Fans and general toric varieties

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Abstract

In this talk, we'll cover the construction of general toric varieties over fans utilizing previous constructions of affine toric varieties with spectrums by discussing disjoint unions as 'gluing' operations, as well as some interesting results about structure preservation between varieties generated from fans belonging to different, but structurally similar lattices. This talk has been typed up following section 1.4 of *Introduction to Toric Varieties* authored by William Edgar Fulton in 1993 [Ful93]: Fans and general toric varieties.

1 Introduction

In the previous talk covering section 1.3 [Ful93]: Affine toric varieties, we discussed how to construct an affine toric variety on a cone σ. We first note that discussing such objects, we impose going forward that our cones are rational (generated by vectors in a lattice), strongly convex (contain no lines through the origin), and polyhedral (a 'polyhedron' that is also a cone in our sense of the form $\{\sum_{i=1}^{n} a_i v_i : a_i \in \mathbb{R}_{\geq 0}, v_i \in \mathbb{R}^n\}$ for 'basis' vectors $\{v_1, v_2, \ldots, v_n\}$, i.e. it geometrically has a polyhedral/polygonal base at suitable dimensions, that are the rays of the cone).

Definition 1.1. A *fan* Δ is a set of cones such that:

1. For all cones $\sigma \in \Delta$, any face τ of σ is a cone τ in Δ.
2. for all cones $\sigma_1, \sigma_2 \in \Delta$, $\sigma_1 \cap \sigma_2$ is a face τ of both σ_1, σ_2.

Remark 1.2. It is trivial that for all cones $\sigma_1, \sigma_2 \in \Delta$, $\sigma_1 \cap \sigma_2$ is a cone in Δ (closed under intersection). We thus proceed with our construction, prefaced with some remarks that tie into the topological interpretations of our work.

2 Zariski topologies and varieties

To interpret our broad work from last time, we first shed light on the nature of varieties.

Definition 2.1. An *affine variety* is the set of solutions to a homogenous system of polynomial equations over a an algebraically closed field $\mathbb{F}$ (if a ring is an abelian group under addition and a monoid under multiplication, a field is an abelian group under addition and multiplication, i.e. it is a ring that also demonstrates multiplicative invertibility for non-zero elements, commutativity, and is with unity). That is, for $S = \{f_1, f_2, \ldots\}$ for polynomials $f \in \mathbb{F}[x_1, \ldots, x_n]$, the affine variety $V(S) = \{v \in \mathbb{F}^n : f(v) = 0 \text{ for all } f \in S\}$. A *subvariety* W of a variety V is a variety such that $W \subset V$.

Example 2.2. If $\mathbb{F} = \mathbb{C}$ and $S = \{z^n - 1 : n \in \mathbb{N}\}$, then the affine variety associated to S is $\cup_{n\in\mathbb{N}}\{e^{2\pi ik/n} : 0 \le k \le n-1\}$ for $k \in \mathbb{N}_0$, that is, it is the set of ALL roots of unity in the complex plane which behaves correspondingly due to the *fundmental theorem of algebra*. Similarly, $S = \{\Phi_n(x) : n \in \mathbb{N}\}$, the set of *cyclotomic polynomials* (real-coefficiented in one variable, but have primitive nth roots of unity as solutions), has the associated affine variety $\cup_{n\in\mathbb{N}}\{e^{2\pi ik/n} : 0 \le k < n, \gcd(k,n) = 1 \text{ or } k = 0\}$

Definition 2.3. A *topology* $\tau \subset \mathcal{P}(X)$ on a set X is a collection of subsets of X that contains $\emptyset$, X, is closed under all arbitrary unions as well as under finite intersections, and the structure (X,τ) is a *topological space*.

Definition 2.4. For an ideal I of a ring R, we consider a 'closed set' $V(I)$ in the following sense: with $V(I) = \{J : J \text{ prime ideal}, I \subset J\}$, the set $\mathcal{T} = \{V(I) : I \text{ ideal of } R\}$ of all such closed sets is the *Zariski topology*, which is generalized to make the set of prime ideals of a commutative ring (called the spectrum of the ring) a topological space.

Remark 2.5. The Zariski topology allows tools from topology to be used to study algebraic varieties, even when the underlying field is not a topological field. Thus, the Zariski topology of an algebraic variety is the topology whose closed sets are the algebraic subsets of the variety. Additionally. the Zariski topology on the set of prime ideals (spectrum) of a commutative ring is the topology such that a set of prime ideals is closed if and only if it is the set of all prime ideals that contain a fixed ideal.

3 Expanding on affine toric varieties

We know that the *affine toric variety* over a cone σ, denoted U_σ, equals $\text{Spec}(\mathbb{C}[S_\sigma]) = \text{Spec}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n])$, that is, the set of prime ideals of the space of complex-coefficient polynomials over elements of the dual cone, latticized.

Remark 3.1. We first note that for variables x, y, $\text{Spec}(\mathbb{C}[x,y]) = (0) \cup \{(x-a, y-b : a, b \in \mathbb{C}\} \cup \{(f) : f(x,y) \text{ irreducible polynomial}\}$. The spectral structure is much harder to investigate for more adjoined variables, but it serves to suffice for the most elementary examples.

Example 3.2. Taking the cone σ generated by $u_1 = (2,0), u_2 = (-1,-1)$, our dual $\sigma^\vee$ is clearly generated by $u_1' = (0,2), u_2' = (-1,1)$, whose lattice S_σ is spanned by these vectors. Then, $\mathbb{C}[S_\sigma] = \mathbb{C}[\vec{x}^{u_1'}, \vec{x}^{u_2'}] = \mathbb{C}[y^2, \frac{y}{x}]$. So, $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) = (0) \cup \{(y^2 - a, \frac{x}{y} - b : a, b \in \mathbb{C}\} \cup \{(f) : f(y^2, \frac{x}{y}) \text{ irreducible polynomial in } y^2, \frac{x}{y}\}$.

Remark 3.3. It turns out that (0) corresponds to the trivial origin, and these prime ideals indeed act analogously to points of the face of each cone in the fan by closure under their generators, which gives way to our gluing.

4 General toric varieties

Definition 4.1. The *toric variety* $X(\Delta)$ of a fan Δ of cones is the structure obtained from the *disjoint union* of our affine toric varieties U_σ over each cone $\sigma \in \Delta$: $\sqcup_{\text{cones } \sigma\in\Delta} U_\sigma = \cup_{\sigma\in\Delta}\{(v,\sigma) : v \in U_\sigma\}$, that is, an adhesion such that for cones $\sigma_1, \sigma_2 \in \Delta$, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

Definition 4.2. A toric variety is *nonsingular* if its cones of maximal dimension are generated by a basis of the lattice over which it adheres.

Remark 4.3. Since each face itself is a cone in the fan, the affine toric variety $U_{\sigma_1 \cap \sigma_2}$ is a *subvariety* of U_{σ_1} and of U_{σ_2} that is also contained in the Zariski topology we expanded upon earlier, with the base $\text{Spec}(R)$ for ring R. Then, for σ a cone in our lattice, we can let σ along all of its faces (which, by definition, are cones themselves) comprise the fan Δ. Then, not only is $X(\Delta)$ simply the fan's affine toric variety U_σ, but only for such a fan construction do we get an affine toric variety from our gluings.

Example 4.4. In just $\mathbb{Z}$, we only have left and right rays along with the origin for our cones, so we get $\mathbb{C}$, $\mathbb{C}^*$, and $\mathbb{P}^1$ (can be visualized as the space of lines through the origin in $\mathbb{C}^2$) as our possible toric varieties depending on the possible fans we choose from these (possible unions generated from the cones).

Example 4.5. Consider in $\mathbb{R}^2$ a fan Δ of two cones σ_1, σ_2 generated by $u_1 = (-1, 0)$ and $u_2 = (0, 1)$ / u_2 and $u_3 = (1, 0)$ respectively, i.e. they share a face u_2. In vein with the above example, we can take, by way of the fans being formed by two rays each, two copies of $\mathbb{C}^2$ associated with our adjoined rings $\mathbb{C}[\frac{1}{x}, y], \mathbb{C}[x, y]$. So, because our gluing (considering the shared face u_2 that is thus also a cone in its own dimension), we consider the faces, themselves cones in Δ by closure, so u_1, u_3 with the origin lead to $\mathbb{P}^1$ (from two opposite direction rays with origin) and u_2 leads to $\mathbb{C}$ (one ray with origin), we see our toric variety $X(\Delta) = \mathbb{P}^1 \times C$. Indeed, if we added $u_4 = (0, -1)$ to our collection of faces thus reflecting σ_1, σ_2 to σ_3, σ_4 additional cones, we see that u_2, u_4 with that origin leads to $\mathbb{P}^1$ itself, so $X(\Delta) = \mathbb{P}^1 \times \mathbb{P}^1$.

Corollary 4.5.1. We see that for Δ, Δ' fans in lattices $\mathbb{Z}^m, \mathbb{Z}^n$ respectively for $m, n \in \mathbb{N}$, we can generate a *product fan* in $\mathbb{Z}^{m+n}$ from the set of all Cartesian products $\sigma \times \sigma'$ for $\sigma \in \Delta$, $\sigma' \in \Delta'$, and indeed due to the reflected structure preserved with our affine toric varieties, the toric variety $X(\Delta \times \Delta') = X(\Delta) \times X(\Delta')$. From this, it is clear we see that the *projective n-space* $\mathbb{P}^n = (\mathbb{C}^{n+1}) * /(\mathbb{C}^*)$, in line with our geometric interpretation, because a collection of $n + 1$ vectors that generate an n-dimensional lattice have a corresponding toric variety (from the fan with cones generated by any proper subset of those vectors) that is 'isomorphic' to $\mathbb{P}^n$, i.e. there exists a one-to-one correspondence (mapping) between two of these groups that preserves operational structure between elements of the groups, expanded upon below.

Example 4.6. We work through toric varieties generated by the fans:

1. What is the toric variety $X(\Delta_1)$ corresponding to the fan Δ_1 generated from cones $\sigma_1, \sigma_2 \in \mathbb{Z}^2$ where σ_1 has faces $u_1 = (0, 1)$, $u_2 = (-1, 0)$ and σ_2 has faces u_2 and $u_3 = (0, -1)$?
2. What is the toric variety $X(\Delta_2)$ corresponding to the fan Δ_2 generated from cones $\sigma_1, \sigma_2 \in \mathbb{Z}^2$ where σ_1 has faces $u_1 = (1, 0)$, $u_2 = (0, 1)$ and σ_2 has faces $u_3 = (-1, 0)$ and $u_4 = (0, -1)$?
3. What is the toric variety $X(\Delta_3)$ corresponding to the fan Δ_3 generated from cones $u_1 = (1, 0)$, $u_2 = (0, 1)$ in $\mathbb{Z}^2$?

Theorem 4.7. Now, we have a *homomorphism* $\varphi : \mathbb{Z}^m \to \mathbb{Z}^n$ for $n, m \in \mathbb{N}$, i.e. there exists a mapping between two groups that preserves operational structure between elements of the groups (an isomorphism is a bijective homomorphism, as above), that is we have an $\varphi : \mathbb{Z}^m \to \mathbb{Z}^n$ such that for all $x, y \in \mathbb{Z}^m$, $\varphi(x+y) = \varphi(x)+\varphi(y)$ (the group operation is addition because we have zero, which would of course not have a multiplicative inverse), i.e *Cauchy's functional equation* is satisfied. Then, for fans of cones $\Delta \in \mathbb{Z}^m$ and $\Delta' \in \mathbb{Z}^n$, we know that for any cone $\sigma' \in \Delta'$, there exists a cone $\sigma \in \Delta$ such that $\varphi(\sigma') = \{\varphi(x) : x \in \sigma'\} \subset \sigma$. That is, a *morphism* (structure-preserving map from one mathematical structure to another one of the same type, analogous to homomorphism but used in references to structures beyond basic algebraic groups or rings that can have embedded topologies, etc.) $U_{\sigma'} \to U_\sigma \subset X(\Delta)$ the toric variety associated to Δ is *induced* by the homomorphism $\varphi : \mathbb{Z}^m \to \mathbb{Z}^n$.

Corollary 4.7.1. In fact, we actually see that in patching together these morphisms $U_{\sigma'} \to X(\Delta)$ as a whole results in the overarching morphism $\varphi_* : X(\Delta') \to X(\Delta)$. That is, toric varieties of fans in homomorphic lattices are themselves similar in structure!

5 Conclusion

Today, we defined many fundamental objects and notions along with our construction which all prove to be deeply useful over more complex varieties like those dealing with more intricate varieties generated from convex *polytopes* (polyhedra generalized over more dimensions) that reflect conic properties. In a nutshell, our examples covered *affine* and *projective toric varieties* derived from *normal* (vis-à-vis dual) cones.

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References

[Ful93] William Fulton. *Introduction to Toric Varieties. (AM-131), Volume 131.* Princeton University Press, Princeton, 1993.