Smooth Toric Surfaces

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October 18th, 2023

1 Two-dimensional Nonsingular Complete Toric Varieties

1.1 Smooth Affine Toric Varieties

Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. The affine toric variety U_σ is considered smooth if and only if σ itself is smooth. Importantly, all smooth affine toric varieties are of this form (Cox, Little, Schenck 40)

1.2 Refresher on Lattices

Definition 1. A *lattice* is a discrete additive subgroup of $\mathbb{R}^n$, that is, it is a subset $\Lambda \subseteq \mathbb{R}^n$ satisfying the following properties:

1. **(subgroup)** Λ is closed under addition and subtraction.

2. **(discrete)** There exists an $\varepsilon > 0$ such that any two distinct lattice points $x \neq y \in \Lambda$ are at a distance at least $||x - y|| \geq \varepsilon$.

It's worth noting that not every subgroup of $\mathbb{R}^n$ is a lattice.

1.3 Specifying a Sequence of Lattice Points

To describe a two-dimensional nonsingular complete toric variety, you must specify a sequence of lattice points $v_0, v_1, \ldots, v_{d-i}, v_d = v_0$ in a counterclockwise order within the lattice $N = \mathbb{Z}^2$. A vital property of this sequence is that any **successive pair in the series should generate the entirety**

of the lattice. This means that for every consecutive pair of points in the sequence, it is possible to represent any other point in $N = \mathbb{Z}^2$ as an linear combination of the two.

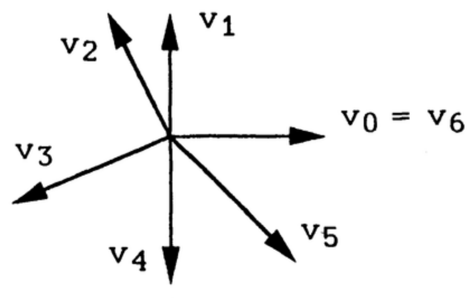
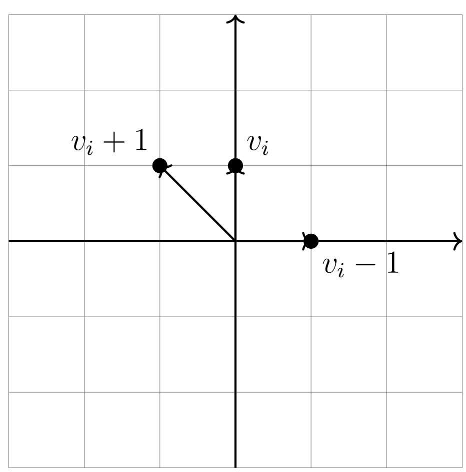


Figure 1: Sequence of lattice points.

Remark: Considering the vectors presented in the diagram, one can deduce that v_0 and v_1 form a basis for the lattice. Similarly, v_1 and v_2 can also be viewed as a basis. This basis formation leads us to the conclusion that $v_2 = -v_0 + a_1 v_1$ for some integer a_1. Following this pattern, in a generalized sense, the relation $a_i v_i = v_{i-1} + v_{i+1}$ holds true for all i such that $1 \leq i \leq d$, where a_i are specific integers.



Example 1: Let's determine the scalar a such that $a\mathbf{v}_i - \mathbf{v}_{i-1} = \mathbf{v}_{i+1}$. With the given vectors $\mathbf{v}_{i-1} = (1, 0)$, $\mathbf{v}_i = (0, 1)$, and the desired result

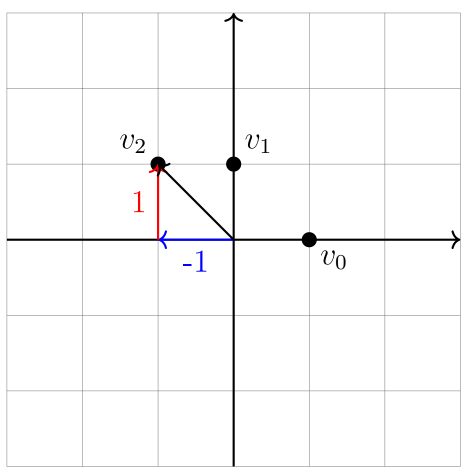
$\mathbf{v}_{i+1} = (-1, 1)$, we set up the following system of equations based on their vector components:

$$a = -1,$$

Solving, we obtain that $\mathbf{v}_{i+1} = -1(1, 0) + 1(0, 1) = (-1, 1)$. Additionally, you can show the following calculations:

$$\begin{aligned} a_i v_i &= v_{i-1} + v_{i+1} \\ v_{i-1} &= (1, 0) \\ a v_i &= a(0, 1) \\ \Rightarrow v_{i+1} &= a(0, 1) - (1, 0) \\ &= (0, a) - (1, 0) \\ &= (-1, a) \end{aligned}$$

Illustration:



From the fact that v_0 and v_1 serve as a basis for the lattice, and similarly v_1 and v_2 are also a basis, we can infer that

$$v_2 = -v_0 + a_1 v_1$$

for some integer a_1. More generally, the relationship

$$a_i v_i = v_{i-1} + v_{i+1}$$

holds for all i such that $1 \leq i \leq d$, where a_i are specific integers.

1.4 Topological Constraints

The possible configurations are topologically constrained. For instance, two of the cones cannot be arranged with v_j in the angle strictly between v_{i+1} and $-v_i$, and $-v_i$ and v_{j+1} in the angle strictly between $-v_i$ and $-v_{i+1}$.

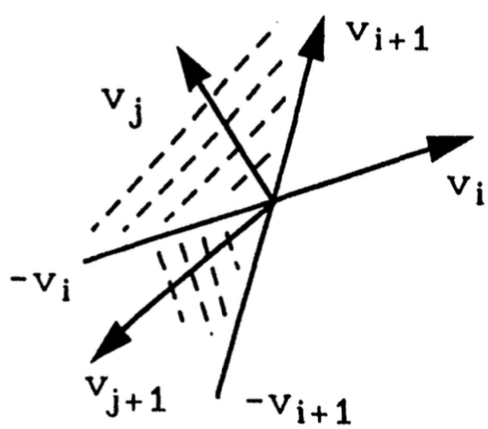


Figure 2: A visual of Topological Constraints found in textbook (Fulton 43)

1.5 Proof of Topological Constraints

Consider the expressions for the vectors:

$$v_j = -av_i + bv_{i+1}$$

and

$$v_{j+1} = -cv_i - dv_{i+1},$$

where a, b, c, and d are positive integers.

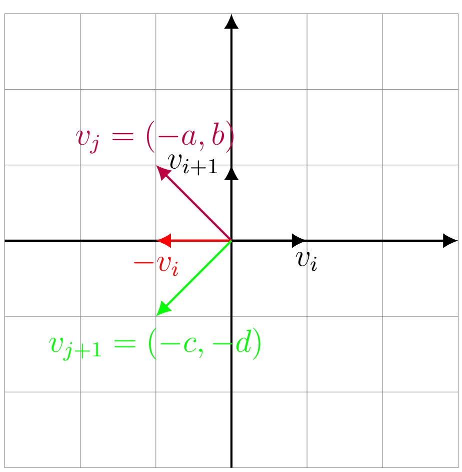
To determine the relationship between these vectors, compute the determinant of the matrix

$$\begin{bmatrix} -a & b \\ -c & d \end{bmatrix}$$

which should be 1. However, computing the determinant, we have $ad+bc \geq 2$, presenting a contradiction.

Remark: The determinant of a matrix represents the scaling factor of the area when a transformation is applied. For our vectors v_i and v_{i+1}, the determinant value should be 1, indicating that the vectors preserve their relative area under the transformation. However, the computed determinant, $ad + bc$, being ≥ 2, suggests that the area is scaled by at least a factor of 2, which is contradictory to our original premise. This establishes the inconsistency of the given vector relations under the specified constraints.

Diagram of the Proof



$$v_j = -av_i + bv_{i+1} = (-a, b)$$
$$v_{j+1} = -cv_i - dv_{i+1} = (-c, -d)$$
$$a, b, c, d \in \mathbb{Z}^+$$

1.6 Classifying Surfaces

We want to classify all of these surfaces. The cases when the number d of edges is small are easy to do by hand.

Exercise: Show that for $d = 3$, the resulting toric variety must be P^2, and for $d = 4$, one gets a Hirzebruch surface F_a, both as constructed in §1.1.

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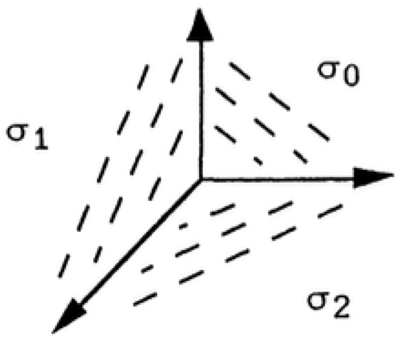


Figure 3: Fulton 6

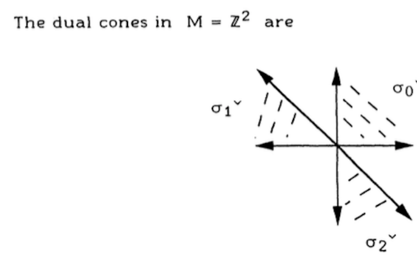


Figure 4: Fulton 7

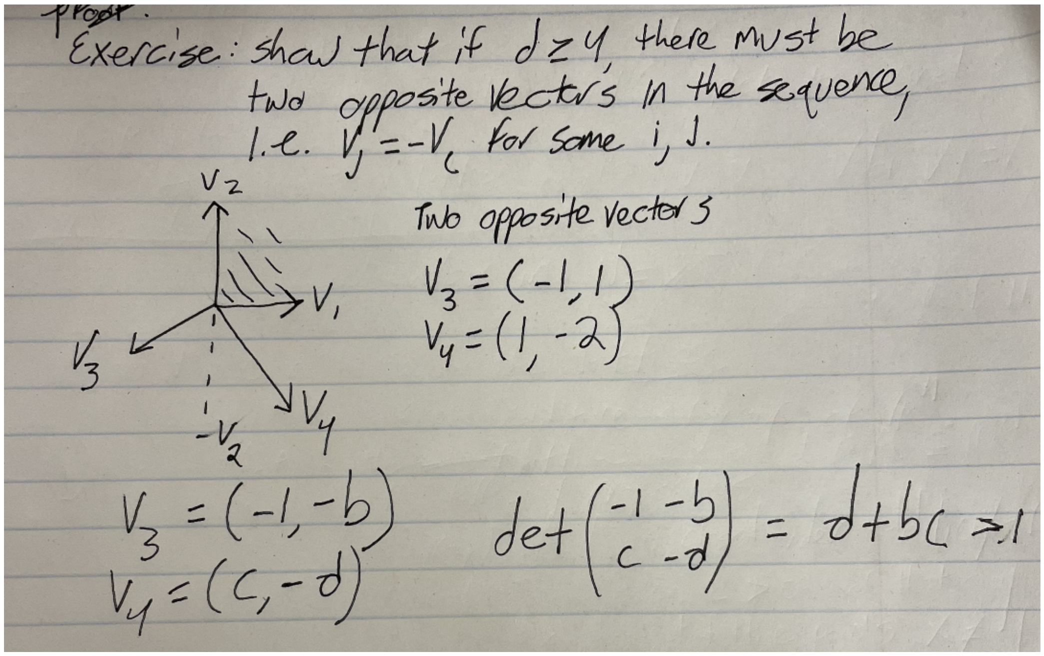
Remark: Each U_{θ_a} is isomorphic to $\mathbb{C}^2$, with coordinates (X, Y) for θ_0. $(X^{-1}, X^{-1}Y)$ for θ_1, and (Y^{-1}, XY^{-1}) for θ_2. These glue together to form the projective plane $\mathbb{P}^2$ in the usual way: if $(T_0 : T_1 : T_2)$ are the homogeneous coordinates on $\mathbb{P}^2$, $X = T_1/T_0$ and $Y = T_2/T_0$.

1.7 Blowing Up

Given one of these toric surfaces, we know how to construct another that is the blow-up of the first at a T_N-fixed point: simply insert the sum of two adjacent vectors.

Proposition: All complete nonsingular toric surfaces are obtained from $\mathbb{P}^2$ or F_a by a succession of blow-ups at T_N-fixed points.

Claim: If $d \geq 5$, there must be some j, $1 \leq j \leq d$, such that v_{j-1} and v_{j+1} generate a strongly convex cone and $v_j = v_{j-1} + v_{j+1}$.



Exercise

Suppose $v_i = -v_0$ and $i \geq 3$. Show that $v_j = v_{j-1} + v_{j+1}$ for some $0 < j < i$.

Hint from the Textbook

Write each v_j as $v_j = -b_j v_0 + b'_j v_1$. Define $c_j = b_j + b'_j$. Given that $c_2 \geq 3$ and $c_i = 1$, find a j such that $c_j > c_{j+1}$ and $c_j \geq c_{j-1}$. Demonstrate that $a_j = 1$ (Fulton 136).

Proof

Expressing each v_j in terms of v_0 and v_1, we have:

$$v_j = -b_j v_0 + b'_j v_1$$

and

$$c_j = b_j + b'_j$$

From the given conditions:

- $c_2 \geq 3$
- $c_i = 1$

Now, we search for a j for which:

1. $c_j > c_{j+1}$
2. $c_j \geq c_{j-1}$

Given the expression $a_j v_j = v_{j-1} + v_{j+1}$, it follows that:

$$a_j c_j = c_{j-1} + c_{j+1}$$

Since c_j is greater than c_{j+1} and roughly equal to or greater than c_{j-1}:

$$a_j c_j < 2c_j$$

Given that a_j must be an integer and it lies in the open interval (0,2), it's evident that:

$$a_j = 1$$

This concludes the proof since we now have shown that $v_j = v_{j-1} + v_{j+1}$ for some $0 < j < i$.

References

1. David Cox, John Little, Hal Schenck. *Toric Varieties.* American Mathematical Society, 2011.
2. William Fulton. *Introduction to Toric Varieties. (AM-131), Volume 131.* Princeton University Press, Princeton, 1993.