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Section 3.4 Line Bundles

line bundle: a vector bundle of dimension 1, a vector bundle whose typical fiber is a line

Pic(X) group of all line bundles, modulo isomorphism, also know as Picard group   Pic(X) = Cartier divisors / linear equivalence

For any irreducible variety X, map Cartier divisor on X → Pic(X) gives a homomorphism

$D \to \mathcal{O}(D)$ (inverse sheaf), ker = group of principal divisors

$A_{n-1}(X)$: group of all Weil divisors modulo subgroup of divisors [div(f)] of rational functions

map: $Pic(X) \to A_{n-1}(X)$ an embedding when X is normal

$D \to [D]$

$$\begin{array}{ccccc} \{\text{principal Cartier divisor}\} & \subset & \{\text{Cartier divisor}\} & \twoheadrightarrow & Pic(X) \\ \shortparallel & & \downarrow & & \downarrow \\ \{\text{principal Weil divisor}\} & \subset & \{\text{Weil divisor}\} & \twoheadrightarrow & A_{n-1}(X) \end{array}$$

X toric variety, $\forall u \in M$ determines a principal Cartier divisor $div(\chi^u)$

homomorphism $M \to Div_T X$ (T-Cartier divisors)

$u \to div(\chi^u)$

where $M = N^\vee = Hom(N, \mathbb{Z})$. if $N \cong \mathbb{Z}^n$, $M \cong \mathbb{Z}^n$

Proposition: $X = X(\Delta)$, Δ is a fan not contained in any proper subspace of $N_{\mathbb{R}} \cong \mathbb{R}^n$

$$\begin{array}{ccccccccc} 0 & \to & M & \to & Div_T X & \to & Pic(X) & \to & 0 \\ & & \shortparallel & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & \bigoplus_{i=1}^{d} \mathbb{Z} \cdot D_i & \to & A_{n-1}(X) & \to & 0 \end{array}$$

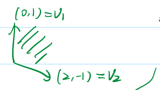
$\nwarrow$ weil divisor

in particular, rank $(Pic(X)) \le$ rank $(A_{n-1}(X)) = d-n$, d = # edges in fan

$Pic(X)$ is free abelian: $\cong \mathbb{Z}^n$ D_i form a basis

Rmk: $A_{n-1}(\cup D_i) = \bigoplus_{i=1}^{d} \mathbb{Z} \cdot D_i \to A_{n-1}(X) \to A_{n-1}(T_N) = 0$

$\bigoplus_{i=1}^{d} \mathbb{Z} \cdot D_i = \{a_i D_i \mid a_i \in \mathbb{Z}\}$ basis with $D_i \cong \mathbb{Z}^d$

Ex: $\sigma =$ 

$(0,1) = v_1$, $(2,-1) = v_2$

only one maximal cone $\Rightarrow$ D principal

$\Rightarrow$ all Cartier divisor $= 0$

$X = X_\sigma$

$Pic(X) = 0$

$A_{n-1} = \mathbb{Z}/2$ even number

$D = a_1 D_1 + a_2 D_2$

$u(\sigma)$ unique $= (u_1, u_2)$ affine toric variety

$\langle u(\sigma), v_1 \rangle = -a_1$ $\langle u(\sigma), (0,1) \rangle = -a_1$ $u_2 = -a_1$

$\langle u(\sigma), v_2 \rangle = -a_2$ $\langle u(\sigma), (2,-1) \rangle = -a_2$ $2u_1 - u_2 = -a_2$

$\Rightarrow 2u_1 = -(a_1 + a_2)$

$a_1 + a_2$ be even

if Cartier, then $D = div(\chi^{u(\sigma)})$

Section 3.3:

Thm: weil divisor $\sum a_i D_i$ is Cartier iff $\forall$ maximal σ, $\exists$ $u(\sigma) \in M$

s.t. $\langle u(\sigma), v_i \rangle = -a_i$ for all $v_i \in \sigma$

$N \cong \mathbb{Z}^n$, $M = N^\vee$, $M \cong \mathbb{Z}^n$

$X = \mathbb{P}^n$ non-singular, so two rows are equal

$$0 \to \underset{\substack{\shortparallel \\ \mathbb{Z}^n}}{M} \xrightarrow[(n+1)\times n]{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ -1 & \cdots & -1 \end{bmatrix}} \underset{\substack{\shortparallel \\ \mathbb{Z}^{n+1}}}{\bigoplus_{i=1}^{n+1} D_i} \xrightarrow[1\times(n+1)]{[1 \ 1 \cdots 1]} \underset{\substack{\shortparallel \\ Pic(X) \\ \shortparallel \\ \mathbb{Z}}}{A_{n-1}(X)} \to 0$$

$\mathbb{P}^n$ has basis $(e_1, e_2, \dots, e_n, -e_1 - e_2 - \dots - e_n)$

$$\underbrace{\underbrace{\begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & \vdots \\ -1 & -1 & \cdots & -1 \end{bmatrix}}_{M} \begin{matrix} 1 \end{matrix}}_{\oplus \mathbb{Z} \cdot D_i}$$

$A_{n-1}(X) = |\oplus \mathbb{Z} \cdot D_i| / M$

$= \mathbb{Z} \cdot \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} / M$

$= \mathbb{Z} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \cong \mathbb{Z}$

why $A_{n-1}(T_N) = 0$?

Unique factorization domain (UFD): commutative ring A s.t. $\forall f \in A$ has a unique factorization into irreducibles

Ex: $\mathbb{C}[x_1 \cdots x_n]$, $\mathbb{Z}$

in $\mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1} \cdots X_n, X_n^{-1}]$, $D \subseteq T_N$ be irreducible prime divisor

$D = V(f)$ vanishing f for some $f \in \mathbb{C}[X_1^{\pm}, X_2^{\pm} \cdots X_n^{\pm}] \subset \mathbb{C}[X_1 \cdots X_n] \Rightarrow D$ is principal $\Rightarrow [D] = 0 \in A_{n-1}(T_N) = 0$

Corollary: if all maximal cones Δ are n-dimentional, then TFAE

1) Δ is simplicial

2) rank $(Pic\, X(\Delta)) = d-n$

Lilah's talk: $M_\sigma = M \cap span(\sigma)$, $M(\sigma) = M/M_\sigma$ linear function of $u(\sigma)$

$\{u(\sigma) \in M/M(\sigma)\}$ for a Cartier divisor D defines a continuous piecewise function ψ_D on the support $|\Delta|$, where $\psi_D(v) = \langle u(\sigma), v \rangle$ for $v \in \sigma$

Conversely, any continuous function on $|\Delta|$ that is linear and integral on each cone comes from a unique T-Cartier divisor

if $D = \sum a_i D_i$, use $\psi_D(v_i) = -a_i \Leftrightarrow [D] = \sum -\psi_D(v_i) D_i$ (weil divisor) to determine $\psi_D(v)$

properties of ψ: $\psi_{D+E} = \psi_D + \psi_E$, $\psi_{mD} = m\psi_D$

rmk: $\psi_{div(\chi^u)}$ is a linear function of $-u$; if D, E are linearly equivalent divisors, then ψ_D and ψ_E differ by a linear function $u \in M$

why?

proved here!

Section 3.3:

Thm: $\sum a_i D_i$ is Cartier iff $\forall$ maximal σ, $\exists$ $u(\sigma) \in M$

s.t. $\langle u(\sigma), v_i \rangle = -a_i$ for all $v_i \in \sigma$

$u \in M$, $div(\chi^u) = \sum \langle u(\sigma), v_i \rangle D_i$, $\langle u(\sigma), v_i \rangle = a_i$

$\psi_{div}(\chi^u)(v_i) = -a_i = -\langle u(\sigma), v_i \rangle = \langle -u, v_i \rangle$

Connection to polytopes (my first talk)

A T-Cartier divisor $D = \sum a_i D_i$ on $X(\Delta)$ also determines a rational convex polyhedron in $M_{\mathbb{R}}$ defined by

$P_D = \{u \in M_{\mathbb{R}} : \langle u, v_i \rangle \ge -a_i \text{ for all } i\} = \{u \in M_{\mathbb{R}} : u \ge \psi_D \text{ on } |\Delta|\}$

When $|\Delta| = N_{\mathbb{R}}$, the variety $X(\Delta)$ is complete, then in toric case means the polyhedron P_D is bounded

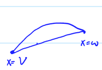
Proposition: if cones in Δ span $N_{\mathbb{R}}$ as a cone, then $P_D \cap M$ is finite

Ex: toric variety in $\mathbb{P}^1$, D_1, D_2 are divisors corresponding to the positive and negative edges

$D = a_1 D_1 + a_2 D_2$ correspond to the function ψ_D on $\mathbb{R}$

Recall: upper convex functions ψ if $\psi(t \cdot v + (1-t) \cdot w) \ge t\psi(v) + (1-t)\psi(w)$

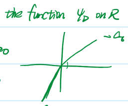
for all $t \in [0,1]$, vectors v, w



$\psi_D(1) = -a_1$

$\psi_D(-1) = -a_2$

$\Rightarrow \psi_D = \begin{cases} -a_1 x, & x \ge 0 \\ a_2 x & x \le 0 \end{cases}$



$-a_1 \le a_2$

$a_1 + a_2 \ge 0$

Convex iff $a_1 + a_2 \ge 0$

in N dimentional, convex means "tent shaped"