2023.09.27 讲课

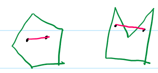
2023年9月17日 星期日 下午4:00

Section 1.5: Toric varietis from polytopes

For this section, we assume to be 2-dimensional, and K includes the origin for simplicity.

Convex polytope (K): Convex hull of finite set of points in a finite dimensional vector space

Convex hull: it Contains line segments connecting each pair of its points. → e.g.



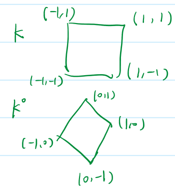
✓ X

(proper) face (F): intersection with a supporting affine hyperplane.

$F = \{v \in K : \langle u, v\rangle = r\}$, where $u \in E^*$ is a function satisfy $\langle u, v\rangle \geq r$ for all $v \in K$

$E = \{[\begin{smallmatrix} a_1 \\ \vdots \\ a_n \end{smallmatrix}]\}$, $E^* = \{[a_1 \cdots a_n]\}$ for computational convenience

Polar set (K°): $K^\circ = \{u \in E^* : \langle u, v\rangle \geq -1$ for all $v \in K\}$ → e.g. K



Counter example

$(2,0) \notin K^\circ$

since $(2,0)\cdot(-1,1) = -2 < -1$

Preposition: K° polar set is a convex polytope, K is the polar of K°, if F is a face of K, then

$F^* := \{u \in K^\circ : \langle u, v\rangle = -1, \forall v \in F\}$ is a face of K°

and the Correspondence between $F \to F^*$ is a one-to-one, order reversing correspondence between

faces of K and faces of K°, with $\dim(F) + \dim(F^*) = \dim(E) - 1$

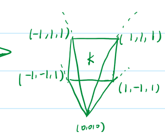
If K is rational (vertices in lattice E), then K° is also rational with vertices in dual lattice

Proof: σ cone in $K \times 1$, dual cone $\sigma^\vee$ consist of those $u \times r$ in $E^* \times \mathbb{R}$ s.t. $\langle u, v\rangle + r \geq 0$

for all $v \in K$. It follows that $\sigma^\vee$ is a cone in $K^\circ \times 1$. Then properties in Section 1.2 can be applied

Recall Section 1.2-(10) If τ is a face of σ, then $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$, with $\underbrace{\dim(\tau)}_{F} + \underbrace{\dim(\sigma^\vee \cap \tau^\perp)}_{F^*} = n = \dim(V)$

This sets up a one to one order reversing correspondence between the faces of σ and faces of $\sigma^\vee$.



$K \subseteq E$, $K \times 1 \subseteq E \times \mathbb{R}$
$(v_1, \cdots v_k)$ $(v_1, \cdots v_k, 1)$

$F =$ line segment $(-1,-1,1), (-1,1,1)$

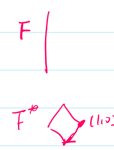
$F_\sigma = \tau = \mathbb{R}_{\geq 0}(1,-1,1) + \mathbb{R}_{\geq 0}(-1,-1,1)$

$\dim \sigma = \dim k + 1$

$\dim F + 1 = \dim F_\sigma$

$\overset{2}{\dim F_\sigma} + \overset{1}{\dim F^*_\sigma} = \overset{2}{\dim E + 1}$
$\quad\shortparallel \qquad\qquad \shortparallel$
$\dim F + 1 \quad \dim F^* + 1$

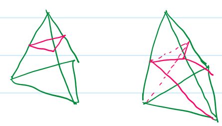
$\underset{1}{\dim F} + \underset{0}{\dim F^*} = \underset{2}{\dim E - 1}$



All results from 1.2 follows. For example, $(K^\circ)^\circ = K$: For face F of K, then τ cone over $F \times 1$, then dual $\sigma^\vee \cap \tau^\perp$ is over $F^* \times 1$, so results holds

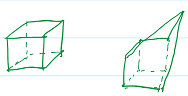
Relationship between fan and polytopes

If K' is a subdivision of boundary of K, then cones over polytopes in K' form a fan   Ex:



Question: Does fans all comes from convex polytope?

No.



It is impossible to find 8 points, each on a vertices,

s.t. for each of 6 cones generated by 4 points, the points are on the

same affine plane.

Toric Varieties in rational polytope P (within dual space $M_\mathbb{R}$)

We assume P be n-dimentional, but not necessarily contain origin

For each face Q of P, there is a cone σ_Q within Δ_P s.t. $\sigma_Q = \{v \in N_\mathbb{R} : \langle u, v\rangle \leq \langle u', v\rangle$ for all $u \in Q, u' \in P\}$

↳ $\langle u' - u, v\rangle \geq 0$

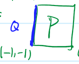
σ_Q is dual to the "angle" at Q consisting vectors from point of Q to point of P

By definition, dual cone $\sigma_Q^\vee$ is generated by $u' - u$

Proposition: ① σ_Q forms a fan Δ_P as Q varies over faces of P

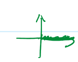
② If P contains origin as an interior point, then Δ_P consist of cones over the faces of polar polytope P°

ex:



$\sigma_Q = \{v \in \mathbb{R}^2 \mid v \cdot u \leq v \cdot u'$ for $u \in Q, u' \in P\}$

$= \{(v_1, v_2) \mid v_1 \geq 0, v_2 = 0\}$



proof: ②: by definition, $\forall v \in$ faces of P°, $u \in \sigma_Q$, $\langle u, v\rangle = 0 \geq 0$ So σ_Q is the cone over dual face Q^* of P°

$\sigma_{(1,1)} = \{v \in \mathbb{R}^2 \mid v \cdot (1,1) \leq v \cdot u'$ for $u \in Q, u' \in P\}$

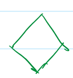
$= \{(v_1, v_2) \mid v_1 \leq 0, v_2 \leq 0\}$

$v_1 + v_2 \leq x v_1 + y v_2$

①: if contain origin, By theorem, $\Delta_{mp+u} = \Delta_P$, $u \in M$ since it only measure the angle

Any P spanning $M_\mathbb{R}$ can be changed to one containing origin ✓

P°



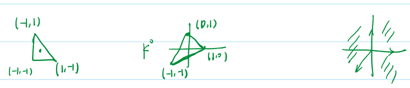
if $v_2 = 0$ we know $x \leq 1$, so $v_1 \leq 0$

Conversely, For a convex polytope K in $N_\mathbb{R} = \mathbb{R}^n$ (with origin as interior), the fan of cones over faces of K is the same as ΔP, where $P = K^\circ$ is its polar polytope

toric variety $X(\Delta P)$ will be denote as X_P

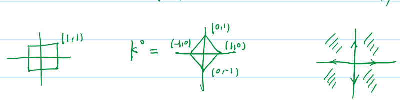
$\Delta P = \mathbb{R}^2$

Examples: (1) $K =$



$X(\Delta P) = \mathbb{P}^2$

(2) $K =$



$X(\Delta P) = \mathbb{P}^1 \times \mathbb{P}^1$