Complex Toric Varieties and Manifolds with Singular Corners

1. Definition of Complex Toric Varieties:
   - Complex toric varieties are primarily defined over integers, replacing the field of complex numbers (C) with the integers (Z). This transformation is denoted as U_g = Spec(Z[Ω^MI), where Ω represents the algebra.

$$\mathrm{Hom}_{sg}(\sigma^{\vee} \cap M, K) \, ,$$

2. Parametrization of K-valued Points:
   - For a field K, the K-valued points of U_g are described as semigroup homomorphisms, where K*U(0) denotes the multiplicative semigroup of K excluding zero.

3. Real Points of Toric Varieties:
   - When K = R or C, the real points of the toric variety can be obtained. This includes the important case of a sub-semigroup of C, particularly the semigroup of nonnegative real numbers $R_\geq$ = R+ U(0).

4. Retraction Using Absolute Value:
   - A retraction is established using the absolute value function, mapping z to |z|, defining a map Ra ⊂ C → $R_{\geq 0}$.

5. Topological Subspace for Any Cone:
   - For any cone o, the retraction determines a closed topological subspace $(U_g)_a$ = Hom_{sg}(Ω^MI, $R_{\geq 0}$) ⊆ U_g = Hom_{sg}(Ω^MI, C).

6. Construction of Closed Subspace X(A):
   - These constructions for various cones A collectively form a closed subspace X(A) of the toric variety X(A). This subspace includes a retraction map U → $(U_g)_a$.

together with a retraction $U_\sigma \to (U_\sigma)_\geq$. For any fan Δ, these fit together to form a closed subspace $X(\Delta)_\geq$ of $X(\Delta)$ together with a retraction

$$X(\Delta)_\geq \subset X(\Delta) \to X(\Delta)_\geq \, .$$

For example, if a is generated by vectors e1,..,ek that
form part of a basis for N, then (Uo) is isomorphic to a product of k copies of R and n-k copies of R.

Thus if X is nonsingular, Xz is a manifold with corners.

When X is singular, the singularities of X can be a little worse.

For the toric variety X = ph, with its usual covering by affine open sets U, = Ug, (U,) consists of points (to:...:1: •itn) with t,≥ 0. Hence

$$\mathbb{P}^n_{\geq} = \mathbb{R}_{\geq}^{n+1} \smallsetminus \{0\} / \mathbb{R}^+$$
$$= \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \quad t_i \geq 0 \text{ and } t_0 + \ldots + t_n = 1\},$$

which is a standard n-simplex. The retraction from $\mathbb{P}^n$ to $\mathbb{P}^n_{\geq}$ is

$$(x_0 : \ldots : x_n) \mapsto \frac{1}{\Sigma|x_i|}(|x_0|, \ldots, |x_n|) .$$

The fiber over a point $(t_0, \ldots, t_n)$ is a compact torus of dimension equal to $\mathrm{Card}\{i : t_i \neq 0\} - 1$.

The algebraic torus In contains the compact torus Sy:

Sy = Hom(M,s1) c Hom(M,C*) = TN,
where s1 = U(1) is the unit circle in C*. So Sy is a product of n circles.
From the isomorphism of C* with S1 x R+ = S1 × R, we have
Tn = SN ×Hom(M,R+) = SN ×Hom(M, R) = SN ×Nr. a product of a compact torus and a vector space.

Moment Map:
Moment maps arise frequently in the context of actions by Lie groups on varieties. Toric varieties offer a broad range of tangible instances. In this segment, we explicitly formulate these maps, establishing a connection to general moment maps.

Consider P as a convex polytope in Mr with vertices in M, leading to a toric variety X = X(p) and a morphism q: X - pr-1 through the sections.

$$\mu : X \to M_{\mathbb{R}}$$

by

$$\mu(x) = \frac{1}{\Sigma |\chi^u(x)|} \sum_{u \in P \cap M} |\chi^u(x)| \, u .$$

Note that μ is S_N-invariant, since, for t in S_N and x in X, $|\chi^u(t \cdot x)| = |\chi^u(t)| \cdot |\chi^u(x)| = |\chi^u(x)|$. It follows that μ induces a map on the quotient space $X/S_N = X_{\geq}$:

$$\bar{\mu} : X_{\geq} \to M_{\mathbb{R}} .$$

Proposition: The moment map establishes a homeomorphism from X onto the polytope P. Importantly, this homeomorphism is achieved by considering any subset of the sections X^(¼), provided that P represents the convex hull of its vertices, i.e., the subset includes the vertices of P.

Proof:
Consider Q as a face of P, with O being the corresponding cone in the fan. Our assertion is that the mapping effectively bijectively maps the subset onto the relative interior of Q.

$$(O_\sigma)_{\geq} \xrightarrow{\cong} \mathrm{Int}(Q) ,$$

Lemma:
Consider a finite-dimensional real vector space V, and let K be the convex hull of a finite set of vectors $U_1, \ldots, U_r$ in the dual space V*. Suppose K is not entirely contained in a hyperplane. For any positive numbers $\varepsilon_1, \ldots, \varepsilon_r$, define a mapping π_i: V → ℝ by the formula

$$\rho_i(x) = \varepsilon_i e^{u_i(x)} / (\varepsilon_1 e^{u_1(x)} + \ldots + \varepsilon_r e^{u_r(x)}) .$$

Then the mapping H: V - V*, M(x) = P1(x)u1 +. . .+Pr(x)Ur, defines a real analytic isomorphism of V onto the interior of K.