abelian group of such line bundles on X with group operation given by tensor product allow as to characterize the Picard group on the variety of X since two Cartier divisors D and D' are clearly linearly equivalent if and only if $\mathcal{O}(D) = \mathcal{O}(D')$. Conversely, a line bundle L determines a Cartier divisor $D(L)$ up to the property that it has a nonzero rational section s of L.

Remark 2.3. Informed by the above characterizations, Cartier divisors can be thought of as the data of a line bundle together with associated nonzero rational section that it has.

Lemma 2.4. For any rational f on X, a *principal Cartier divisor* $div(f)$ is yielded by choosing a cover $\{U_\alpha\}$ and defining $f_\alpha = f|_{U_\alpha}$ so the image $[div(f)]$ of this divisor under its map to $Z_{n-1}X$ is the *Weil principal divisor*. So, with $\text{Pic}\, X$ the group of Cartier divisors modulo linear equivalence, the above shows that the map $Div(X) \to Z_{n-1}X$ descends to a map $\text{Pic}\, X \to A_{n-1}X$ affine. The map $D \to [D]$ embeds Cartier divisors within Weil divisors, and as such we denote a divisor $div(f)$ determined by a nonzero rational function f to be a *principal divisor* when the local equation in each open set is f

Definition 2.5. The *support* $|D|$ of a Cartier divisor D is the union of codimension one subvarieties V of X such that f_α is not a unit for U_α that nontrivially intersects V (that is, $ord_V(D) \neq 0$). So, a Cartier divisor $D = (\{U_\alpha, f_\alpha\})$ if all functions f_α are regular (admit no poles).

Example 2.6. Consider affine toric variety $X = U_\sigma$ where $\dim(\sigma) = n \in \mathbb{N}_0$ and D a divisor preserved by torus T corresponding to the *fractional ideal* $I = \Gamma(X, \mathscr{O}(D))$ (non-trivial subset of a fraction field $\text{Frac}\, R$ over a commutative ring R for which a $0 \neq r \in R$ exists so that $rI \subset R$ is an ideal in R:

Proposition 2.7. *I is generated by a function χ_u for a unique $u \in \sigma^{\checkmark} \cap M$. It follows that such a unique u exists with $i = A_\sigma \cdot \chi^u$.*

Lemma 2.8. A general T-Cartier divisor on U_σ has the form $div(\chi^u)$ for a unique $u \in M$ lattice.

Theorem 2.9. For $u \in M$ and v the first lattice point along an edge τ, $ord_{V(\tau)}(div(\chi^u)) =< u, v >$ yields that Weil divisor $[div(\chi^u)] = \sum_i < u, v_i > D_i$.

Proof. The order on the open set $U_\tau \cong \mathbb{C} \times (\mathbb{C}^*)^{n-1}$ is apparent, on which $V(\tau)$ is associated to $\{0\} \times (\mathbb{C}^*)^{n-1}$. Therefore we reduce our computation to the case where $N = \mathbb{Z}$ a single-dimensional lattice so τ generated by $v = 1$ and $u \in M = \mathbb{Z}$. Easily, χ^u is the resultant monomial of X^u with *order of vanishing* being u at origin 0.

Example 2.10. Take cone $\sigma \subset \mathbb{Z}^2$ generated by $v_1 = (2, -1), v_2 = (0, 1)$ so U_σ admits two T-Weil divisors D_1 and D_2 that are simply straight lines on the cone. Then, for $u = (p, q) \in M = \mathbb{Z}^2$, it is clear that $div(\chi^u) = (2p - q)D_1 + qD_2$ and thus $2D_1, 2D_2$ are Cartier divisors even though D_1 and D_2 are not!

Exercise 2.11. For $\sigma \subset \mathbb{Z}^2$ generated by $v_1 = (2, -1), v_2 = (-1, 2)$ corresponding to divisors D_1, D_2, show that $a_1 D_1 + a_2 D_2$ is a Cartier divisor on U_σ if and only if $a_1 \equiv a_2 \mod 3$.

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References

[Ful93] William Fulton. *Introduction to Toric Varieties. (AM-131), Volume 131.* Princeton University Press, Princeton, NJ, 1993.

[Ful98] William Fulton. *Intersection Theory.* Springer, New York, NY, 1998.