Let V be the vector space $\mathbb{R}^n$

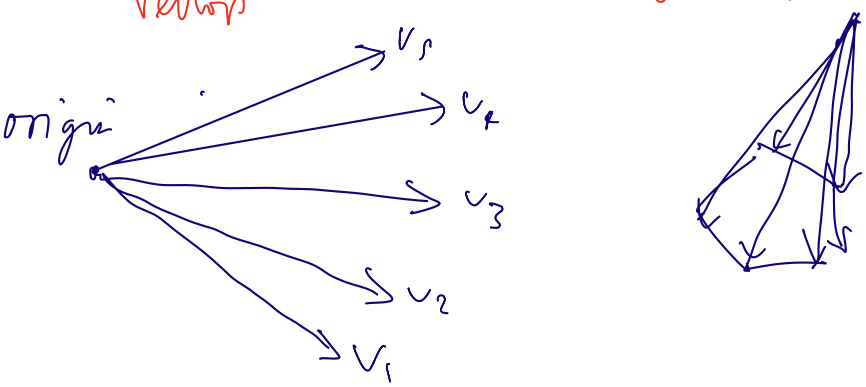
A convex polyhedral cone is a set generated by any finite set of vectors $v_1, \dots, v_r$ in V.

sigma $\sigma = (r_1 v_1 + \dots + r_s v_s \in V : r_i \geq 0)$

linear comb. w. non neg. coefficients

vectors

fig 1



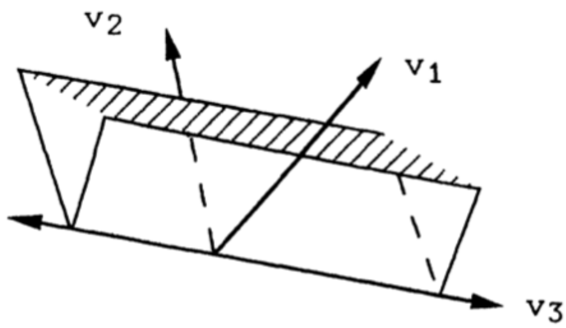
As per P's defn last week

A strongly convex polyhedral cone has the additional property

sigma

if $0 \neq v \in \sigma$, then $-v \notin \sigma$

Fig 2



These positive multiples of some v_i are called generators for the cone σ.

[You can also describe cones as intersections of half spaces]

The dimension $\dim(\sigma)$ of σ is the dimension of the linear space

$\mathbb{R}\cdot\sigma = \dim(\mathrm{Span}(\sigma))$ spanned by σ.

The dual cone $\sigma^\vee$ of any set σ is the set of eq^{n}s of supporting hyperplanes.

e.g. $\sigma^\vee = (u \in V^* : \langle u, v \rangle \geq 0$ for all $v \in \sigma)$

dot product

$V^* := V = \mathbb{R}^n$

(*)

If σ is convex polyhedral cone & $v_0 \notin \sigma$ then there is some $u_0 \in \sigma^\vee$ with $\langle u_0, v_0 \rangle < 0$

where dot product is less than zero.

This fact is important because consequences include:

1) duality theorem:

$$(\sigma^\vee)^\vee = \sigma$$

"dual of your dual of your cone is the cone"

A face τ of σ is the intersection of σ with any supporting hyperplane

$\perp$ = perp

$\tau = \sigma \cap u^\perp = \{v \in \sigma : \langle u, v \rangle = 0\}$ for some u in $\sigma^\vee$. A cone is regarded as a face of itself.

That will be when $\tau = \{\vec{0}\}$, then we get σ is a face of itself. The others are called proper faces.

NB Any linear subspace of a cone is contained in every face of the cone.

(show line in fig 2.)

eg. line / full plane.

2) *Any face is also a convex polyhedral cone.*

Given $x \in \sigma$, $a_1 v_1 + \dots + a_n v_n = x$ $\quad \sigma = \langle v_1, \dots, v_n \rangle$.

Let $u \in \sigma^\vee$ and consider $\tau = \sigma \cap u^\perp$

Then $x \in \tau$ implies

$$\begin{aligned}\langle u, x \rangle &= \langle u, a_1 v_1 + \dots + a_n v_n \rangle \\ &= \underbrace{a_1}_{\geq 0} \underbrace{\langle u, v_1 \rangle}_{\geq 0} + \dots + a_n \langle u, v_n \rangle \\ &= 0\end{aligned}$$

Thus either $a_1 = \dots = a_n = 0$ or at least one of $\langle u, v_i \rangle = 0$

If $v_{i_1}, \dots, v_{i_k}$ are those yielding 0

$\tau := \langle v_{i_1}, \ldots, v_{i_k} \rangle$

So there are finite subsets of τ & therefore finitely many faces.

3) *Any intersection of faces is also a face.*

$\underbrace{\cap (\sigma \cap u_i^\perp)}$

$x \in \sigma$ and

let $x \in$, then $\langle x, u_i \rangle = 0$ $(\forall i)$

$u = \sum u_i$ $\qquad \langle x, u \rangle = \langle x, \sum u_i \rangle = \sum \langle x, u_i \rangle = 0$

$\cap (\sigma \cap u_i^\perp) \supseteq \underbrace{\sigma \cap (\sum u_i)^\perp}_{y \in}$

$y \in \sigma$ and $\langle y, \sum u_i \rangle = 0$

we know $u_1, \ldots, u_n \in \sigma^\vee$ and $y \in \sigma$

so $\langle y, u_i \rangle \geq 0$ $(\forall i)$

thus if $\langle y, u_1 \rangle > 0$, then $\langle y, \sum u_i \rangle$

WLOG

$= \underbrace{\langle y, u_1 \rangle}_{> 0} + \underbrace{\langle y, \sum_{i \geq 2} u_i \rangle}_{\geq 0}$

CONTRA.

4) ***Any face of a face is a face.***

In fact, if $\tau = \sigma \cap u^\perp$ and $\gamma = \tau \cap (u')^\perp$ for $u \in \sigma^\vee$ and $u' \in \tau^\vee$, then for large positive p, $u' + pu$ is in $\sigma^\vee$ and $\gamma = \sigma \cap (u' + pu)^\perp$.

It's easier to be a dual of τ because there are fewer conditions to satisfy compared to being a dual of sigma so therefore the set is potentially larger.

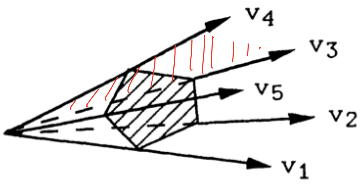
The large positive is used to overcome that.

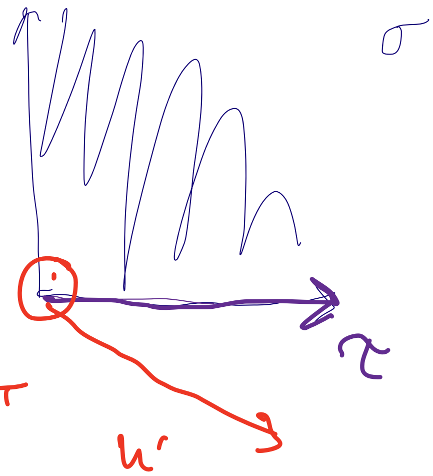
WTS $u' + pu \in \sigma^\vee$ let $v \in \sigma$

$\langle u' + p, v \rangle = \underbrace{\langle u', v \rangle}_{\text{if } \geq 0, \text{ done}} + \underbrace{p \langle u, v \rangle}_{\geq 0 \text{ b } u \in \sigma^\perp}$

if < 0 then must be large enough to force sum ≥ 0

A facet is a face of codimension one





face

$\tau \cap (u')^{\top}$

$v \in \sigma \rightarrow v = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad x, y \geq 0$

$$\langle \overbrace{u' + p u}^{\in \sigma^\vee}, v \rangle \qquad p \geq 0$$

$$= \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ p \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle$$

$$= \underset{\geq 0}{x} + (p-1)\underbrace{y}_{\geq 0} \geq 0 \qquad p - 1 \geq 0$$

$$p \geq 1$$

5) Any proper face is contained in some facet. ① – maybe just say this and the proof involves quotient spaces.

$\dim(\underset{\text{cone}}{\sigma}) = \dim(\underset{\text{space spanned}}{V})$

$\dim(\tau) = \dim(W)$

we don't need to worry about when the co. d = 1 because there facet = proper face

Assume $\underbrace{\dim(\sigma) - \dim(\tau)}_{\text{co dimension}} \geq 2$

The images $\bar{v}_i$ in V/W of the generators of σ are contained in a half-space determined by u.

$u \in \sigma^\vee$
and $\tau = \sigma \cap u^\perp$

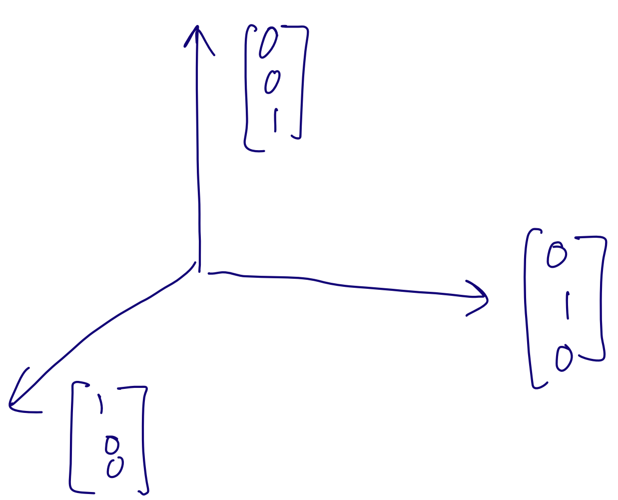
If $\sigma = \text{gen}\{v_1, \dots, v_n\}$

then V/W will contain $\bar{v}_1, \dots, \bar{v}_n$ in the half-space made by u: $\langle \bar{v}_i, \bar{u} \rangle \geq 0$

At least two are $\neq \bar{0}$ (i.e. two or more $v_i \notin W$)

In fact any face of codimension two is the intersection of exactly two facets.

(6) Any proper face is the intersection of all facets containing it.



$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

σ = first octant
$= \text{gen}\{v_1, v_2, v_3\}$

$\tau_1 = \text{gen}\{v_1, v_2\}$ $\tau_3 = \text{gen}\{v_2, v_3\}$

$\tau_2 = \text{gen}\{v_1, v_3\}$

$$\tau_1 \cap \tau_2 = \gamma_1 = \text{gen}\{v_1\}$$

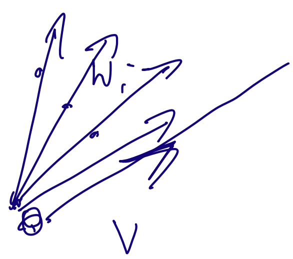
$$\tau_2 \cap \tau_3 = \gamma_2 = \text{gen}\{v_2\}$$

$$\gamma_1 \cap \gamma_2 = \{0\} \leftarrow \text{proper face codim} = 3$$

$$\{0\} = \underbrace{\tau_1 \cap \tau_2 \cap \tau_3}_{3 \text{ facets}}$$

Indeed, if τ is any face of codimension larger than two, from (5) we can find a facet γ containing it; by induction τ is the intersection of facets in γ, and each of these is the intersection of two facets in σ, so their intersection τ is an intersection of facets.

(7) *The topological boundary of a cone that spans* V *is the union of its proper faces (or facets).*



$w_i \longrightarrow v \qquad w_i \notin \sigma$

By $*$ — (*) *If* σ *is a convex polyhedral cone and* $v_0 \notin \sigma$, *then there is some* $u_0 \in \sigma^\vee$ *with* $\langle u_0, v_0 \rangle < 0$.

$$\text{seq.}(w_i) \longrightarrow v$$

$$\text{seq}(u_i) \longrightarrow u_0 \in \sigma^\vee$$

$$\langle w_i, u_i \rangle < 0 \quad (\text{all } i)$$

$$\langle v, u_i \rangle \geq 0 \quad \text{by def. of } \sigma$$

Then by continuity:

$$\underbrace{\langle w_i, u_i \rangle}_{\text{neg.}} \longrightarrow \langle v, u_0 \rangle = 0$$

$$\underbrace{\langle v, u_i \rangle}_{\text{pos.}}$$

result is

$\therefore$ v is in a face

(8) (8) *If σ spans V and $\sigma \neq V$, then σ is the intersection of the half-spaces $H_\tau = \{v \in V : \langle u_\tau, v\rangle \geq 0\}$, as τ ranges over the facets of σ.*

(not giving proof)

This is helpful for finding generators for the dual cone

dim $(V) = n$ σ spans V

$\sigma \neq V$

$\sigma = \text{gen}\{v_1, \ldots, v_m\rangle$ $(m \geq n)$

PROCEDURE

get a lin. indep. subset of size $n-1$ ← gets a facet in σ

(There are $\binom{m}{n-1}$ of these)

Check each set for Linear Independence

$a_1 v_1 + \ldots + a_{n-1} v_{n-1} = 0$

WTS $a_1 = \ldots = a_{n-1} = 0$

and complete perp. subspace in V (which will have dim. 1)

Choose the generator u_τ that has

$\langle v, u_\tau\rangle \geq 0 \quad \forall v \in \sigma$

Get all u_τ to find the generator list.

$\sigma^\vee = \text{gen}\{u_{\tau_1}, \ldots, u_{\tau_k}\}$

Farkas' Theorem :

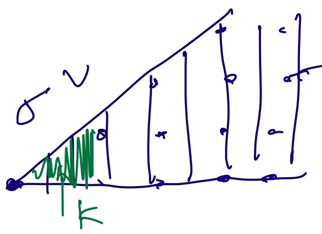
(9) *The dual of a convex polyhedral cone is a convex polyhedral cone.*

The corollary is the generators of the perp. subspaces are the generators for $\sigma^\vee$ (just choose the generators with gen.s with non neg. dot product w. elements of σ)

If we now suppose σ is *rational*, meaning that its generators can be taken from ~~N~~, then $\sigma^\vee$ is also rational; indeed, the above procedure shows how to construct generators u_i in $\sigma^\vee \cap M$.

rational means v_i has integer coordinates

$\mathbb{Z}^n$



$\mathbb{Z}^n$

call it $\mathbb{Z}^n$

integer coordinate vectors

Proposition 1. (Gordon's Lemma) *If σ is a rational convex polyhedral cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.*

main point

A semigroup: $(S, *)$ $\quad * : S \times S \to S$

set

and $*$ is associative

$a * (b * c) = (a * b) * c$

Proof: • Let σ be a rational convex poly. cone

• Then $\sigma^\vee$ is also RCPC

• Let $\{u_1, \dots, u_r\} \subseteq \sigma^\vee \cap \mathbb{Z}^n$ be a generating set for $\sigma^\vee$ as a cone in $\mathbb{R}^n$

• Let $K = \{\sum t_i u_i : 0 \le t_i \le 1\}$

• Then K is compact (closed & bounded)

• Thus $K \cap \mathbb{Z}^n$ is finite ✩

• Let $u \in \sigma^\vee \cap \mathbb{Z}^n$; write $u = \sum r_i u_i$, $r_i \in \mathbb{R}_{\geq 0}$

- Take $t_i = r_i - \lfloor r_i \rfloor \in [0,1)$ ← greatest integer less r_i
- Set $m_i = \lfloor r_i \rfloor$
- $u = \sum r_i u_i = \sum (m_i + t_i) u_i = \underbrace{\sum m_i u_i}_{\in \mathbb{Z}_n} + \underbrace{\sum t_i u_i}_{\in K}$
- If u has integer coordinates & $\sum m_i u_i \in \mathbb{Z}^n$, then $\sum t_i u_i \in \mathbb{Z}^n$ as well so $\sum t_i u_i \in K \cap \mathbb{Z}^n$
- Therefore u is generated by elements of $K \cap \mathbb{Z}^n$

It is often necessary to find a point in the *relative interior* of a cone σ, i.e., in the topological interior of σ in the space $\mathbb{R}\cdot\sigma$ spanned by σ. This is achieved by taking any positive combination of $\dim(\sigma)$ linearly independent vectors among the generators of σ. In particular, if σ is rational, we can find such points in the lattice.

Any point in the relative interior can be found by taking a positive combination of $\dim(\sigma)$ L.I vectors among the generators of σ.

(10) *If* τ *is a face of* σ, *then* $\sigma^\vee \cap \tau^\perp$ *is a face of* $\sigma^\vee$, *with* $\dim(\tau) + \dim(\sigma^\vee \cap \tau^\perp) = n = \dim(V)$. *This sets up a one-to-one order-reversing correspondence between the faces of* σ *and the faces of* $\sigma^\vee$. *The smallest face of* σ *is* $\sigma \cap (-\sigma)$.

eg. origin on σ would map to the whole cone $\sigma^\vee$

$\vec{v} \in \tau$ (a face of σ) s.t. $\vec{v}$ is in τ's interior,

then $\sigma^\vee \cap v^\perp = \sigma^\vee \cap (\tau^\vee \cap v^\perp) = \sigma^\vee \cap \tau^\perp$

perp to $\vec{v} \Rightarrow$ perp to everything in τ

- define $\tau^* = \underbrace{\sigma^\vee \cap \tau^\perp}_{\text{faces of } \sigma^\vee}$

$F : \text{Faces}(\sigma) \longrightarrow \text{Faces}(\sigma^\vee)$

$F(\tau) = \sigma^\vee \cap \tau^\perp$

$\tau \subseteq \sigma \cap (\sigma^\vee \cap \tau^\perp)^\perp = (\tau^*)^*$

From this $\tau^* = ((\tau^*)^*)^*$ so bijective

and this implies

$$
\begin{aligned}
(\sigma^\vee)^* &= (\sigma^\vee)^\vee \cap (\sigma^\vee)^\perp \\
&= \sigma \cap (\sigma^\vee)^\perp \\
&= (\sigma^\vee)^\perp \\
&= \underbrace{\sigma \cap (-\sigma)}_{\substack{\text{subspace in } \mathbb{R}^n \\ \text{contained in } \sigma}}
\end{aligned}
$$

have $\sigma \subseteq (\sigma^\vee)^\perp$

and $-\sigma \subseteq (\sigma^\vee)^\perp$

Subset of σ and $-\sigma$

(11) If $u \in \sigma^\vee$, and $\tau = \sigma \cap u^\perp$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u)$.

$\tau^\vee = \sigma^\vee = \mathbb{R}_{\geq 0} \cdot (-u)$

- $(\tau^\vee)^\vee = \tau$
- $(\sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u))^\vee = \sigma \cap (-u)^\vee$

$= \sigma \cap u^{\perp}$

Proposition 2

Proposition 2. *Let σ be a rational convex polyhedral cone, and let u be in $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n$. Then $\tau = \sigma \cap u^\perp$ is a rational convex polyhedral cone. All faces of σ have this form, and*

$$S_\tau = S_\sigma + \mathbb{Z}_0 \cdot (-u)$$

Faces of RCPC are themselves RCPC.

Proof: If τ is a face, then $\tau = \sigma \cap u^\perp$ for any u in the relative interior & u can be in $\mathbb{Z}^n$ since $\sigma^\vee \cap \tau^\perp$ is rational.

Know it's rational from (9.5). σ rational $\Rightarrow \sigma^\vee$ rational.

Given $w \in S_\tau$ then $w + p \cdot u$ is in $\sigma^\vee$ for large positive p (4)

and taking p to be an integer shows that

w is in $S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u)$

(12) Separation Lemma

If σ and σ' are convex polyhedral cones τ is a face of each, then there is a u in $\sigma^\vee \cap (-\sigma')^\vee$ with

$$\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$$

- Look at cone $\gamma = \sigma - \sigma' = \sigma + (-\sigma')$
- We know that for any u in the relative interior of $\gamma^\vee$, $\gamma \cap u^\perp$ is the smallest face of γ.

"$u^\perp$ makes smallest face"

$$\gamma \cap v^{\perp} = \gamma \cap (-\gamma) = (\sigma - \sigma') \cap (\sigma' - \sigma)$$
(10)

don't show

$\sigma \subseteq \gamma$: $w \in \gamma$, $\exists v \in \sigma$, $\exists v' \in \sigma'$
s.t. $w = v + v'$
Consider when $v' = 0$: $w = v \in \sigma$
so $\forall v \in \sigma$, $v \in \gamma \Rightarrow \sigma \subseteq \gamma$

- Since σ is contained in γ, u is contained in $\sigma^{\vee}$ & since τ is contained in $\gamma \cap -\gamma$, τ is contained in $\sigma \cap u^{\perp}$.
- If $v \in \sigma \cap u^{\perp}$ then v is in $\sigma' - \sigma$ so if $w' \in \sigma'$, $w \in \sigma$ $v = w' - w$.
- $v + w = w'$ $v + w \in \sigma'$ $v + w \in \sigma$ $\tau = \sigma \cap \sigma'$

$\Rightarrow v + w \in \tau$ the sum of 2 elements in a face can be in a face only if the summands ~~are~~ in the face (because all the coef.s are pos. or equal to zero in a cone)

$\Rightarrow v \in \tau$

This shows that $\tau = \sigma \cap u^{\perp}$
& same argument can be applied for $-u$ to give $\sigma' \cap u^{\perp} = \tau$.

Proposition 3

Proposition 3. *If* σ *and* σ' *are rational convex polyhedral cones whose intersection* τ *is a face of each, then*

$$S_\tau = S_\sigma + S_{\sigma'}.$$

Proof

$$\tau \subseteq \sigma \cap \sigma'$$

$$\Rightarrow (\sigma \cap \sigma')^\vee \subseteq \tau^\vee$$

$$\sigma^\vee + (\sigma')^\vee \subseteq \tau^\vee$$

$$(\sigma^\vee + (\sigma')^\vee) \cap \mathbb{Z}^n \subseteq \tau^\vee \cap \mathbb{Z}^n$$

$$S_\sigma + S_{\sigma'} \subseteq S_\tau$$

For the other way around by (12) we can say u in $\sigma^\vee \cap (-\sigma')^\vee \cap \mathbb{Z}^n$ so that $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$ By proposition 2 & that $-u$ is in $S_{\sigma'}$ we have $S_\tau \subset S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u) \subset S_\sigma + S_{\sigma'}$

13) For a convex polyhedral cone σ the following conditions are equivalent

i) $\sigma \cap (-\sigma) = \{0\}$ the origin

ii) σ contains no non linear subspace

If $0 \neq v \in \sigma$, then $-v \notin \sigma$

iii) there is a u in $\sigma^\vee$ with $\sigma \cap u^\perp = \{0\}$

iv) $\sigma^\vee$ spans $\mathbb{R}^n$

A cone is called *strongly convex* if it satisfies the conditions of (13). Any cone is generated by some minimal set of generators. If the cone is strongly convex, then the rays generated by a minimal set of generators are exactly the one-dimensional faces of σ (as seen by applying (*) to any generator that is not in the cone generated by the others); in particular, these minimal generators are unique up to multiplication by positive scalars.

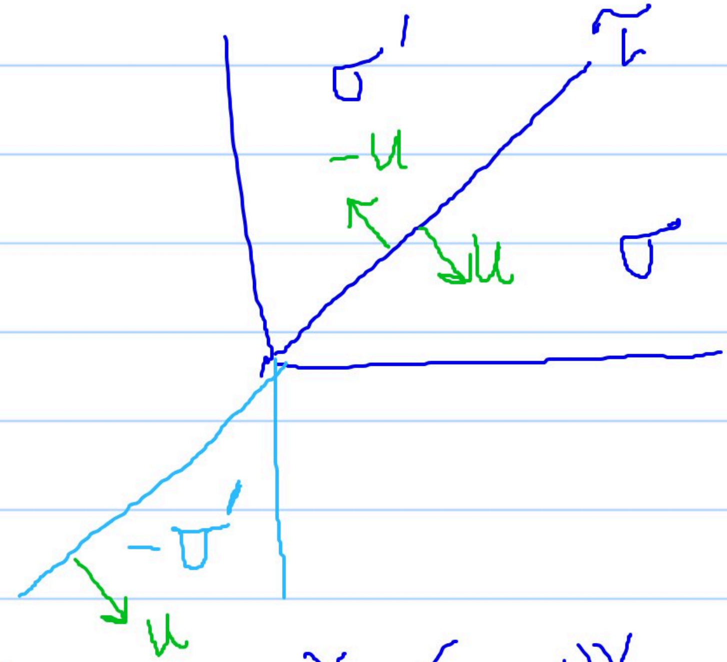
In future lectures we will just call them cones.

$$\sigma \begin{cases} x \geq 0 \quad \text{and} \quad y \leq x \\ x - y \geq 0 \\ \Rightarrow (x,y)\cdot\overbrace{(1,-1)}^{u} \geq 0 \end{cases}$$

$u \longrightarrow u \in \sigma^\vee$

$$\sigma' \begin{cases} y \geq 0 \quad \text{and} \quad x \leq y \\ y - x \geq 0 \\ \Rightarrow (x,y)\cdot(-1,1) \geq 0 \\ \Rightarrow (x,y)\cdot\underbrace{-(1,-1)}_{-u \in (\sigma')^\vee} \geq 0 \end{cases}$$

$$\Rightarrow u \in (-\sigma')^\vee$$



$$u \in \sigma^\vee \cap (-\sigma')^\vee$$

wow!
amazing!