Quantum Information Theory Seminar – Chapter 4

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1 4.1 Hilbert Spaces

Mathematical setting for a quantum system: **Hilbert space** $\mathcal{H}$ = vector space with an inner product.

The result of any setup of the Hilbert space is then represented by some **unit** vector $|\psi\rangle \in \mathcal{H}$. Test: $|e\rangle \in \mathcal{H}$.
The inner product of these vectors, $\langle e|\psi\rangle$, gives us the probability that the system, prepared in state $|\psi\rangle$, will pass the test for being in state $|e\rangle$. We get this probability by squaring the absolute value of the inner product.

$$|\langle e|\psi\rangle|^2 = \langle \psi|e\rangle \langle e|\psi\rangle .$$

After a test, the system collapses into the state corresponding to the test it passed. This means that if we immediately measure the system again, we'll find it in the same state with probability 1. For instance, let the object forget about its previous state $|\psi\rangle$, then it is now in state $|e\rangle$.
That is, if we immediately measure the object again, we will find it to still be in state $|e\rangle$ with probability 1. This is known as **quantum collapse** which we will come back to.

A more comprehensive test involves multiple states, forming an orthonormal basis $|e_1\rangle, \ldots, |e_n\rangle$ in $\mathcal{H}$, and such states are distinct. The probability that the system, initially in state $|\psi\rangle$, will be found in state $|e_k\rangle$ is $\langle e_k|\psi\rangle$.
The probability amplitude that the system in state $|\psi\rangle$ will be found in state $|e_k\rangle$ is $\langle e_k|\psi\rangle$ and, given that the vectors $|e_k\rangle$ span the whole vector space, the system will be always found in one of the basis states. This basis provides a complete set of ”measuring tools.” Measuring the system in any basis state gives a definite outcome, and the sum of probabilities over all possibilities is always 1:

$$\sum_k |\langle e_k|\psi\rangle|^2 = 1.$$

a **complete measurement** is defined by selecting an orthonormal basis in $\mathcal{H}$ represented by $\{|e_i\rangle\}$. These bases form a possible complete measurement, a notion that we will touch upon next.

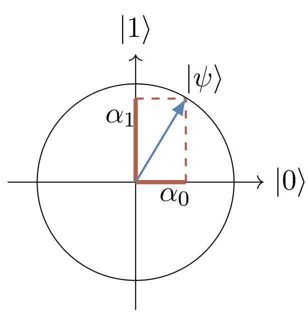
2 4.2 Complete Measurements

A **projector** is a Hermitian operator which is **idempotent** ($P^2 = P$).
Reminder: Hermitian: ($P = P^\dagger$)
The **rank** of P is given by $\mathrm{tr}(P)$.
Using the notation we have been using: If $|e\rangle$ is a unit vector, then $|e\rangle\langle e|$ is a projector where rank $= 1$ on the subspace spanned by $|e\rangle$ acting on any vector $|v\rangle$.
This is done by $(|e\rangle\langle e|)\,|v\rangle = |e\rangle\,\langle e|v\rangle$.
We can also see that in matrix form, we get the following:

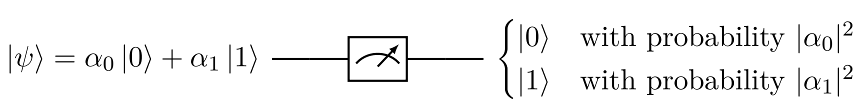
$$|e\rangle\langle e| = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \begin{pmatrix} \bar{e}_1 & \bar{e}_2 \end{pmatrix} = \begin{pmatrix} e_1\bar{e}_1 & e_1\bar{e}_2 \\ \bar{e}_1 e_2 & e_2\bar{e}_2 \end{pmatrix} = \bar{e}_1 \begin{pmatrix} e_1 & 0 \\ e_2 & 0 \end{pmatrix} + \bar{e}_2 \begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix} = |e_1|^2 + |e_2|^2 = 1$$

The **Standard measurement**: $\{|0\rangle, |1\rangle\}$. This is the most common measurement used in Quantum Information theory.
When we draw circuit diagrams, we assume that this measurement is performed on each qubit.

To show us how we might define our basis:



We can also include a measurement explicitly in the diagram as a special quantum gate as follows:



Or, in alternative notation:

$$|\psi\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle \;\;\&\&\; |k\rangle - Dk \text{ with probability } |\alpha_k|^2 \qquad (k = 0, 1).$$

If the qubit is prepared in state $|\psi\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle$ and then measured in the standard basis, then the outcome is $|k\rangle$ (for $k = 0, 1$) with probability

$$\begin{aligned}
|\alpha_k|^2 &= |\,\langle k|\psi\rangle\,|^2 \\
&= \underbrace{\langle\psi|k\rangle}_{\alpha_k^\star}\underbrace{\langle k|\psi\rangle}_{\alpha_k} \\
&= \langle\psi|\;\underbrace{|k\rangle\langle k|}_{\text{projector}}\;|\psi\rangle \\
&= \langle\psi|P_k|\psi\rangle
\end{aligned}$$

where $P_k = |k\rangle\langle k|$ is the projector on $|k\rangle$.

When we make a measurement in quantum systems, the result we obtain is the state $|k\rangle$. This changes the system's state irreversibly from its original state $|\psi\rangle$ to either $|0\rangle$ or $|1\rangle$, which is known as a **collapse** or **reduction** of the state.

This abrupt change may seem distinct from the continuous evolution of quantum states described by unitary transformations. However, measurements follow the same laws of physics.

A measurement is a physical process where a complex system, such as a measuring device or observer, interacts with the system being measured, becoming correlated with it. We'll explore this further later.

Quantum collapse essentially represents the irreversible interaction between a quantum system and its classical environment. It basically means we use projectors instead of unitary operators to describe measurements and observations.

3 4.3 The projection rule and incomplete measurements

As we have already seen in quantum measurements, we often associate measurements with sets of orthonormal bases or projectors. An orthonormal basis satisfies two conditions:

Orthonormality: $\langle e_k|e_l\rangle = \delta_{kl}$ meaning the unit vectors here are perpendicular
Completeness: $\sum_k |e_k\rangle\langle e_k| = \mathbf{1}$. i.e. The sum of projectors onto basis vectors equals the identity operator.

Given a quantum system in state $|\psi\rangle$ such that $|\psi\rangle = \sum_k \alpha_k |e_k\rangle$:

$$\begin{aligned}
|\psi\rangle &= \mathbf{1}\,|\psi\rangle \\
&= \sum_k (|e_k\rangle\langle e_k|)\,|\psi\rangle \\
&= \sum_k |e_k\rangle\,\langle e_k|\psi\rangle \\
&= \sum_k |e_k\rangle\,\alpha_k \\
&= \sum_k \alpha_k\,|e_k\rangle
\end{aligned}$$

This tells us that any vector in $\mathcal{H}$ can also be written as the sum of orthogonal projections on the $|e_k\rangle$.
The measurement in basis $\{|e_i\rangle\}$ gives the outcome labelled by e_k with probability:

$$|\,\langle e_k|\psi\rangle\,|^2 = \langle\psi|e_k\rangle\,\langle e_k|\psi\rangle$$

and leaves the system in state $|e_k\rangle$.
This is a complete measurement that allows us to attempt resolving state vectors in the basis states. However, when we do not want our measurement to distinguish all elements of an orthonormal basis, we can consider the following example which outlines incomplete measurements.

A complete measurement in a four-dimensional Hilbert space has four distinct outcomes: $|e_1\rangle$, $|e_2\rangle$, $|e_3\rangle$, and $|e_4\rangle$. Alternatively, we can combine these if we decide we want to only distinguish between certain ones. For instance, we can combine them as follows: $\{|e_1\rangle, |e_2\rangle\}$, and $\{|e_3\rangle, |e_4\rangle\}$.
Incomplete measurements are a result of this, i.e. measurements which can be less disruptive than complete measurements, but allow us to distinguish subspaces from each other without separating vectors in the same

subspace.

Now, rather than just focusing on projecting onto one-dimensional subspaces defined by vectors from an orthonormal basis, we can break down our Hilbert space into different-sized, mutually orthogonal subspaces and perform projections onto them.

A full system of projectors satisfies two conditions:

Orthogonality: $P_k P_l = P_k \delta_{kl}$ **Completeness**: $\sum_k P_k = \mathbf{1}$

For the decomposition of the identity into these orthogonal projectors P_k (which is ensured by the completeness condition), we have a corresponding measurement. This measurement takes a quantum system initially in state $|\psi\rangle$, yields an outcome labeled k with a probability calculated from $\langle\psi|P_k|\psi\rangle$, and leaves the system in the state $P_k|\psi\rangle$ after normalization. This normalization ensures that the state remains properly scaled.
So, essentially what is happening is:

$$|\psi\rangle \mapsto \frac{P_k|\psi\rangle}{\sqrt{\langle\psi|P_k|\psi\rangle}}.$$

4 4.4 Example of an incomplete measurement

Consider a three-dimensional quantum system with basis vectors $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$. Let's say we have two orthogonal projectors:

$$P = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$$
$$Q = |e_3\rangle\langle e_3|$$

Together, they decompose the identity: $P + Q = \mathbf{1}$.

Suppose we have a physical system in a state $|\psi\rangle = \alpha_1|e_1\rangle + \alpha_2|e_2\rangle + \alpha_3|e_3\rangle$. Ideally, we'd want to measure it completely, distinguishing all three basis states. However, imagine our equipment isn't perfect and can only tell the difference between two subspaces: those linked with projectors P and Q.

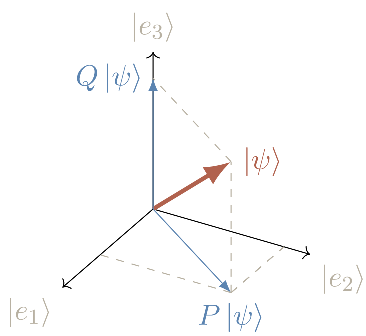
During this incomplete measurement, the apparatus might detect the system in the subspace tied to P. This happens with a probability equal to

$|\alpha_1|^2 + |\alpha_2|^2$. This can be derived as follows:

$$\begin{aligned}\langle\psi|P|\psi\rangle &= \langle\psi|e_1\rangle\langle e_1|\psi\rangle + \langle\psi|e_2\rangle\langle e_2|\psi\rangle \\ &= |\alpha_1|^2 + |\alpha_2|^2,\end{aligned}$$

Immediately after this, the state becomes the normalized vector $P|\psi\rangle$, which looks like $\frac{\alpha_1|e_1\rangle+\alpha_2|e_2\rangle}{\sqrt{|\alpha_1|^2+|\alpha_2|^2}}$.

Alternatively, the measurement might locate the system in the subspace linked with Q, with a probability of $|\alpha_3|^2$. This leads to the post-measurement state $|e_3\rangle$.



5 4.5 Observables

An **observable** A represents a measurable physical property, such as spin, position, momentum, or energy, with a numerical value. It extends to any basic measurement where each outcome has an associated numerical value. If λ_k is the numerical value associated with outcome $|e_k\rangle$, then the observable A is **represented** by the operator

$$A = \sum_k \lambda_k |e_k\rangle\langle e_k| = \sum_k \lambda_k P_k,$$

where λ_k now corresponds to the eigenvalue of the eigenvector $|e_k\rangle$ or the projector P_k.

We've encountered various types of operators:

Type	Property
normal	$AA^\dagger = A^\dagger A$
unitary	$A^\dagger = A^{-1}$
Hermitian (or **self-adjoint**)	$A^\dagger = A$
positive semi-definite	$\langle v\|A\|v\rangle \geq 0$ for all $\|v\rangle$

According to the **spectral theorem**, an operator A is normal **iff** it's **unitarily diagonalizable**. This means that $\exists$ a unitary matrix U and a diagonal matrix D such that $A = U^\dagger D U$. Note that unitary, Hermitian, and positive semi-definite operators are all normal.

Since $(|a\rangle \langle b|)^\dagger = |b\rangle \langle a|$, the projectors $P_k = |e_k\rangle \langle e_k|$ are Hermitian, and so they are normal. As a result, it follows that A is also a normal operator.

On another note, when we have a normal operator A, we can link it to a measurement defined by its eigenvectors, forming a special set of vectors that are both orthogonal and normalized. We use the corresponding eigenvalues to identify the outcomes. If these eigenvalues are real, then the operator A is Hermitian. Let's take an example: imagine measuring a single qubit using what's known as the Z-measurement. It's tied to the Pauli Z operator, which can be broken down into the standard basis as $Z = (+1)\,|0\rangle\langle 0| + (-1)\,|1\rangle\langle 1|$. The outcomes are associated with $+1$ and -1 labels respectively. While these labels can be arbitrary, real number labels are often favored, which is why Hermitian operators are preferred.

Now, let's delve into the concept of the **expected value**, also known as the **mean**. This value represents the average of the numerical values λ_k weighted by their probabilities. It's an important quantity and can be calculated using the operator A and the system's state $|\psi\rangle$. The formula is as follows:

$$\begin{aligned}
\langle A\rangle &= \sum_k \lambda_k P_k \\
&= \sum_k \lambda_k |\langle e_k|\psi\rangle|^2 \\
&= \sum_k \lambda_k \langle \psi|e_k\rangle \langle e_k|\psi\rangle \\
&= \langle\psi| \left(\sum_k \lambda_k |e_k\rangle\langle e_k|\right) |\psi\rangle \\
&= \langle\psi|A|\psi\rangle\,.
\end{aligned}$$

This value is a statistical average derived from many measurements of the observable A on quantum objects that are prepared in the state $|\psi\rangle$.

6 4.6 Compatible observables and the uncertainty relation

Now that we've explained how observables relate to normal operators, let's explore what happens when matrix multiplication doesn't commute ($AB \neq BA$). We'll try to understand when two operators A and B do or don't commute, focusing on eigenvectors for clarity.

Consider an eigenbasis of A, where each vector $|e_k\rangle$ is an eigenvector. If A and B commute ($AB = BA$), any vector $B|e\rangle$ is also an eigenvector of A, meaning A and B share an eigenbasis. Conversely, if A and B share an eigenbasis, they commute ($AB = BA$). We call A and B compatible if they commute, and incompatible otherwise.

When A and B are compatible, measurements of both observables can be made simultaneously in their shared eigenbasis. But if they're incompatible, measuring one observable affects the outcome of the other, illustrating quantum uncertainty. Let us consider the expected values of A and B as $\langle A\rangle$ and $\langle B\rangle$, then we can call their respective standard deviations σ_A and σ_B. The interesting, purely quantum, phenomena, however, comes when A and B are incompatible: we can prove that the standard deviations cannot both be made arbitrarily small, i.e., we can't assume that they are 0. Heisenberg's uncertainty principle quantifies this, stating $\sigma_A\sigma_B \geq \left|\frac{1}{2i}\langle[A,B]\rangle\right|$, where $[A,B] = AB - BA$ is the commutator.

This principle highlights a fundamental aspect of quantum physics, where $\hbar$ (Planck's constant) determines the discreteness of quantum systems. Taking $\hbar \to 0$ recovers classical physics. Conversely, quantization theory aims to derive quantum versions of classical theories.

Incompatible operators also lead to intriguing phenomena. Suppose we have operators A, B, and C. Calculating probabilities of outcomes reveals that $[A,B] = 0$ or $[B,C] = 0$ only if the outcomes are independent, demonstrating compatibility. Otherwise, outcomes are entangled, showcasing the peculiar nature of quantum systems.

This discussion sets the stage for later exploration, including Bell's theorem and the quantum Venn diagram paradox, revealing deeper insights into quantum mechanics.