# Stabilisers

Sam Sangyoon Kim

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## Introduction

In today's lecture, we'll delve deeper into abstract mathematics to lay the groundwork for our exploration of Pauli stabilisers in quantum computing. Our focus will be on understanding the concepts of normal subgroups and their relevance in the context of Pauli operators.

## 1 Normal Subgroups

**Definition:** A subgroup $H$ of a group $G$ is said to be a *normal subgroup*, denoted as $H \triangleleft G$, if it is invariant under conjugation by all elements of $G$. This means that for any $g \in G$ and $h \in H$, $ghg^{-1} \in H$. Unlike regular subgroups, normal subgroups maintain their structure under conjugation, making them significant in group theory.

### 1.1 Cosets and Lagrange's Theorem

**Definition:** Given a subgroup $H$ of a group $G$, a *coset* of $H$ in $G$ is a subset of $G$ of the form $gH = \{gh \mid h \in H\}$ for some fixed $g \in G$. These cosets partition $G$ into subsets of equal size.

**Lagrange's Theorem:** For a finite group $G$ and a subgroup $H$, the order of $H$ divides the order of $G$. Mathematically, this is expressed as:

$$|G| = |G : H| \cdot |H|$$

where $|G|$ is the order of $G$, $|H|$ is the order of $H$, and $|G : H|$ is the number of cosets of $G$ given by $H$. Lagrange's theorem is fundamental in group theory and has significant implications for subgroup analysis.

## 2 Normalisers

**Definition:** The normaliser of a subgroup $H$ in $G$, denoted as $N_G(H)$, is the largest subgroup of $G$ in which $H$ is a normal subgroup. Formally, it is defined as:

$$N_G(H) = \{g \in G \mid ghg^{-1} \in H \text{ for all } h \in H\}$$

Normalisers play a crucial role in understanding the structure of groups and subgroups, particularly in the context of symmetry and transformation.

# 3 Pauli Normalisers

In this section, we explore the concept of Pauli normalisers and their significance in quantum computing.

## 3.1 Introduction

When considering a stabiliser $\mathcal{S}$ in the Pauli group $\mathcal{P}_n$, two important subgroups emerge: the centraliser and the normaliser.

### 3.1.1 Centraliser of a Stabiliser

The centraliser $Z(\mathcal{S})$ consists of elements in $\mathcal{P}_n$ that commute with every element of $\mathcal{S}$. Formally, it is defined as:

$$Z(\mathcal{S}) = \{g \in \mathcal{P}_n \mid gsg^{-1} = s \text{ for all } s \in \mathcal{S}\}$$

In the context of Pauli groups, the centraliser coincides with the normaliser due to the specific properties of Pauli operators.

### 3.1.2 Normaliser of a Stabiliser

The normaliser $N(\mathcal{S})$ comprises elements in $\mathcal{P}_n$ that map elements of $\mathcal{S}$ back into $\mathcal{S}$ under conjugation. Formally, it is defined as:

$$N(\mathcal{S}) = \{g \in \mathcal{P}_n \mid gsg^{-1} \in \mathcal{S} \text{ for all } s \in \mathcal{S}\}$$

For Pauli stabilisers, the normaliser serves as a central concept for error correction and fault tolerance in quantum systems.

## 3.2 Relation between Centraliser and Normaliser

In Pauli groups, the centraliser and normaliser coincide due to the specific properties of Pauli operators. This is because $gsg^{-1} = s'$ implies that $sg = gs'$, resulting in $s' = s$ or $s' = -s$. Since $s' = -s$ contradicts the properties of stabilisers, we conclude that $s' = s$.

## 3.3 Quotient Groups and Error Correction

Given a stabiliser $\mathcal{S}$, the normaliser $N(\mathcal{S})$ provides insight into error correction mechanisms in quantum systems. By forming quotient groups such as $N(\mathcal{S})/\mathcal{S}$ and $\mathcal{P}_n/N(\mathcal{S})$, we can identify error syndromes and construct logical operators for error correction.

### 3.3.1 Normality of the Normaliser

The normality of the normaliser $N(\mathcal{S})$ in $\mathcal{P}_n$ is established by showing that $gng^{-1} \in N(\mathcal{S})$ for any $g \in \mathcal{P}_n$ and $n \in N(\mathcal{S})$. This property is crucial for error correction schemes based on Pauli stabilisers.

## 3.4 Counting Elements

Counting the elements of $N(\mathcal{S})$ and the quotient groups $N(\mathcal{S})/\mathcal{S}$ and $\mathcal{P}_n/N(\mathcal{S})$ provides valuable insights into the structure of Pauli stabilisers and their role in error correction. Specifically, the cardinality of these groups sheds light on the complexity and effectiveness of error correction strategies in quantum computing.

# Conclusion

Pauli normalisers play a fundamental role in quantum error correction, providing a framework for identifying error syndromes and constructing logical operators. Understanding the relationship between stabilisers, centralisers, and normalisers is essential for developing robust error correction schemes in quantum computing.

# 4 Clifford Walks on Stabiliser States

There are essentially two ways to define stabiliser states of $n$ qubits. We have already seen how we can describe them as simultaneous $+1$ eigenstates of $n$ generators of some stabiliser group $\mathcal{S} \leq \mathcal{P}_n$, but it turns out that we could also define them as the states that are reachable from the $0^{\otimes n}$ state using only the gate, the Hadamard $H$, and the phase gate $S = 1i$. If you start playing around with these three gates, you'll soon notice that you tend to reach certain discrete states, and never anything in between them.

For example, in the single qubit case (so with just the $H$ and $S$ gates), you'll be able to go between 0, 1, $\pm$, and $\pm i$, but never anything like, say, $\sqrt{\frac{1}{3}}0 + \sqrt{\frac{2}{3}}1$. When you have two or more qubits, you might also notice that whenever you create an $n$-qubit superposition that assigns non-zero amplitudes to strings in some set $A \subset \{0,1\}^n$, it's always an equal superposition over $A$ (though possibly with $\pm 1$ or $\pm i$ phases), and $|A|$ is always some power of 2.

For example, you can generate states such as $\frac{1}{\sqrt{2}}(000 + 111)010$ or $\frac{1}{2}(000 + i100 + 011 - i111)010$, but never states such as $\frac{1}{\sqrt{3}}(001 + 010 + 100)010$.

## 4.1 Clifford Group

Circuits composed of only , $H$, and $S = P_{\pi/2}$ are special: they effect unitaries that map stabiliser states to stabiliser states.

The $n$-qubit **Clifford group** $\mathcal{C}_n$ is the group generated by these three unitaries, and it happens to be exactly the normaliser of the $n$-qubit Pauli group inside the group of all $(2^n \times 2^n)$ unitary matrices:

$$\mathcal{C}_n = \{U \in \mathrm{U}(2^n) \mid UPU^\dagger \in \mathcal{P}_n \text{ for all } P \in \mathcal{P}_n\} = N_{\mathrm{U}(2^n)}(\mathcal{P}_n).$$

It's a confusing (but immutable) matter of terminology that **Clifford gates** (i.e. gates made from only unitaries in the Clifford group) are sometimes called **stabiliser gates**, and **Clifford circuits** (i.e. circuits made from only Clifford gates) are sometimes called **stabiliser circuits**, but stabiliser states are *never* called "Clifford states".

So if we have an $n$-qubit stabiliser state, described by $n$ Pauli generators, then any unitary in the Clifford group $\mathcal{C}_n$ will map each of the $n$ Pauli generators to another Pauli generator, and the set of these $n$ new generators will define a new stabiliser state. Indeed, suppose we have some vector space $V$ stabilised by the group $\mathcal{S}$, and we apply some unitary operation $U$. If $\psi$ is an arbitrary element of $V$, then, for any element $S$ of $\mathcal{S}$,

$$\begin{aligned} U\psi &= US\psi \\ &= US(U^\dagger U)\psi \\ &= (USU^\dagger)U\psi \end{aligned}$$

and so the state $U\psi$ is stabilised by $USU^\dagger$, from which we deduce that the vector space

$$UV\{U\psi \mid \psi \in V\}$$

is stabilised by the group

$$U\mathcal{S}U^\dagger\{USU^\dagger \mid S \in \mathcal{S}\}.$$

Furthermore, if $G_1, \ldots, G_r$ generate $\mathcal{S}$, then $UG_1U^\dagger, \ldots, UG_rU^\dagger$ generate $U\mathcal{S}U^\dagger$, so to compute the change in the stabiliser we need only compute how it affects the generators of the stabiliser.

Since the Clifford group is generated by only three elements, we can easily work out how each of these gates acts by conjugation on the Pauli group. For instance, we have previously seen that the Hadamard gate performs the following transformation:

$$\begin{aligned} X &\mapsto HXH = Z \\ Z &\mapsto HZH = X. \end{aligned}$$

Given that $Y = iXZ$, there is no need to specify the action of $H$ on $Y$, since we can calculate that

$$\begin{aligned} Y &\mapsto i(HXH)(HZH) \\ &= iZX \\ &= -Y. \end{aligned}$$

## 5 Conclusion

In conclusion, the study of Clifford walks on stabiliser states provides valuable insights into the behavior and manipulation of quantum systems. The Clifford group, generated by the, Hadamard, and phase gates, plays a fundamental role in stabiliser-based quantum computation. Through the simple rules governing the action of Clifford gates on stabiliser states, we can efficiently simulate stabiliser circuits and compute physical observables.

However, while stabiliser circuits are powerful and efficient for certain tasks, they do not fully capture the computational power of quantum computing. The inclusion of non-Clifford gates, such as the $T$ gate, is necessary for achieving universal quantum computation. Despite this limitation, stabiliser computation remains a central aspect of quantum computing, particularly in the context of quantum error correction and fault-tolerant computation.

In summary, Clifford walks on stabiliser states provide a framework for understanding and implementing quantum algorithms, highlighting the interplay between gate operations, state manipulation, and computational efficiency in quantum information processing.