Proposition: If X is a nonsingular toric variety, and $D_1, \dots, D_d$ are the irreducible T-divisors on X, then $\underbrace{-\sum D_i}_{\text{negative of the sum of the } T\text{-divisor.}}$ is a canonical divisor.

divisor of zeros and poles of rational differential form $\quad \omega = \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_n}{X_n}$

rational differential form has pole in all D_i.

Example: Let X be a non-singular complete surface. $(D_i \cdot D_i) = -a_i$

The canonical divisor $K = -\sum D_i$ has self-intersection number

$$(K \cdot K) = \sum (D_i \cdot D_i) + 2d = -\sum a_i + 2d = \underbrace{-(3d-12)}_{\text{from 2.5 Jake's first talk.}} + 2d = 12 - d$$

Since there are d 2-dimensional cones, we also know that

topological Euler Characteristic $\to$ $\chi(X) = d$. Hence:

$$\underbrace{\frac{(K \cdot K) + \chi(X)}{12} = \frac{(12-d) + d}{12} = 1}_{\text{Noether's formula for the surface } X.}$$

Canonical divisor K_X.

Anti-Canonical divisor $-K_X$.

Definition X is Fano if $-K_X$ is Cartier and ample.

- Thus every Fano variety is projective.

Example: $-K_{\mathbb{P}^2} = D_1 + D_2 + D_3 = 3\underbrace{H}_{\text{class of a line}}$ is ample (strictly convex in this case)

because

$$0 \to \underset{\substack{\| \\ \mathbb{Z}^2}}{M} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}} \bigoplus_{i=1}^{3} \mathbb{Z} \cdot D_i \xrightarrow{[1\ 1\ 1]} \operatorname{Pic} \mathbb{P}^2 \to 0$$

$D = a_1 \underset{(1,0)}{\underset{\uparrow}{D_1}} + a_2 \underset{(0,1)}{\underset{\uparrow}{D_2}} + a_3 \underset{(-1,-1)}{\underset{\uparrow}{D_3}}$ ample $\Leftrightarrow a_1 + a_2 + a_3 > 0$.

$3 > 0 \Rightarrow$ ample.

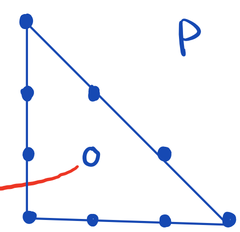
therefore $\mathbb{P}^2$ is a Fano variety.

$\Rightarrow$ classification of 2-dimensional Fano toric varieties.

The standard fan for $\mathbb{P}^2 = X_\Sigma$ has minimal generators $u_0 = -e_1 - e_2$, $u_1 = e_1$ and $u_2 = e_2$. The polytope corresponding to the anticanonical divisor of $\mathbb{P}^2$ is $P = \{m \in \mathbb{R}^2 \mid \langle m, u_i \rangle \geq -1,\ i = 0,1,2\}$.

we can check that: $P = \underset{\text{convex hull}}{\underset{\downarrow}{\operatorname{Conv}}}(-e_1 - e_2, 2e_1 - e_2, -e_1 + 2e_2)$

lattice polygon:



P

origin, unique interior lattice point of P.

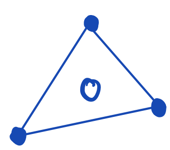
if $v_1, \dots, v_n \in \mathbb{R}^n$

$\operatorname{Conv}(v_1, \dots, v_n) = \{\sum a_i v_i \mid 0 \leq a_i \leq 1, \sum a_i = 1\}$.

the dual polytope $P^\circ = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 1 \text{ for all } m \in P\}$ is given by

dual $P^\circ = \operatorname{Conv}(e_1, e_2, -e_1 - e_2)$

lattice polygon:



P°.

Example: the weighted projective space $\mathbb{P}(q_0, \dots, q_n)$ is Fano iff $q_i \mid q_0 + \cdots + q_n$ for all $0 \le i \le n$.

Fano Toric Varieties and Reflexive Polytopes.

- A lattice polytope in $M_{\mathbb{R}}$ is reflexive if its facet presentation is $P = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \ge -1 \text{ for all facets } F \}$.
- It follows that if P is reflexive, the origin is the unique interior lattice point of P. Since $a_F = 1$ for all facets F, the dual polytope is $P^\circ = \mathrm{Conv}(u_F \mid F \text{ is a facet of } P)$. $\rightarrow$ inward point in normal.
- Finally, P° is a lattice polytope and is reflexive.

Theorem: let X be a toric variety. If X is a projective Fano variety, then the polytope associated to the anticanonical divisor $-K_X = \sum_\rho D_\rho$ is reflexive. Conversely, if X_P is the projective toric variety associated to a reflexive polytope P, then X_P is a Fano variety.

Toric varieties $\xrightarrow{\text{not injective}}$ Reflexive polytopes

Fano toric varieties $\xleftrightarrow{\text{bijection}}$ reflexive polytopes.

proof.

($\Rightarrow$) X projective Fano variety $\Rightarrow$ anticanonical divisor $-K_X = \sum_\rho D_\rho$ cartier and ample $\Rightarrow$ polytope has facet presentation

$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -1$ for all facets $\rho\}$. $\Rightarrow$ reflexive.

($\Leftarrow$). P is a reflexive polytope in $M_{\mathbb{R}}$

the facet presentation has $a_F = 1$ for every facet F of P.

Cartier divisor correspond to p is $D_P = \sum_F D_F = -K_{X_P}$.

D_P is ample $\Rightarrow$ $-K_{X_P}$ ample.

Hence X_P is Fano.

$\uparrow$

Classification: By theorem

Classifying toric Fano varieties is equivalent to classifying the

reflexive polytopes P in $M_{\mathbb{R}}$.

Since reflexive polytopes contain the origin as an interior point, "classify" means up to

invertible linear maps of $M_{\mathbb{R}}$ induced by isomorphisms of M.

This is called <u>lattice equivalence</u>.

- the lattice points of P are the origin and the lattice points on the boundary.
- any boundary lattice point is primitive
- Given a reflexive polgon P and a primitive element $m \in M$. there is a projective map

  $\pi_m : M_{\mathbb{R}} \longrightarrow M_{\mathbb{R}} / \mathbb{R}m$.

  whose image is a polytope whose vertices lie in $M / \mathbb{Z}m$.

<u>Lemma</u> Let P be a reflexive polytope in $M_{\mathbb{R}} \simeq \mathbb{R}^n$ and let m be a lattice point in the boundary of P. Then $\pi_m(P)$ is a lattice polytope in $M_{\mathbb{R}} / \mathbb{R}m$ containing the origin as an interior lattice point, and $\pi_m(P) = \pi_m\left(\bigcup_{m \notin F \text{ facet of } P} F\right)$

proof...

<u>Lemma</u>: Let m, m' be distinct lattice points on the boundary of a reflexive polytope P. Then exactly one of the following holds:

(a) m and m' lie in a common edge of P.

(b) $m+m'=0$, or

(c) $m+m'$ also lie in a common edge of P.

2-dimensional case that classifies reflexive polygons in the plane $M_{\mathbb{R}} \simeq \mathbb{R}^2$.

<u>Theorem</u>: there are exactly 16 equivalence classes of reflexive polygons in the plane

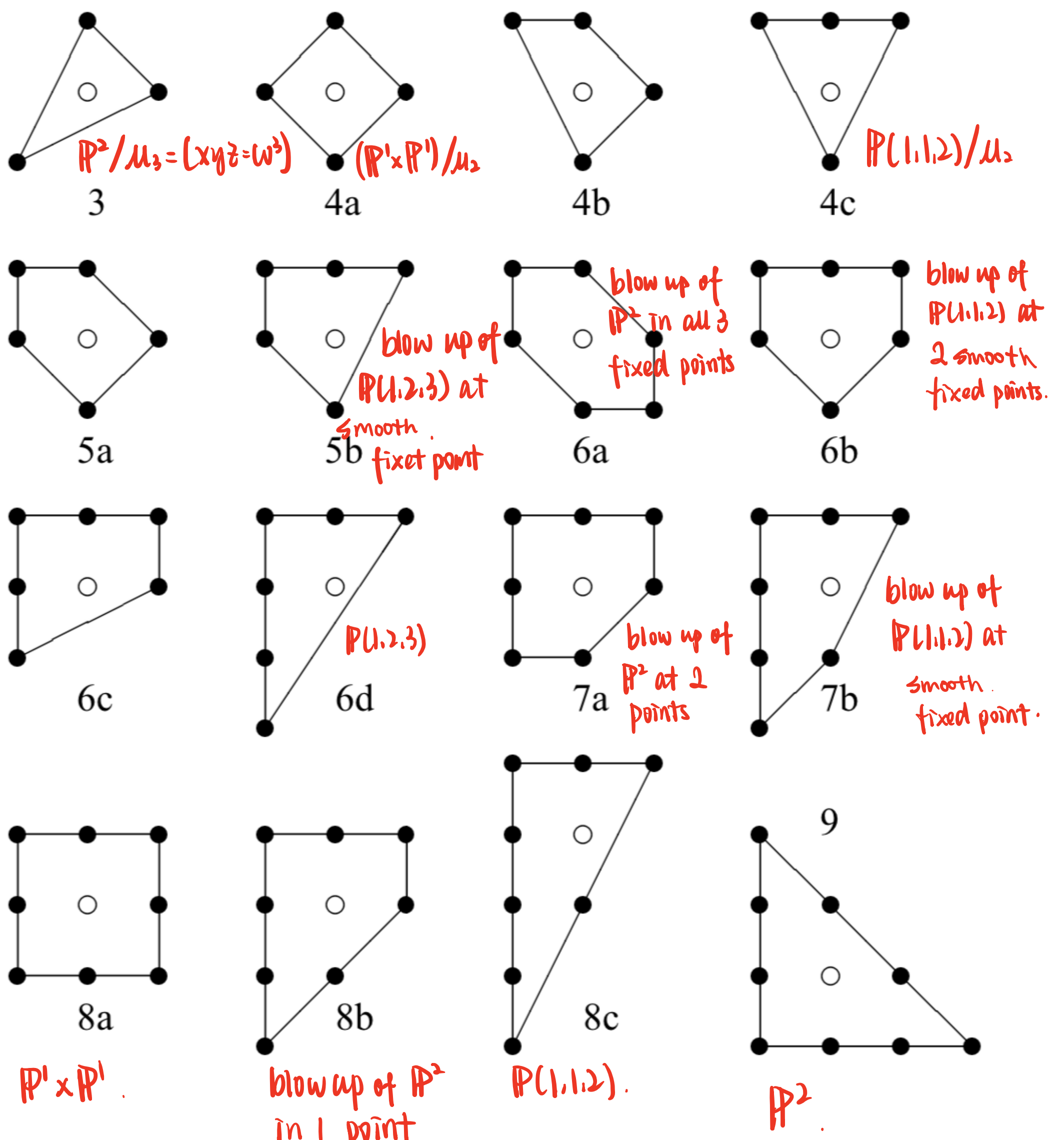


Figure 2. The 16 equivalence classes of reflexive lattice polygons in $\mathbb{R}^2$

dual

$3 \leftrightarrow 9$

$4a \leftrightarrow 8a$

$4b \leftrightarrow 8b$

$4c \leftrightarrow 8c$

$5a \leftrightarrow 7a$

$5b \leftrightarrow 7b$

$6a \leftrightarrow 6a$

$6b \leftrightarrow 6b$

$6c \leftrightarrow 6c$

$6d \leftrightarrow 6d$.