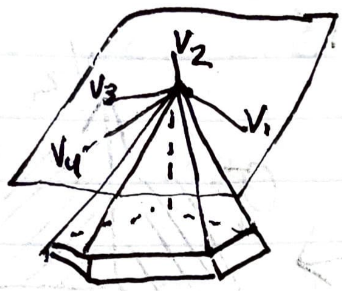
Line Bundles part 2 Talk (Jake Bernstein)

Start:



The convex function ψ is called <u>strictly convex</u> if the graph of ψ on the complement of σ lies strictly under the graph of $u(\sigma)$, for all n-dimensional cones σ

<u>Proposition</u>: Assume all maximal cones in Δ are n-dimensional. Let D be a T-cartier divisor on $X(\Delta)$. Then $\mathcal{O}(D)$ is generated by its sections if and only if ψ_D is convex

proof: on any toric variety X $\mathcal{O}(D)$ is generated by its sections if and only if any cone σ, there is a $u(\sigma) \in M$ such that

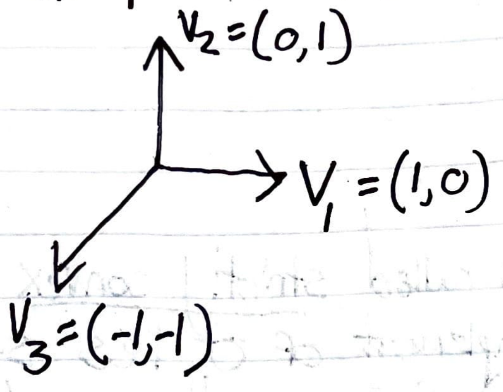
(i) $\langle u(\sigma), v_i \rangle \geq -a_i$ for all i, and

(ii) $\langle u(\sigma), v_i \rangle = -a_i$ for those i for which $v_i \in \sigma$

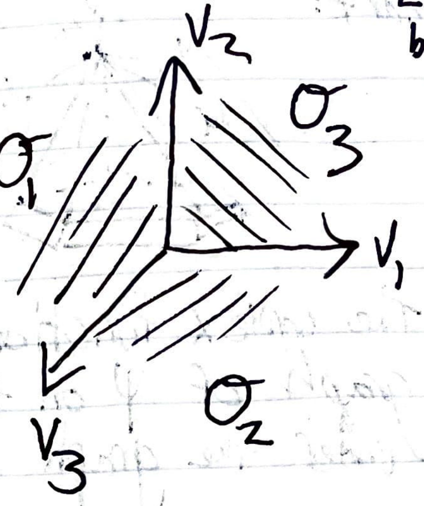
Ex: $D = \sum a_i D_i$

(Similar to Lilah's talk but rotating it)

Example in 2-Dimensions



$\rightarrow$



$D = D_1 + D_2 + D_3$

$$\left[\begin{array}{l} \sigma_3 : \langle u(\sigma_3), v_1 \rangle \geq -1 \\ \quad\ \langle u(\sigma_3), v_2 \rangle \geq -1 \\ \quad\ \langle u(\sigma_3), v_3 \rangle \geq -1 \end{array}\right]$$

Note: The inner product with each of the generators satisfies the inequality

~~We know that $u(\sigma_3) = (-1,-1)$~~

We can compute the dot products

$\sigma_3 : \langle (-1,-1), (1,0) \rangle = -1$ (meets the condition)

$\cdot \langle (-1,-1), (0,1) \rangle = -1$ (meets the condition)

$\cdot \langle (-1,-1), (-1,-1) \rangle = 2$ (meets the condition)

we can thus look at (ii): $\langle u(\sigma), v_i \rangle = -a_i$
for those i for which $v_i \in \sigma$

we can create the inequalities → ii) $\langle u(\sigma_3), v_1 \rangle = -1$

$\langle u(\sigma_3), v_2 \rangle = -1$

Note: If D is a Cartier then there is already a $u(\sigma)$ satisfying these constraints given by (ii)

$$D = \sum a_i D_i \rightarrow D = D_1 + D_2 - 3D_3$$

What do these conditions tell us?

- (i) → $\langle u(\sigma), v \rangle \geq -a_i$ for all i is the condition for $u(\sigma)$ to be in the polyhedron P_D that determines global sections
- (ii) - states that $\chi^{u(\sigma)}$ generates $\mathcal{O}(D)$ on U_σ
  - The function φ_D is determined by its restrictions to the n-dimensional cones, where its values are given by the constraints in (ii)
- Lastly, the convexity of φ_D is equivalent to (i)

If $\mathcal{O}(D)$ is generated by its sections, and all maximal cones of the fan are n-dimensional, we reconstruct D (equivalent to function ψ_D) from the polytope P_D:

$$\psi_D(v) = \min_{u \in P_D \cap M} \langle u, v \rangle = \min \langle v_i, v \rangle$$

where v_i are the vertices of P_D

Lemma: If $|\Delta| = N_{\mathbb{R}}$, the mapping φ_D is an embedding, i.e., D is very ample, if and only if ψ_D is strictly convex for every n-dimensional cone σ the semigroup S_σ is generated by

$$(u - u(\sigma) : u \in P_D \cap M).$$

Proof: - First we take corresponding homogeneous coordinates T_u on $\mathbb{P}^{r-1}$ indexed by the lattice points u in P_D

- Next, let σ be an n-dimensional cone in the fan and $u(\sigma)$ be the corresponding element of $P_D \cap M$

$\hookrightarrow \chi^{u(\sigma)}$ generates $\mathcal{O}(D)$ on U_σ

|Δ| complete
D very ample
if ψ_D strictly
convex &
S_σ generated
by the
lemma

- It follows ~~of~~ easily (as the textbook pretentiously puts it) from the strict convexity of ψ_D that the inverse image by φ_D of the set $\mathbb{C}^{r-1} \subset \mathbb{P}^{r-1}$ where $T_{U}(\sigma) \neq$ is the open set U_σ

- The restriction on this open set $U_\sigma \to \mathbb{C}^{r-1}$ is given by the functions $\chi^{U-U(\sigma)}$ $\chi^{v-u(\sigma)}$
- ~~They~~ generate S_σ means that the map of rings is surjective
  ↳ mapping is a closed embedding

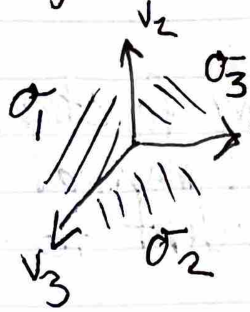
Proposition: on a complete toric variety, a T-cartier divisor D is ample (some pos multiple of D is very ample) if & only if its function ψ_D is strictly convex

proof:

- $\psi_{mD} = m\psi_D$ → follows from the lemma
- conversely → replacing D by m·D replaces the polytope P_D by $m \cdot P_D = (u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geq -m \cdot a_i$ for all $i)$
- For any $u \in S_\sigma$, $\langle u(\sigma), v_i \rangle > -a_i$ ~~for~~ if $v_i \notin \sigma$ that $u + m \cdot u(\sigma)$ is in $m \cdot P_D$ for large m
- S is a generated semigroup, S is generated by elements $u - m \cdot u(\sigma)$ as u runs through $m \cdot P_D \cap M$ for m sufficiently large (thus → mD is very ample)

Exercise: If $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are generated by their sections, show that $P_{D+E} = P_D + P_E$

$D = D_1 + D_2 + D_3$

1 

$P_D = \{\langle u, v_i \rangle \geq -1 \text{ for all } i\}$

1) $\langle u, v_1 \rangle \geq -1$

2 2) $\langle u, v_2 \rangle \geq -1$

3) $\langle u, v_3 \rangle \geq -1$

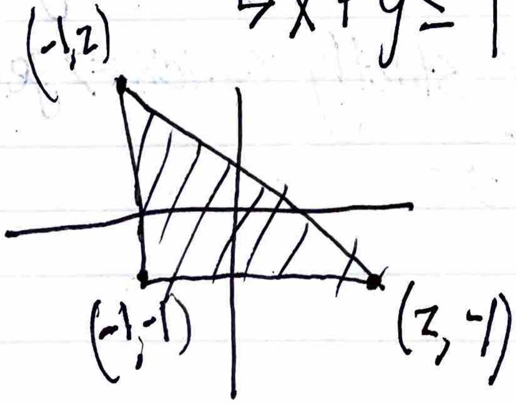
3 $v_1 = (1,0),\ v_2 = (0,1),\ v_3 = (-1,-1)$

1) $x \geq -1$

4 2) $y \geq -1$

3) $-x - y \geq -1$

$\hookrightarrow x + y \leq 1$

5 

6 $\Psi_D(v) = \min(\langle(-1,-1),v\rangle, \langle(-1,2),v\rangle, \langle(2,-1),v$

Exercise:

- If $X(\Delta)$ is complete and nonsingular, show that a T-divisor is ample if and only if it is very ample

- Note: If the corresponding function is given by $u(\sigma)$ on the maximal cone σ, both are equivalent to the condition:

$$\langle u(\sigma), v_j \rangle > -a_j$$

whenever $v_j \notin \sigma$

$|\Delta|$ complete and D very ample
- iff ψ_D strictly convex + S_σ generated by $\{u - u(\sigma) \mid u \in P_D \cap M\}$ for any max-cone σ
- D ample iff (if and only if) ψ_D strictly convex

$X(\Delta)$ nonsingular strictly convex $\Rightarrow$ generation condition for S_σ