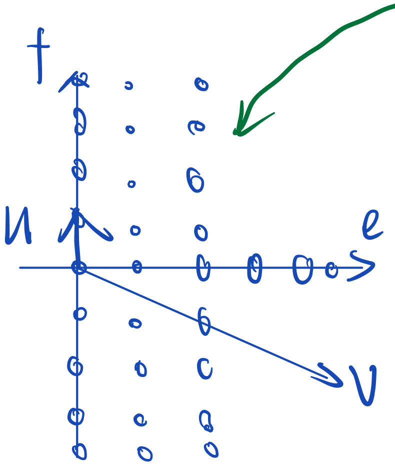
Resolution of singularities (in higher dimension $n \geq 3$).

- lose some of the nice properties from the surface case.

Key idea: bring a cone into normal form. (in lower dimension).



Definition: Fix a lattice basis (e, f). Given a cone $\sigma = \mathrm{Cone}(u, v)$ for primitive vectors (u, v), we say that σ is in normal form, with respect to the basis, if

$$\begin{cases} u = f \text{ and} \\ v = \lambda e - \mu f, \text{ for natural numbers } 0 \leq \mu < \lambda. \end{cases}$$

- this concept does not translate to dimensions ≥ 3.

Problem: Let Δ be a fan in $N_{\mathbb{R}}$ (vector space) where $\dim(N_{\mathbb{R}}) = n$. We want to find a smooth fan $\tilde{\Delta}$ refining Δ such that the smooth cones in Δ are in the refinement $\tilde{\Delta}$.

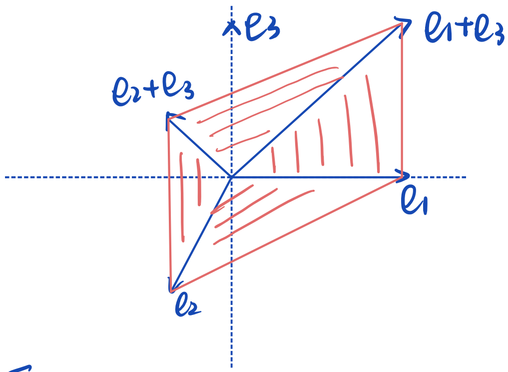
To find such a smooth refinement: stellar refinements.

Definition (Stellar refinement). Let $\sigma \in N_{\mathbb{R}}$ be a cone and let ρ be any ray in $N_{\mathbb{R}}$. Then we define

$$\sigma^*(\rho) = \begin{cases} \sigma & \text{if } \rho \notin \sigma \\ \{\underbrace{\rho + \tau}_{\text{sum as subsets.}} \mid \underbrace{\tau < \sigma}_{\tau \text{ is a face of } \sigma.} \mid \rho \notin \tau\} & \text{if } \rho \leq \sigma. \end{cases}$$

Examples: ...

Consider the 3-dimensional cone σ.



$\sigma \rightsquigarrow \{ \sigma, \langle e_1, e_1+e_3 \rangle, \langle e_1+e_3, e_2+e_3 \rangle, \langle e_2+e_3, e_2 \rangle, \langle e_1, e_2 \rangle, \langle e_1 \rangle, \langle e_1+e_3 \rangle,$

faces of σ. $\langle e_2+e_3 \rangle, \langle e_2 \rangle, \langle 0 \rangle \}$.

$\sigma^*(\rho_1) = \{ \underbrace{\langle e_1, e_2+e_3, e_1+e_3 \rangle, \langle e_1, e_2, e_2+e_3 \rangle}_{\text{two new 3D cones.}}, \dots \}$.

$\rho_1 = \mathbb{R}_{\geq 0} \cdot e_1$.

• Introducing of rays in the algorithm is an instance of stellar refinement.

Two steps:

- We first find a simplicial fan Δ' refining Δ
- Next we find a smooth fan $\tilde{\Delta}$ refining Δ'.

1) Simplicializing: We first find a simplicial refinement of Δ.

We can refine in such a way that the simplicial cones are unchanged.

Definition: (Simplicial cone): A cone $\sigma \subset N_{\mathbb{R}}$ is called simplicial if its minimal generators are linearly independent over $\mathbb{R}$.

Remark: It follows that a smooth cone (generated by a lattice basis.) is a simplicial cone. Also a cone $\sigma \in N_{\mathbb{R}}$ is simplicial if the number of edges equals $\dim \sigma$.

• A simplicial cone need not to be smooth.

Now if σ is a k-dimensional simplicial cone, and $v_1, \ldots, v_k$ are the first lattice points along the edges of σ, the *multiplicity* of σ is defined to be the index of the lattice generated by the v_i in the lattice generated by σ:

$$\text{mult}(\sigma) = [N_\sigma : \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_k].$$

Note that U_σ is nonsingular precisely when the multiplicity of σ is one.

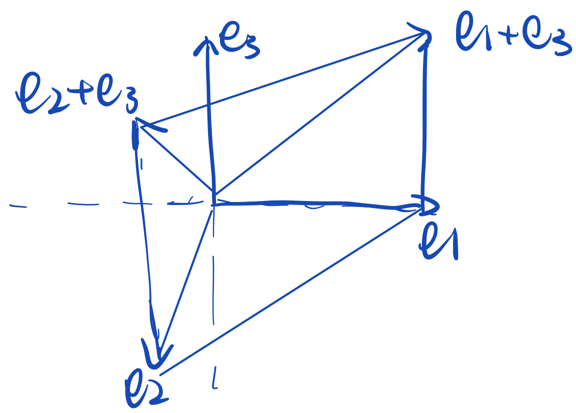
Remark: The variety U_σ is smooth when $\text{mult}(\sigma)=1$.

Lemma Every fan Δ has a simplicial refinement Δ' such that the simplicial cones of Δ are also cones in Δ'. This can be achieved by iterated stellar refinement.

proof.

Definition: A ray ρ is a splitting edge of a cone σ if there is a complementary facet τ of σ, i.e. a facet such that $\sigma = \tau + \rho$. A cone is stout if it has no splitting edges.

example: Stout cone



none of the four edges has a complementary facet.

Recall: a d-dimensional cone is simplicial if and only if it has exactly d edges and the corresponding ray generators are linearly independent over $\mathbb{R}$. This means that all edges are splitting edges. Furthermore, each face of a simplicial cone has splitting edges.

Therefore:

Remark A cone is simplicial iff it does not include any stout face. Hence a fan that does not contain any stout cones is a simplicial fan.

Continued:

Proof ...

idea: • use stellar refinements to lower the number of stout cones.

• claim that no simplicial cone in Δ get subdivided

i.e. the simplicial cones of Δ are still in the refinement Δ'.

2) Smoothening: Let Δ be a simplicial fan. Then we can refine in such a way that the smooth cones are unchanged.

Lemma Every simplicial fan Δ has a smooth refinement $\tilde{\Delta}$, such that the smooth cones of Δ are also cones in $\tilde{\Delta}$. This can be achieved by iterated steller refinement.

Recall: A (simplicial) cone is smooth whenever its multiplicity is 1.

So a fan Δ is smooth when $\operatorname{mult}(\sigma)=1$ for all cones $\sigma \in \Delta$.

To find a smooth refinement of a simplicial fan: lower the multiplicity of cones in the fan.

idea: subdivide a cone of mult >1 into cones of lower multiplicity.

Useful properties of the multiplicity of a cone:

Proposition: Let $\sigma \subset N_{\mathbb{R}}$ be a simplicial cone with minimal generators $u_1, \dots, u_d$ and let $e_1, \dots, e_d$ be a basis for $N_\sigma = \operatorname{span}(\sigma) \cap N$. When we write $u_i = \sum_j a_{ij} e_j$, then we have $\operatorname{mult}(\sigma) = |\det(a_{ij})|$

proof. linear algebra: determinant is the index of the sublattice $\mathbb{Z}u_1+\cdots+\mathbb{Z}u_d$ inside N_σ.

Proposition: Let $\sigma \subset N_{\mathbb{R}}$ be a simplicial cone with minimal generators $u_1,\dots,u_d$. Then the generators span the fundamental parallelotope

$$P_\sigma = \left\{ \sum_{i=1}^{d} \lambda_i u_i \,\middle|\, \lambda_i \in \mathbb{R} \text{ and } 0 \le \lambda_i < 1 \right\}.$$

and the multiplicity of the cone is the number of lattice points inside the parallelotope: $\mathrm{mult}(\sigma) = |P_\sigma \cap N|$.

Corollary For cones $\sigma < \tau$, we have $\mathrm{mult}(\sigma) \le \mathrm{mult}(\tau)$.

Lemma (shows that a steller refinement may lower the multiplicity of a cone)

Let $\sigma \subset N_{\mathbb{R}}$ be a cone which has multiplicity >1. Denote $u_1,\dots,u_d$ for the minimal generators of σ and assume we have a lattice point in the parallelotope: $u_\rho = \sum_{i=1}^{d} \lambda_i u_i \in P_\sigma \cap N$ for $0 \le \lambda_i < 1$

Assume u_ρ nonzero.

Let $\sigma^*(\rho)$ be the steller refinement with the ray through u_ρ as a center. Then we have $\mathrm{mult}(\tau+\rho) < \mathrm{mult}(\sigma)$, for cones in $\sigma^*(\rho)$.

$$\Downarrow$$

$$\mathrm{mult}(\tau+\rho) \le \mathrm{mult}(\delta_j+\rho)$$

$\downarrow$

facet of σ not containing u_j
generated by other minimal generators.

Recall: ρ is the ray generated by $u_\rho = \sum_{i=1}^{d} \lambda_i u_i$, for $0 \le \lambda_i < 1$

Compare $\delta_i+\rho$ and σ: $\mathrm{mult}(\delta_j+\rho) = \lambda_j \,\mathrm{mult}(\sigma)$.

$\lambda_j < 1 \Rightarrow \mathrm{mult}(\tau+\rho) < \mathrm{mult}(\sigma)$.

Conclude that:

$\Delta^*(\rho)$ is a refinement of Δ such that

- either $\operatorname{mult}(\Delta^*(\rho)) < m$
- or $\operatorname{mult}(\Delta^*(\rho)) = m$ but $\Delta^*(\rho)$ has less cones of maximal multiplicity.

Proposition. *For any toric variety* $X(\Delta)$, *there is a refinement* $\widetilde{\Delta}$ *of* Δ *so that* $X(\widetilde{\Delta}) \to X(\Delta)$ *is a resolution of singularities.*