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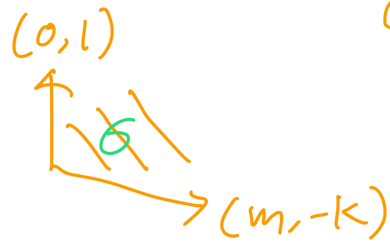
Recall Tane's Lecture:

all 2-dim cones $\xrightarrow{\text{transform}}$



$0 \le k < m$

$\gcd(k,m) = 1$.

$\begin{cases} m=1 : \text{non-sing} \\ m \neq 1 : \text{sing} \end{cases}$

$\updownarrow$

2-dim Affine toric Variety

$\mathbb{C}^2/\mu_m := \mathrm{Spec}(\mathbb{C}[x,y]^{\mu_m})$

μ_m: mth roots of unity

μ_m acts on $\mathbb{C}^2$ w/ weights $(1,k)$

$\mu_m \ni z$, $z(x,y) = (zx, z^k y)$

Now, want $\uparrow$ dimension & add more cones.

$\Downarrow$

The class of examples: the construction of the Weighted Projective Space $\mathbb{P}(d_0, \dots, d_n)$, $d_i \in \mathbb{Z}_{>0}$

- Start w/ the same fan used in the construction of projective space
  - cones generated by proper subsets of $\{e_0, e_1, \dots, e_n\}$ where $e_0 + e_1 + \dots + e_n = 0$.
- BUT take our lattice N to be generated by the vectors $\frac{1}{d_i} \cdot e_i$, $0 \le i \le n$.

$\Rightarrow$ Resulting toric Variety is the variety

$$\mathbb{P}(d_0, \dots, d_n) = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by $\mathbb{C}^* \times \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}$

$$(c, (x_0, \dots, x_n)) \longmapsto (c^{d_0} x_0, \dots, c^{d_n} x_n)$$

Side note: $\mathbb{P}(d_0, \dots, d_n) = \mathbb{P}^n / ((\mu_{d_0} \times \dots \times \mu_{d_n}) / \mu_{\gcd})$

Each chart $\cong \mathbb{C}^n / \mu_{d_i}$ w/ weights $(d_0, \dots, \hat{d_i}, \dots, d_n)$

2.3 One-parameter subgroups; limit points

Goal: one-parameter subgroups of the torus

+

their limit points in toric varieties

$\Downarrow$

to recover the fan from the torus action.

[Reminder: algebraic group = group + variety

Example: $GL_n(\mathbb{C})$.]

Want: recover the lattice N from the torus T_N.

↳ Look at one-parameter subgroups (1-PS)

Def: In the theory of algebraic groups,

1-PS is a hom $\varphi: \mathbb{C}^* \to G = T_N = (\mathbb{C}^*)^n$

$\varphi(ab) = \varphi(a)\varphi(b)$.

$\forall k \in \mathbb{Z}$, $\exists$ homomorphism $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^k$.

In fact, $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$: All of such form.

Given lattice N, w/ dual M $\dim M = \dim N = n$

$\Rightarrow$ corresponding torus $T_N = \mathrm{Hom}(M, \mathbb{C}^*) = (\mathbb{C}^*)^n$

Taking a basis for $N \Rightarrow$

$\mathrm{Hom}(\mathbb{C}^*, T_N) = \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) = \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*)^n$
$= \mathbb{Z}^n = N$

↳ Remark: every one-parameter subgroup $\lambda: \mathbb{C}^* \to T_N$

is given by an unique v in N.

↳ denote as λ_v. $\lambda_v: \mathbb{C}^* \to T_N = (\mathbb{C}^*)^n$

$t \mapsto (t^{v_1}, \dots, t^{v_n})$

Note that $T_N = (\mathbb{C}^*)^n$ has an action on X_Δ

Gluing U_σ, $\sigma \in \Delta \rightsquigarrow X_\Delta \Rightarrow$ it suffices to define its action on each affine piece.

i.e. gluing respects this toric action.

$S_\sigma \subset M = \mathbb{Z}^n \Rightarrow$ induces an action of $T_N = (\mathbb{C}^*)^n$ on $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$, where $\forall I \in U_\sigma$, let $f_1, \dots, f_k$ be the generators of I

i.e. $I = (f_1, \dots, f_k)$, then

$(\mathbb{C}^\times)^n \cdot I = ((\mathbb{C}^\times)^n \cdot f_1, \dots, (\mathbb{C}^*)^n \cdot f_n)$, where

$$\psi$$

$$(t_1, \dots, t_n) \cdot (x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = (t_1^{a_1} \cdots t_n^{a_n})(x_1^{a_1} \cdots x_n^{a_n})$$

Exm: $(t_1, t_2) \cdot x^3 = t_1^3 x^3$; $(t_1, t_2) \cdot xy^2 = t_1 t_2^2 x y^2$.

Def: The character $\chi^m : T \to \mathbb{C}^*$ associated with the lattice point m is defined by

$$\chi^m(t) = t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$$

Remark: $\chi^m(t \cdot s) = \chi^m(t)\chi^m(s)$

$\Rightarrow \chi^m$ is a group homomorphism

Then $\forall u \in \sigma^\vee \subset M$, the dual of N,

we have
$$\begin{aligned} \lambda_v(z)(u) &= \chi^u(\lambda_v(z)) \\ &= \chi^u((z^{v_1}, \dots, z^{v_n})) \\ &= z^{v_1 u_1} \cdot z^{v_2 u_2} \cdot \dots \cdot z^{v_n u_n} \\ &= z^{\langle v, u \rangle} \end{aligned}$$

$\to$ inner product ($u \in M \cong \mathbb{Z}^n$)

Want: recover σ from the Toric embedding $T_N \subset U_\sigma$

$\hookrightarrow$ look at $\lim_{z \to 0} \lambda_v(z)$ for various $v \in N$

Exa: σ generated by part of a basis $e_1, \dots, e_k$ for N, so U_σ is $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$

For $(m_1, \dots, m_n) \in \mathbb{Z}^n$, $\lambda_v(z) = (z^{m_1}, \dots, z^{m_n})$

$\lim_{z \to 0} \lambda_v(z)$ exists in U_σ

$\iff m_i \geq 0 \ \forall i$ and $m_i = 0 \ \forall i > k$

$\iff v \in \sigma$.

$\lim_{z \to 0} \lambda_v(z) = (\delta_1, \dots, \delta_n)$, where $\delta_i = 1$ if $m_i = 0$ and $\delta_i = 0$ if $m_i > 0$.

By Will's talk, know these limit points are the distinguished point x_τ for some face(s) τ of σ.

$U_\tau = U_{\tau'} \times (\mathbb{C}^\times)^{n-k}$

τ': viewed inside the span of itself.

i.e. $\tau' = \tau$ viewed as a top cone

$$x_\tau = \left(\begin{array}{c} \text{the unique fixed point} \\ \text{when acted on by } (\mathbb{C}^*)^k \end{array}, \underbrace{1, \dots, 1}_{n-k} \right)$$

Two claims:

Claim 1: If v is in $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma$, and τ is the cone of Δ that contains v in its relative interior, then $\lim_{z \to 0} \lambda_v(z) = x_\tau$

$$\begin{tikzcd} \underset{z}{\mathbb{C}^*} \arrow[r, "\lambda_v"] \arrow[rr, bend right, "\lambda_v"'] & (\mathbb{C}^*)^n \arrow[r, "i"] & X_\Delta \end{tikzcd}$$

Claim 2: If v is not in any cone of Δ, then $\lim_{z\to 0} \lambda_v(z)$ does not exist in $X(\Delta)$.

Two definition from 2.4

Def: $X(\Delta) = X_\Delta$ is compact iff $|\Delta| = \mathbb{R}^n$

Rahul's lecture: if
$$\begin{array}{ccc} N' & \xrightarrow{\varphi} & N \\ \| & & \| \\ \mathbb{Z}^n & & \mathbb{Z}^n \end{array}$$
maps Δ' into Δ $\rightsquigarrow$ $\varphi^*: X_{\Delta'} \to X_\Delta$

def φ proper if $\varphi^{-1}(|\Delta|) = |\Delta'|$

Ex: