Orbits

Key

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

N = lattice

σ = rational convex polyhedral cone in $N_{\mathbb{R}}$

Σ = fan in $N_{\mathbb{R}}$

T_N = torus $N \otimes_{\mathbb{Z}} \mathbb{C}^* = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ associated to N and M

X_Σ = toric variety of a fan Σ in $N_{\mathbb{R}}$

U_σ = affine toric variety of a cone $\sigma \subseteq N_{\mathbb{R}}$

$\tau \preceq \sigma$ = τ is a face of cone σ

Recall from Casey's talk:

$t \in \mathbb{C}^*$, $\lambda(t) = (t^{u_1}, \ldots, t^{u_n})$

$(u_1, \ldots, u_n) \in \mathbb{Z}^n = N$

$$\begin{tikzcd} \lambda: \mathbb{C}^* \arrow[r] \arrow[d, hook] & T_N \arrow[d, hook] \\ \mathbb{C} \arrow[r, dashed] & X_\Sigma \end{tikzcd}$$

dashed arrow exists $\iff \lim_{t \to 0} \lambda(t)$ exists in X_Σ

Ex (where dashed arrow does not exist)

Let $\Sigma = \{0\} \subseteq \mathbb{R}$

Then $X_\Sigma = \mathbb{C}^* = T_N$

$$\begin{tikzcd} \lambda: \mathbb{C}^* \arrow[r, "id"] \arrow[d, hook] & \mathbb{C}^* = T_N \arrow[d, "id"] \\ \mathbb{C} & \mathbb{C}^* = X_\Sigma \end{tikzcd}$$

dashed arrow does not exist

$\Updownarrow$

$\lim_{t \to 0} \lambda(t)$ does not exist in X_Σ

For our purposes (when $\lambda(t)$ is not constant), if the limit exists in X_Σ, it does not exist in T_N.

Recall from Will's talk:

Def A *distinguished point* $x_\sigma \in U_\sigma$

x_σ is defined by the semigroup homomorphism:

$$\begin{aligned} S_\sigma &\longrightarrow (\mathbb{C}, \times) \\ v &\longmapsto \begin{cases} 1 & \text{if } v \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Now we get the surjective mapping:

$$\begin{aligned} \mathbb{C}[S_\sigma] &\twoheadrightarrow \mathbb{C} \\ x^v &\longmapsto \begin{cases} 1 & \text{if } v \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus, we get the bijection:

point of U_σ $\longleftrightarrow$ semigroup homomorphism

$$S_\sigma \longrightarrow (\mathbb{C}, *)$$

must be multiplication

Recall from Casey's talk:

For $\sigma \in \Sigma$ and $u \in N$,

$u \in \mathrm{RelInt}(\sigma) \iff \lim_{t \to 0} \lambda^u(t)$ exists in X_Σ AND

is equal to x_σ

Now we can define an orbit.

Let G be a group, and X be a set.

Suppose G acts on X.

<u>Def</u> For any $x \in X$, the <u>orbit</u> of x under the action of G is the set

$$\{g \cdot x \mid g \in G\}$$

In our setting,

$G = T_N = (\mathbb{C}^*)^n$

X_Σ

$\dim X_\Sigma = n$

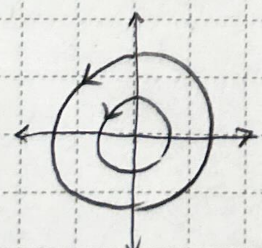
<u>Fact</u> There is an action of T_N on X_Σ

Why is it called "orbit"?

<u>Ex</u> $G = S^1 = \left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\}$

$X = \mathbb{R}^2 = \left\{\begin{pmatrix} x \\ y \end{pmatrix}\right\}$

action: matrix multiplication



orbits are centered at origin.

oriented counterclockwise.

Def The orbit of point x_σ under the action of T_N is

$$O(\sigma) := T_N \cdot x_\sigma$$

Goals

1. Show that these are all of the orbits
2. Find $\dim O(\sigma)$
3. Describe U_σ as a union of orbits
4. Show the relationship between $O(\sigma)$ and $O(\tau)$, where $\tau \preceq \sigma$

Main result #1

a. $\exists$ bijection $\{\text{cones } \sigma \in \Sigma\} \longleftrightarrow \{T_N \text{ orbits in } X_\Sigma\}$

$$\sigma \longleftrightarrow O(\sigma)$$

b. **Def** $N_\sigma = N \cap \operatorname{span}(\sigma)$ $\qquad \dim N_\sigma = \dim \sigma$

$N(\sigma) = N/N_\sigma$ $\qquad \operatorname{rank}(N(\sigma)) = \operatorname{rank}(N) - \operatorname{rank}(N_\sigma)$

$$= n - \dim(\sigma)$$

Lemma (Fulton 3.2.5)

$O(\sigma) = T_{N(\sigma)} = \operatorname{Hom}(\sigma^\perp \cap M, \mathbb{C}^*)$

In particular, $\operatorname{rank} N = n \Rightarrow \dim(O(\sigma)) = n - \dim(\sigma)$

c. $U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$

d. τ is the face of $\sigma \Longleftrightarrow O(\sigma) \subseteq \overline{O(\tau)}$ AND

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$$

Now you may refer to the example up until the introduction of Star(τ).

Main result #2

$\overline{O(\tau)}$ is a toric variety.

So now we can produce its associated fan.
Using our definitions from main result 1.b,

$$N_\tau = N \cap \operatorname{span}(\tau)$$
$$N(\tau) = N/N_\tau$$

Thus, we get the surjective mapping:

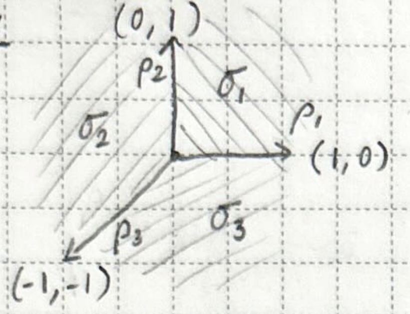
$$\begin{array}{ccc} N_{\mathbb{R}} & \twoheadrightarrow & N(\tau)_{\mathbb{R}} \\ \cup & & \cup \\ \sigma & \longrightarrow & \overline{\sigma} = \text{image of } \sigma \text{ under the above map} \end{array}$$

Note that $\overline{\sigma}$ is still a strongly convex rational polyhedral cone.
Remember that σ contains both the generators of $\tau \preceq \sigma$ and also other generators (not of $\tau \preceq \sigma$).
This map sends all generators of $\tau \preceq \sigma$ to 0, thus, leaving us with only the other generators.

__Def__ $\quad \operatorname{Star}(\tau) = \{\overline{\sigma} \mid \tau \preceq \sigma\}$ is a fan in $N(\tau)_{\mathbb{R}}$

__Thm__ $\quad \overline{O(\tau)} \cong X_{\operatorname{star}(\tau)}$

Ex



Σ

$X_\Sigma = \mathbb{P}^2$

Cones: $\{0\}$, $\rho_1 = \mathbb{R}_{\geq 0} \cdot (1,0)$, $\rho_2 = \mathbb{R}_{\geq 0} \cdot (0,1)$, $\rho_3 = \mathbb{R}_{\geq 0} \cdot (-1,-1)$,
$\sigma_1 = \langle (1,0), (0,1) \rangle$, $\sigma_2 = \langle (0,1), (-1,-1) \rangle$, $\sigma_3 = \langle (-1,-1), (1,0) \rangle$

Recall from main result 1.b that $\dim(O(\sigma)) = n - \dim(\sigma)$. So,

$\dim = 2 : O(\{0\})$
$\dim = 1 : O(\rho_1), O(\rho_2), O(\rho_3)$
$\dim = 0 : O(\sigma_1), O(\sigma_2), O(\sigma_3)$

Recall from main result 1.c that $U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$. So,

$U_{\{0\}} = O(\{0\})$
$U_{\rho_1} = O(\{0\}) \cup O(\rho_1)$
$U_{\sigma_1} = O(\{0\}) \cup O(\rho_1) \cup O(\rho_2) \cup O(\sigma_1)$

Recall from main result 1.d that $\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$. So,

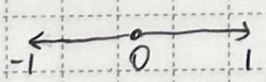
$\overline{O(\{0\})} = X_\Sigma$
$\overline{O(\rho_1)} = O(\rho_1) \cup O(\sigma_1) \cup O(\sigma_3)$
$\overline{O(\sigma_1)} = O(\sigma_1)$

Having defined $\mathrm{Star}(\tau)$, we can complete the example.

$N_{\{0\}} = \{0\}$
$N(\{0\}) = N$
$\mathrm{Star}(\{0\}) = \Sigma$

$N_{\rho_1} = \{(a,0) \mid a \in \mathbb{Z}\}$
$N(\rho_1) = \mathbb{Z}$
$\mathrm{Star}(\rho_1) = \{\underset{\{b \geq 0\}}{\overset{\|}{\overline{\sigma_1}}}, \underset{\{0\}}{\overset{\|}{\overline{\rho_1}}}, \underset{\{b \leq 0\}}{\overset{\|}{\overline{\sigma_3}}}\}$

$$\begin{array}{ccc} N & \longrightarrow & N(\rho_1) \\ \| & & \| \\ \mathbb{Z}^2 & \twoheadrightarrow & \mathbb{Z} \\ \Downarrow & & \Downarrow \\ (a,b) & \longmapsto & b \end{array}$$



← the fan $\mathrm{Star}(\rho_1)$

Fact $X_{\mathrm{Star}(\rho_1)} = \mathbb{P}^1$

$N_{\sigma_1} = N$
$N(\sigma_1) = 0$
$\mathrm{Star}(\sigma_1) = \{\{0\}\}$
$X_{\mathrm{Star}(\sigma_1)} = \mathrm{pt.}$