Toric Surfaces

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\newline **Recall:**\newline
σ = cone\newline
$\sigma^{\vee}$ = dual\newline
U_σ = affine variety of σ\newline
$A_\sigma = C[S_\sigma]$ = ring ↩\newline
lattices N and M with primes N' and M'\newline
and our map of semigroups $S_\sigma = \sigma^{\vee} \cap M$\newline\newline
Recall our definition of non-singularity: U is non-singular if and only if it is generated by a basis of N.
Note that if $m >= 2$, the cone is not nonsingular (and therefore singular). Geometrically, this means that U_σ is singular.\newline
If G is a finite group acting on A_σ, define U_σ/G to be $Spec(A_\sigma^G)$ (the ring of invariants).

We will now look at two examples of 2-dimensional toric varieties.
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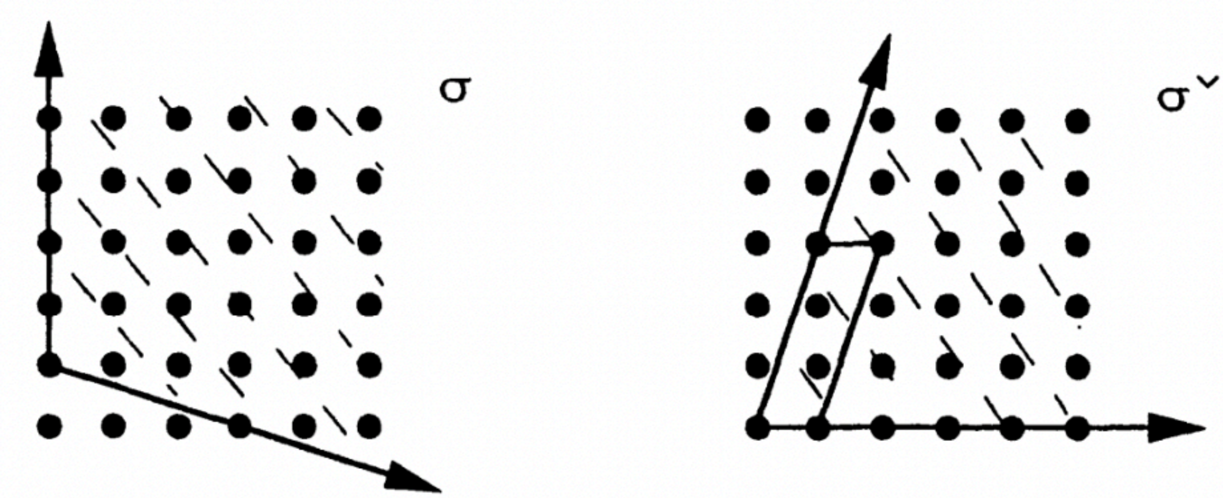
Example Consider the case where N =$\mathbf{Z}^2$ and σ is generated by e_2 and $me_1 - e_2$, generalizing the case m = 2 that we looked at earlier:



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Then $A_\sigma = \mathbf{C}[S_\sigma] = \mathbf{C}[X, XY, XY^2, \ldots XY^m]$.
If we variable substitute $X = U^m$, $Y = V/U$, we can alternatively rewrite this as:
$A_\sigma = \mathbf{C}[U^m, U^{m-1}V, \ldots, UV^m] \subset \mathbf{C}[U, V]$.
\newline
U_σ= Spec(A_σ) and the inclusion of A_σ in $\mathbf{C}[U, V]$ corresponds to a mapping $\mathbf{C}^2 \to U_\sigma$.
\newline
The group $G = \mu_m$ =m^{th} roots of unity $\cong \mathbf{Z}/m\mathbf{Z}$ acts on $\mathbf{C}^2$ by $\zeta * (u, v) = (\zeta u, \zeta v)$. By this action, $A_\sigma = \mathbf{C}[U, V]^G = \mathbf{C}[U, V]^{\mu_m}$ is the ring of invariants. \newline

(Optional: to see why $A_\sigma = \mathbf{C}[U, V]^G$, understand that the invariants are s.t. $\zeta. f = f$; in other words, if we let $f(u, v) = \sum_{k,l>=0} a_{kl} u^k v^l$,
$\zeta. f(u, v) = f(u, v) <=> \sum_{k,l>=0} a_{kl} (\zeta u)^k (\zeta v)^l = \sum_{k,l>=0} a_{kl} u^k v^l <=> \zeta^{k+l} = 1$ or $a_{kl} = 0$ for all k,l
$<=>$ m divides (k+l) or $a_{kl} = 0$ for all k,l.
\newline
Therefore, A_σ is generated by all monomials of degree m, which are listed above.)
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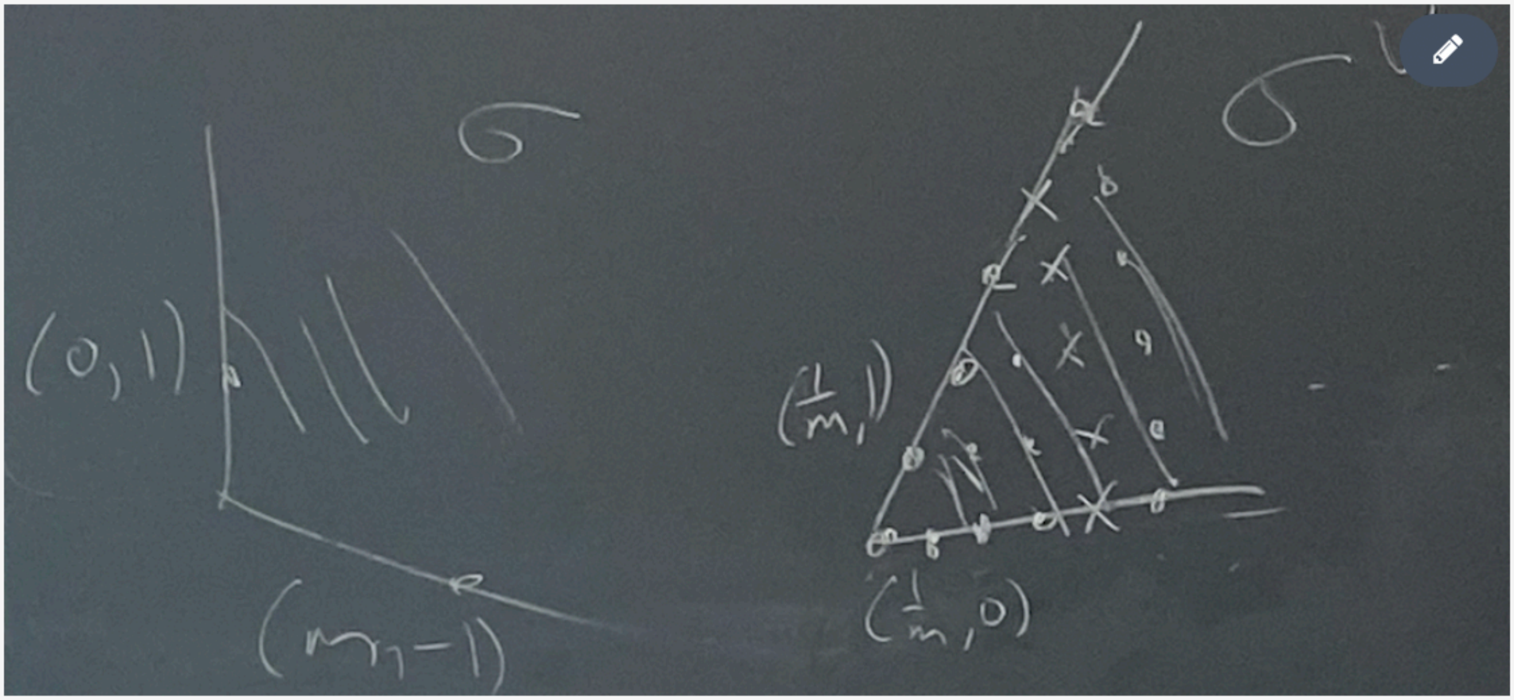
\newline
We can obtain a similar result through a toric structure. Lets let N' $\subset$ N be the lattice generated by the e_2 and $me_1 - e_2$ that generate σ and let σ' be the same cone as σ, but regarded in N'. We will show a similar form for $A_{\sigma'}$. To start, of course, we know N and M are generated by (1,0) and (0,1). Since σ' is generated by two generators for N', U'_σ =C^2, and the inclusion of N' in N gives a map $C^2 = U'_\sigma$->U_σ. The claim is that this is the same as the map constructed by hand above. To see this, note that N' is generated by (m,0) and (0,1) and M' is all vectors whose dot product with anything in N' is an integer, so it is generated by (1/m,0) and (0,1).
\newline
e_2 is already orthogonal to (m,0), so the dual vector for e_2 is (m,0),
$me_1 - e_2$ is orthogonal to (0,1), so the dual vector for $me_1 - e_2$ is (0,1), thus we have:\newline
$S_{\sigma'}$ is generated by (1/m,0) and (1/m,1), so $A_{\sigma'}$ =$C[x^{1/m}, x^{1/m}y]$. Recall $A_\sigma = C[x, xy, \ldots, xy^m]$; thus, if we variable substitute $u = x^{1/m}$ and $v = x^{1/m}y$, we arrive once more at
$A_\sigma = \mathbf{C}[U^m, U^{m-1}V, \ldots, UV^m] \subset \mathbf{C}[U, V]$.



(Note: $\sigma^\vee$ obtained by taking perpendiculars to σ generators)

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We can more or less generalize this approach to any two-dimensional affine toric variety. One thing we like to in common practice with any such example is to put generators of the cone into standard form (0,1), (m,k). To demonstrate this, we can take any generators (x_1, y_1) and (x_2, y_2), but one thing we will assume is that gcd(x_1, y_1) = 1. We apply the following matrix M to these generators: \newline
M = $\begin{pmatrix} y_1 & -x_1 \\ a & b \end{pmatrix}$
\newline
$\begin{pmatrix} y_1 & -x_1 \\ a & b \end{pmatrix} * \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1y_1 - x_1y_1 \\ ax_1 + by_1 \end{pmatrix}$
Here, we see the top term evalutes to 0, and we use the following to see that the bottom evaluates to 1.\newline
Bézout's lemma: Let a and b be integers with greatest common divisor d. Then there exist integers x and y such that $ax + by = d$\newline
Since we assumed gcd(x_1, y_1)=1, we have: \newline

$$\begin{pmatrix} x_1y_1 - x_1y_1 \\ ax_1 + by_1 \end{pmatrix}$$

=

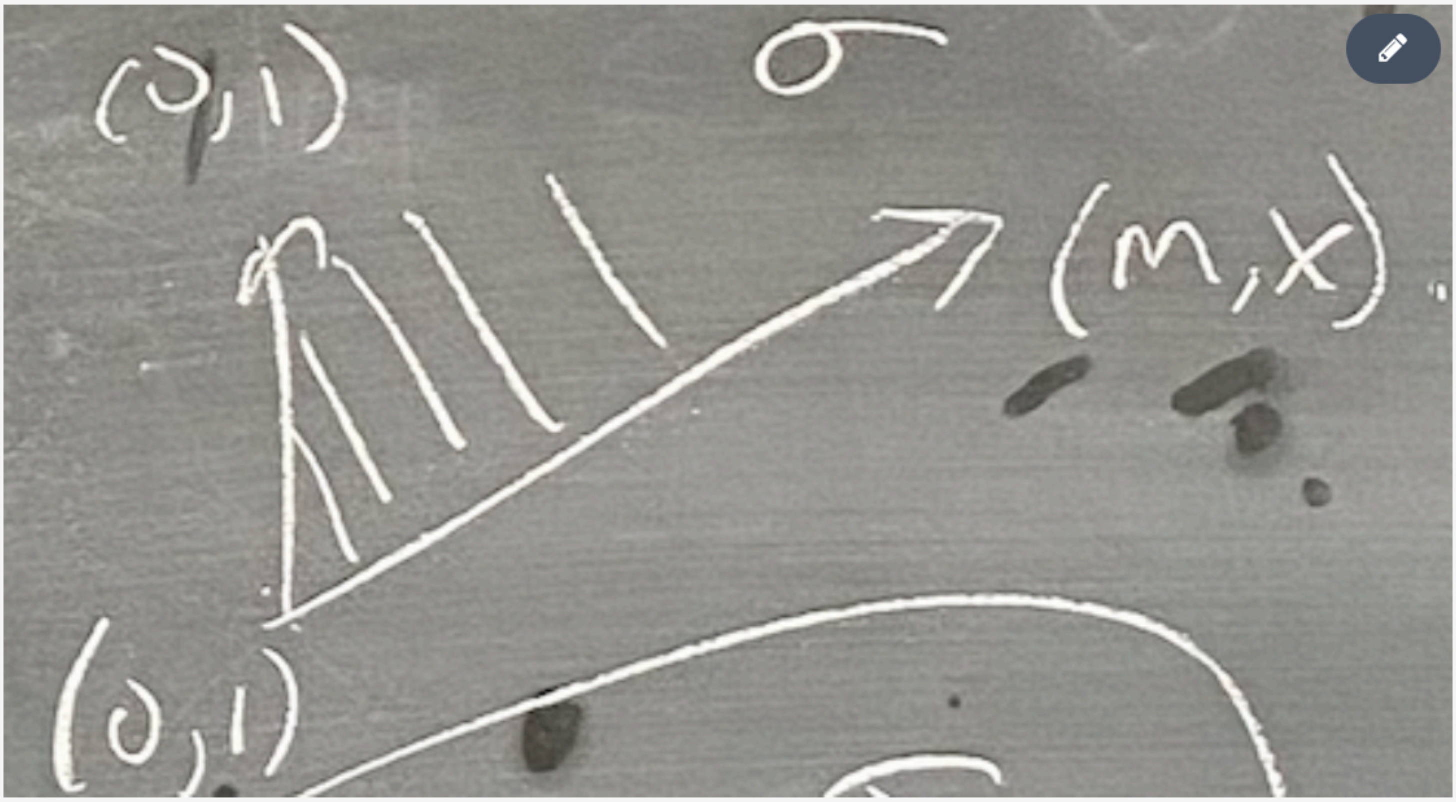
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can assign some $(m, x) := M * \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

(Note that x is different from x_1 or x_2).

\newline

These generators produce σ as such:\newline



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Finally, we can apply the following shearing function $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ to the lattice. (Note that this is an automorphism.) When we do this $(0, 1)$ remains fixed and the second term of (m, x) is arbitrarily stretched by cm: \newline

$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} * (m, x) = (m, x + cm)$

\newline

We will now represent this as

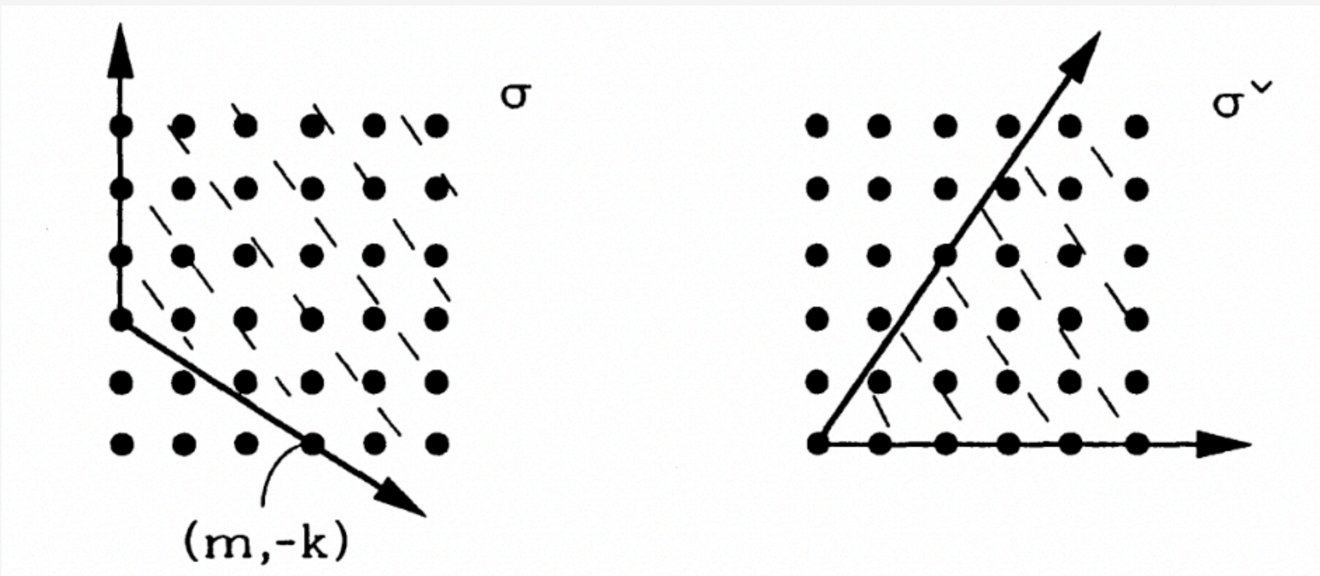
$(m, -k)$, where $cm + x = -k$, $0 <= k < m$, and k and m are relatively prime since $(m, -k)$ is a minimal generator along the edge of σ.

\newline

Meanwhile, $(0, 1)$ remains fixed:

$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} * (0, 1) = (0, 1)$

Any minimal generator along an edge of σ is part of a basis for $N = \mathbf{Z}^2$, so if we take $(0,1)$ and $(m,-k)$ as these two minimal generators, we can reasonably assume σ has form:



(Note: $\sigma^\vee$ obtained by taking perpendiculars to σ generators)

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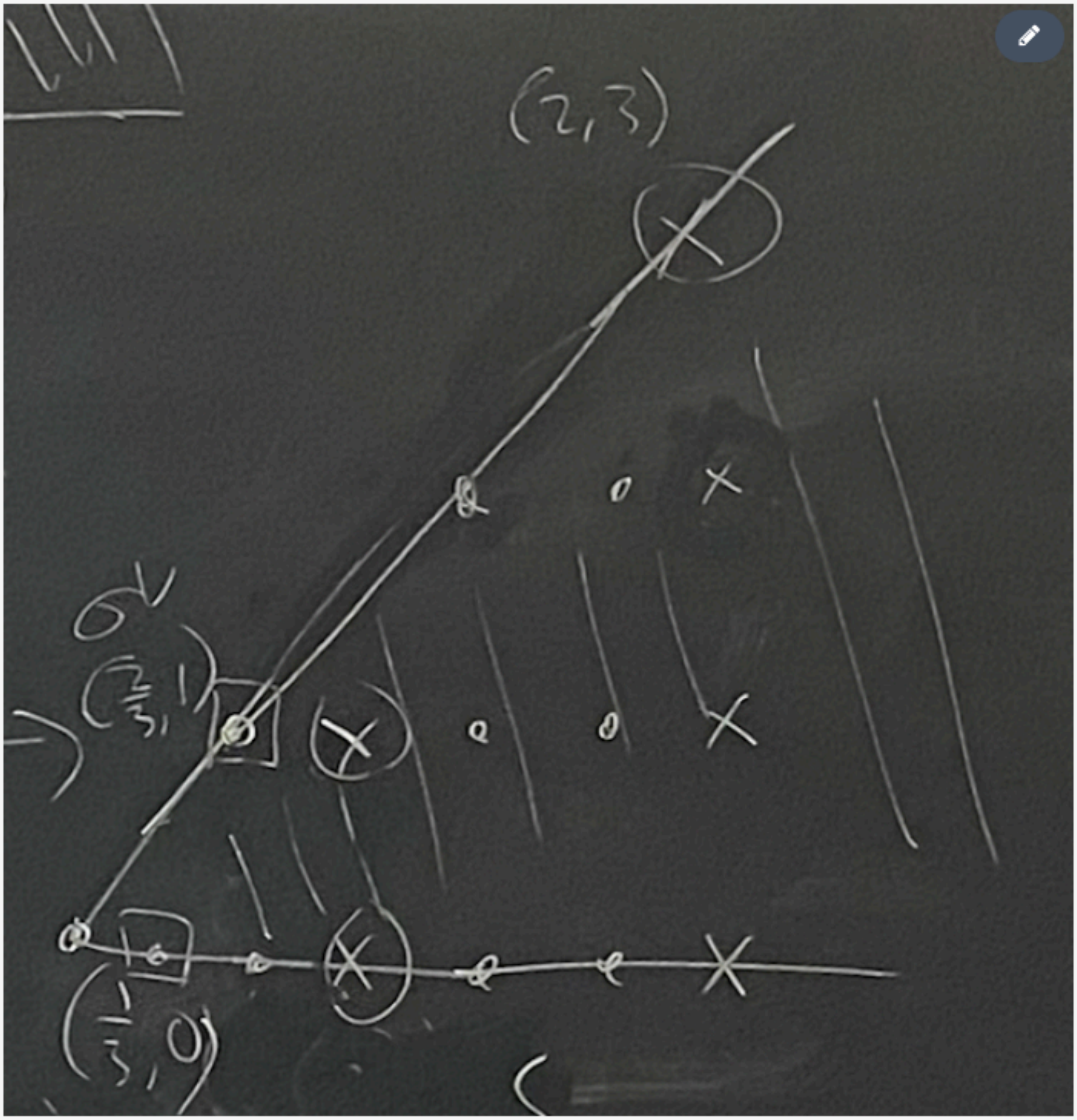
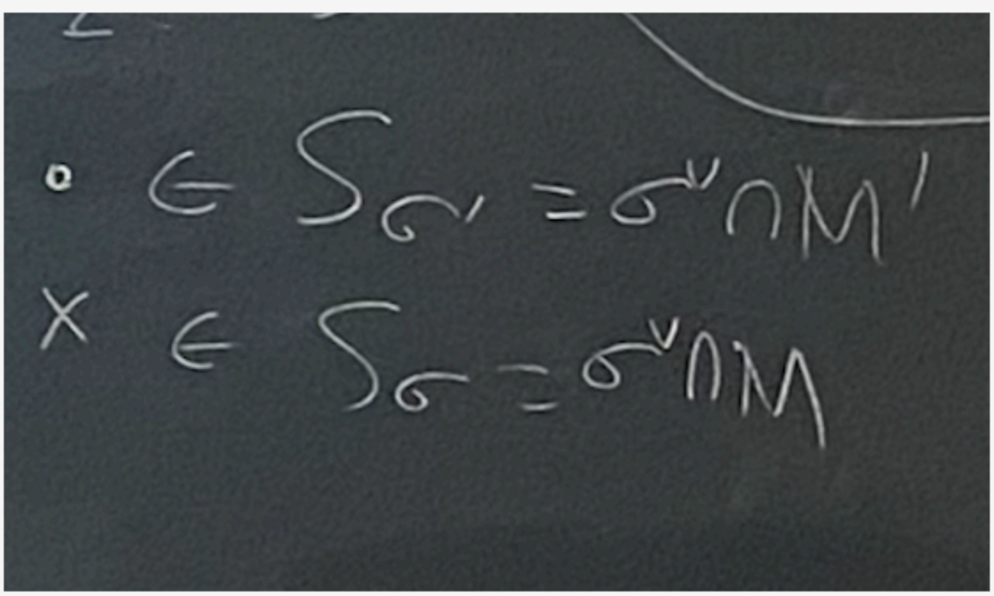
\newline
Let's do the specific example depicted in the figure, **m=3, k=2**:
\newline
To start, of course, we know N and M are generated by (1,0) and (0,1). Now let N' $\subset$ N be the lattice generated by the e_2 and $3e_1 - 2e_2$ that generate σ. Also let σ' be the same cone as σ, but regarded in N'. N' is generated by (3,0) and (0,1) and M' is all vectors whose dot product with anything in N' is an integer, so it is generated by ($\frac{1}{3}$,0) and (0,1). The generators for S'_σ are $(\frac{2}{3}, 1)$ and $(\frac{1}{3}, 0)$.\newline


(2,3)
σ∨
(2/3, 1)
(1/3, 0)


∘ ∈ $S_{\sigma'} = \sigma'^\vee \cap M'$
× ∈ $S_\sigma = \sigma^\vee \cap M$

S_σ is generated by $(1,0),(1,1),(2,3)$. This implies that A_σ=$C[x,xy,x^2y^3]$.\newline
We can variable substitute to get $A_\sigma = C[u^3,uv,v^3]$.\newline
Alternatively, we can look at A_σ as a quotient of a polynomial ring in C^3 by substituting $C[s,t,w]/(sw-t^3)$.\newline
So, the ring $C[s,t,w]/(sw-t^3)$ is the quotient ring formed by taking the polynomial ring in s, t, and w and factoring it by the ideal generated by the polynomial equation $sw-t^3$.
\newline
In general, U_σ, the Spec(A_σ) = $\mathbf{C}^2/\mu_m$ where μ_m, the m^{th} roots of unity, acts on $\mathbf{C}^2/\mu_m$ by $\zeta * (u,v) = (\zeta u, \zeta^k v)$\newline
Here, U_σ, = $\mathbf{C}^2/\mu_3$ where μ_3 acts on $\mathbf{C}^2/\mu_3$ by $\zeta * (u,v) = (\zeta u, \zeta^2 v)$.
\newline
(Sidenote: U_σ is isomorphic the locus $(sw - t^3 = 0)$ in C^3.
\newline \newline
At this point, we have defined A_σ, the ring of invariants, in infinite collections of generators of the form: X^iY^j where $j <= \frac{m}{k}i$. In a future talk, we will learn how to find the minimal generators.