Seminar 4.7-4.11

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1 4.7 Quantum Communication

- Alice and Bob are our characters for this story, the two need to communicate with each other, Alice sends quantum states, **carriers**, and Bob tries to correctly identify them using the correct measurements.
- The carriers are are described by state vectors in a 2^ndimensional Hilbert space. This makes it such that they can encode at most n-bits of information.
- This is the same in the classical situation in which the space of binary stings of length n is the dimension of 2^n where each index can be described as picking a 0 or 1 for n digits.
- If Alice chooses to send one of the 2^n states from a pre-chosen orthonormal basis of

  $$\{|e_k >\}_{k=1,2,..,2^n}$$

  , by choosing the same basis Bob can choose distinguish them reliably.
- If Alice sends more than n bits of information per carrier encoding them in states from

  $$|s_1 >, ..., |s_N >$$

  where

  $$N > 2^n$$

  then Bob can not choose a clever measurement and can not distinguish between the states.

2 4.8 Basic Quantum Coding and Decoding

- You can choose projector P on subspace spanned by signal space

  $$|s_1 >, |s_2 >, ..., |s_N >$$

  then

  $$P_1, P_2, P_3, ..., P_N$$

  are the measurement defined by P

- Identifying the right signal state when $N \geq 2^n$ is given as: '

$$P(success) = \frac{1}{N}\sum < s_k|P_k|s_k >$$

- $\sum < s_k|P_k|s_k >$ represents the probability that the selected signal state is correctly identified over all signal states (Bob's probability choosing the right one over all k)
- $\frac{1}{N}$ represents the uniform distribution probability.
- Use the identity that

$$< \psi|A|\psi >= tr(A|\psi >< \psi|) = tr((\psi >< \psi|A)$$

$$P(success) = \frac{1}{N}\sum < s_k|P_k|s_k >$$

$$= \frac{1}{N}\sum < s_k|PP_kP|s_k >$$

$$= \frac{1}{N}\sum tr(PP_kP|s_k >< s_k|)$$

- By fact that B is a positive semi-definite operator and P is a projector then:

$$trBP \leq trB$$

let

$$Q = 1 - P$$

$$trB = trB(P + Q)$$

$$trB = trBP + trBQ$$

$$trB \geq trBP$$

- By inequality above:

$$= \frac{1}{N}\sum tr(PP_kP|s_k >< s_k|) \leq= \frac{1}{N}\sum tr(PP_kP)$$

$$= \frac{1}{N}tr(P\sum PP_kP)$$

$$= \frac{1}{N}tr(P^3)$$

$$= \frac{d}{N}$$

- By encoding N equally likely messages as states that is in Hilbert dimension,d, and performing a measurement and inferring the message as a result then the probability of success is bounded by $\frac{d}{p}$
- If N possible signals exceeds the dimension then you can not reliably distinguish the signals
- One qubit can store at most one bit of information that can be reliably read

3 4.9 Non-Orthogonal States

- Say Alice sends Bob $|s_1>, |s_2>$ that are non-orthogonal and they are equally chance of being chosen
- The goal is to be able to tell them apart, and we should be able to do that by how close they are to being orthogonal, i.e. their angle
- The closer they are from being orthogonal the better chance we have from distinguishing them perfectly. If they are co-linear we can not distinguish them. If they are orthogonal you can distinguish them perfectly.
- To find this we then have

$$Pr(success) = \frac{1}{2}(<s_1|P_1|s_1> + <s_2|P_2|s_2>)$$

$$= \frac{1}{2}(trP_1|s_1><s_1| + trP_2|s_2><s_2|$$

$$= \frac{1}{2}(trP_1|s_1><s_1| + tr(1-P_1)|s_2><s_2|$$

$$= \frac{1}{2}(1 + trP_1(|s_1><s_1| - |s_2><s_2|)$$

The function $(|s_1><s_1| - |s_2><s_2|)$ is the Hermitian which can be represented as

$$\lambda(|d_+><d_+| - |d_-><d_-|$$

$$= \frac{1}{2}(1 + trP_1\lambda(|d_+><d_+| - |d_-><d_-|$$

$$\leq \frac{1}{2}(1 + \lambda(<d_+|P_1|d_+>$$

- This is bounded by $\frac{1}{2}(1+\lambda)$
- The goal now is to find λ for operator D, since we know the $trD^2 = 2\lambda^2$ then

$$trD^2 = tr((|s_1><s_1| - |s_2><s_2|)^2)$$

$$2\lambda^2 = 2 - 2|<s_1|s_2>|^2$$

$$\lambda = \sqrt{1 - 1 < s_1|s_2>|^2}$$

$$Probability of Success \leq \frac{1}{2}(1 + \sqrt{1 - |<s_1|s_2>|^2}$$

$$Pr(success) \leq \frac{1}{2}(1 + sin\alpha)$$

4 4.11 Quantum Theory Formally: Five Axioms

There's the question why this formalism involving Hilbert spaces, unitary opeartors, and the Born rule is correct way to do it. Besides the fact it just works it can also been seen that to the five axioms below it is the only choice that makese sense. Let K degrees and freedom and for N-dimension for the following statements

1. Probabilities: The relative frequencies of observed outcomes from measuring an ensemble of n systems tend to a well defined value when n tends to ∞
2. Simplicity: Integer K is a function of N and takes the minimum possible values consistent with these axioms for each N
3. Sub-spaces: If a system all lies within the M-dimensional subspace where M¡N then it behaves like system M
4. Composite Systems" If a composite system behaves multiplicatively then $N = N_a N_) b$ and $K = K_a N_b$
5. Continuity: For two pure states there are continuous reversible transformations of the systems sending one to the other

4.1 Quantum States

We have it such that for an **isolated quantum system that can be prepared in n perfectly distinguishable states** then we can have

- a Hilbertspace, H of dimension n
- vector $|v >\in H$ is of unit length represents a quantum state of the system.
- where the inner product $< u|v >$ is the probability amplitude that a quantum state of the system in state $|v >$ will be found in state $|u >$
- If states u and v are orthogonal vectors then $< u|v >= 0$ which mean they are perfectly distinguishable

4.2 Quantum Evolutions

- Any physically admissible evolution of an isolated quantum system is represented by a unitary operator which are typically derived from the Schrodinger equation

4.3 Quantum Circuits

- We assume that there are already certain unitary operators and use this pre-selected elementary quantum operators as quantum logic gates or quantum circuits to help understand how they act on qubits. If you have unitaries, U and V followed by one another then we have $----U--$ $-V--$ where the qubit is is carried through from left to right has the qubit moves through the operators.

4.4 Measurements

- A complete measurement in quantum theory is by the choice of an orthonormal basis in H

$$|e_1, ..., |e_n \in H$$

. States that follow an orthonormal basis can be perfectly distinguishable