# Quantum Information Theory - Quantum Channels

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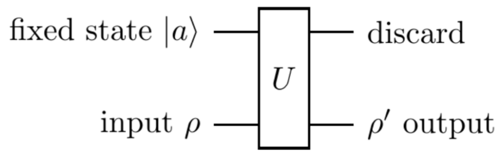
## 1 Introduction

In contrast to classical channels, which simply convey information, quantum channels are influenced by the principles of quantum mechanics, such as superposition and entanglement. These channels don't just carry information; they do so in a way that can be affected by quantum noise and environmental interactions, making the study of their properties and behaviors both fascinating and crucial for the development of quantum technologies. As we explore quantum channels, we must consider the concept of unitary evolution—wherein quantum states evolve in a reversible manner in closed systems—and how this ideal is challenged in open systems where noise is inevitable. The implications of these concepts are profound, not only for our theoretical understanding but also for practical applications in quantum computing and secure communication.

## 2 Everything is Unitary

The Quantum Mantra is: there is only unitary evolution, and if there is any other evolution then it has to be derived from a unitary evolution. All evolutions become unitary when you make your system large enough! But how? The short answer is: by adding (via tensoring) and removing (via partial trace) physical systems. A typical combination of these operations is shown in the following diagram:

The process for a quantum channel is as follows:




1. Prepare the input state $\rho$.

2. Enlarge the system by tensoring $\rho$ with an auxiliary state $\alpha\alpha$, forming the extended state $\rho_{ext} = \rho \otimes \alpha\alpha$.

3. Allow the extended system to undergo closed unitary evolution with a unitary operator $U$, resulting in $U(\rho_{ext})U^\dagger$.

4. Apply a trace operation to remove the auxiliary system, yielding the final output state $\rho'$.

The transformation can be described by the equation:

$$\rho \rightarrow \rho' = \sum_i E_i \rho E_i^\dagger \tag{1}$$

where $\{E_i\}$ are the Kraus operators satisfying the completeness relation $\sum_i E_i^\dagger E_i = I$. This defines a completely positive trace-preserving (CPTP) map, also known as a quantum channel.

The essential properties of quantum channels are trace preservation and positivity:

**Trace preserving:** For the trace-preserving property, the map must satisfy the condition

$$\sum_i E_i^\dagger E_i = 1. \tag{2}$$

Given this, the trace of the output state remains unchanged:

$$tr\left(\sum_k E_k \rho E_k^\dagger\right) = tr\left(\sum_k E_k^\dagger E_k \rho\right) = tr(\rho). \tag{3}$$

**Positivity preserving:** Since $\rho$ is a positive operator, the channel must also preserve this positivity. For any set of operators $\{E_k\}$, the following holds:

$$\sum_k E_k \rho E_k^\dagger = \sum_k (E_k \sqrt{\rho})(\sqrt{\rho} E_k^\dagger). \tag{4}$$

An operator is positive if it can be expressed as $XX^\dagger$ for some $X$. Thus, each term $(E_k\sqrt{\rho})$ contributes to the sum as a positive operator, ensuring the map is positivity preserving.

*Note:* These conditions are necessary for density operators to maintain physical legitimacy through the map. However, they are not sufficient. Quantum channels require not only positive but completely positive maps.

**Why this matters:**

- Quantum Simulations and Cryptography: The unitary evolution principle is crucial for simulating complex quantum systems and securing information via quantum cryptography, leveraging quantum channels to surpass classical limitations in computation and communication security.
- Enhancements in Quantum Sensing: It underpins advancements in quantum sensing, where precision and accuracy are enhanced through controlled quantum channels, leading to breakthroughs in fields ranging from navigation to medical diagnostics.
- Quantum Networks and Entanglement: This concept is foundational for the development of quantum networks, enabling robust quantum communication and exploration of quantum entanglement, with quantum channels acting as vital conduits for entangled state transmission and quantum teleportation.

# 3 Random Unitaries

In the initial exploration of quantum systems, consider a two-qubit setup with a controlled-NOT gate. The unitary operation for this gate can be described as: U $= 0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & X \end{bmatrix}$

where 1 is the identity operator and $X$ is the bit-flip operator.

The transformation of the target qubit is conditional on the state of the control qubit:

- When the control qubit is in state 0, the target qubit experiences the identity operation, evolving as 1
- When the control qubit is in state 1, the target qubit undergoes a bit-flip, evolving as $X$.

- For control qubit states that are superpositions of 0 and 1, the evolution of the target qubit is not represented by a single unitary operator.

It is important to note that the behavior of the target qubit is inherently linked to the state of the control qubit in a controlled-NOT operation.

To justify this last point, note that, if the control qubit is in the state $\alpha_0|0\rangle + \alpha_1|1\rangle$ and the target qubit is in some state $|\psi\rangle$, then the output state can be written as

$$\alpha_0|0\rangle \otimes \mathbf{1}|\psi\rangle + \alpha_1|1\rangle \otimes X|\psi\rangle$$

which shows that the control and the target become entangled. The target qubit alone ends up in the statistical mixture of states $|\psi\rangle$ with probability $|\alpha_0|^2$ and $X|\psi\rangle$ with probability $|\alpha_1|^2$.

We can verify this by expressing the above output state of the two qubits as the density matrix

$$\begin{array}{rcl} |\alpha_0|^2\,|0\rangle\langle 0| \otimes \mathbf{1}|\psi\rangle\langle\psi|\mathbf{1} & + & |\alpha_1|^2\,|1\rangle\langle 1| \otimes X|\psi\rangle\langle\psi|X \\ +\alpha_0\alpha_1^\star|0\rangle\langle 1| \otimes \mathbf{1}|\psi\rangle\langle\psi|X & + & \alpha_0^\star\alpha_1|1\rangle\langle 0| \otimes X|\psi\rangle\langle\psi|\mathbf{1} \end{array}$$

and then tracing over the control qubit, which gives

$$|\alpha_0|^2\,\mathbf{1}|\psi\rangle \left\langle \psi|\mathbf{1} + |\alpha_1|^2\,X \mid \psi \right\rangle \langle\psi|X.$$

Then we can say that the input state of the target qubit evolves either according to the identity operator (with probability $|\alpha_0|^2$ ) or according to the $X$ operator (with probability $\left.|\alpha_1|^2\right)$.

In quantum control systems, it's possible to extend the analysis of conditional dynamics to more than two qubits. Consider a system where each state $|i\rangle$ in a control system's orthonormal basis is associated with a distinct unitary operation $U_i$ acting on a target system. The combined operation for the system can be represented by a block-diagonal matrix:

$$U = \sum_i |i\rangle\langle i| \otimes U_i$$

Given the initial state of the control system as $\sum_i \alpha_i|i\rangle$ and the target system in state $|\psi\rangle$, the final combined state is:

$$\sum_i \alpha_i|i\rangle \otimes U_i|\psi\rangle$$

From this, we can deduce the target system's evolution. The target's final state $\rho'$ evolves according to:

$$\rho' = \sum_i |\alpha_i|^2 U_i \rho U_i^\dagger$$

Here, $U_i$ modifies the state of the target system, selected randomly with a probability $p_i = |\alpha_i|^2$.

One key point of interest is that each unitary $U_i$ is reversible. This reversibility offers a way to undo the action of the channel. If we know which $U_i$ was applied by measuring the control system in the $|i\rangle$ basis, we can reverse the evolution using $U_i^{-1}$.

If access to the control system is unavailable, we face a limitation: we can't discern which unitary was applied by only examining the target system. In this case, we can guess and apply the inverse of the most likely unitary to try and recover the input state, though this method only offers a chance of success. For improved results, one must consider different quantum channels.

First though, a **fundamental example** of a random unitary evolution: A single-qubit Pauli channel applies one of the Pauli operators, $X, Y$ or $Z$, chosen randomly with some prescribed probabilities $p_x, p_y$ and $p_z$, giving

$$\rho \longmapsto p_0 \mathbf{1} \rho \mathbf{1} + p_x X \rho X + p_y Y \rho Y + p_z Z \rho Z.$$

The Pauli operators represent quantum errors: bit-flip $X$, phase-flip $Z$, and the composition of the two $Y = iXZ$.

**Why this matters:**

- Quantum Error Correction: Random unitaries provide the mathematical backbone for understanding and developing quantum error correction techniques. By modeling how quantum systems interact with their environment through stochastic processes, we can design quantum channels that correct errors introduced by noise, a crucial step for realizing fault-tolerant quantum computing.

- Quantum Communication Protocols: The study of random unitaries paves the way for more robust quantum communication protocols. By leveraging the randomness in unitary evolution, quantum channels can be optimized for secure transmission of information over long distances, enhancing the reliability of quantum key distribution (QKD) and other quantum cryptography methods.

- Quantum Computing Algorithms: Understanding random unitaries is essential for the development of novel quantum computing algorithms. By exploiting the probabilistic nature of quantum mechanics, researchers can design algorithms that perform tasks with greater efficiency or solve problems that are currently intractable, thereby expanding the frontier of computational possibilities.

# 4 Random Isometries

Isometries are pivotal in various quantum mechanics applications, including quantum communication and quantum error correction, as they enable encoding the quantum state from a smaller system into a state within a larger system. They can be thought of as a generalization of unitaries. Isometries maintain inner products, thus also conserving the norm and the metric induced by the norm, even when mapping between Hilbert spaces of different dimensions.

For two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ where the dimension of $\mathcal{H}$ is less than or equal to that of $\mathcal{H}'$, an isometry $V$ is a linear transformation from $\mathcal{H}$ to $\mathcal{H}'$ satisfying $V^\dagger V = I_\mathcal{H}$, where $V^\dagger$ is the adjoint of $V$, and $I_\mathcal{H}$ is the identity operator on $\mathcal{H}$.

A full isometry $V$ from $\mathcal{H}$ to $\mathcal{H}'$ effectively maps the entire space $\mathcal{H}$ onto a subspace of $\mathcal{H}'$. The representation of an isometry in matrix form is a rectangular matrix obtained by choosing a subset of columns from a unitary matrix. For instance, starting with a unitary matrix $U$, an isometry $V$ can be constructed by selecting appropriate columns from $U$.

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{bmatrix} \longmapsto V = \begin{bmatrix} U_{12} & U_{14} \\ U_{22} & U_{24} \\ U_{32} & U_{34} \\ U_{42} & U_{44} \end{bmatrix}$$

Isometries are key in preserving the inner product structure of quantum states, defined by the condition $V^\dagger V = I\mathcal{H}$ for a Hilbert space $\mathcal{H}$. This condition is sufficient for isometries and does not necessitate that $VV^\dagger = I\mathcal{H}$, which would imply that $V$ is unitary. The operator $VV^\dagger$ is in fact a projection onto the image of $\mathcal{H}$ under $V$.

Expressed in Dirac notation, an isometry $V$ takes the form:

$$V = \sum_i |b_i\rangle \langle a_i|$$

where $\{|a_i\rangle\}$ is an orthonormal basis for $\mathcal{H}$ and $\{|b_i\rangle\}$ are orthonormal vectors in $\mathcal{H}'$, which do not necessarily span $\mathcal{H}'$. When $V$ is unitary, $\{|b_i\rangle\}$ spans $\mathcal{H}'$. It is evident from this representation that $V^\dagger V$ is the identity on $\mathcal{H}$ and $VV^\dagger$ is a projection on the subspace of $\mathcal{H}'$ spanned by $\{|b_i\rangle\}$.

Despite isometries being more general than unitaries, they are physically realizable operations. They can be implemented by tensoring two systems together and applying unitary operations to the enlarged system. For instance, combining a system $A$ in state $|\psi\rangle$ with another system $B$ in a fixed state $|b\rangle$, and then applying a unitary $U$ to $A \otimes B$, creates an isometry from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}_B$ defined by the transformation:

$$V : |\psi\rangle \mapsto U(|\psi\rangle|b\rangle)$$

Isometries also act as quantum channels, transforming any quantum state $|\psi\rangle$ or density operator $\rho$ according to:

$$|\psi\rangle \mapsto V|\psi\rangle$$

$\rho \mapsto V\rho V^\dagger$ with the normalization condition for isometries being $V^\dagger V = 1$.

The significance of isometries is particularly pronounced in quantum error correction, as they allow for precise manipulations of quantum information without altering the underlying quantum state's integrity.

**Why this matters:**

- Quantum State Transfer and Communication: Random isometries play a crucial role in the encoding and transmission of quantum states across quantum channels. This process is foundational for quantum communication, enabling the transfer of quantum information with minimal loss, which is essential for the development of quantum internet.

- Quantum Computing Scalability: The concept of random isometries is key to scaling quantum computers. By embedding quantum states from smaller to larger Hilbert spaces without losing information, random isometries facilitate the design of scalable quantum architectures, ensuring that quantum computers can handle increasingly complex calculations as they evolve.

- Enhancing Quantum Error Correction: Random isometries contribute to the advancement of quantum error correction schemes by allowing for the embedding of logical qubits into higher-dimensional spaces. This capability is critical for protecting quantum information against errors and decoherence, thus maintaining the integrity of quantum computations and communications.

# 5 Evolution of Open Systems

In the broader context of quantum systems, not all interactions can be neatly described as control-target relationships. Instead, let's consider two interacting quantum systems, A and B, without this assumption. System A will act as an auxiliary system or ancilla, and we'll concentrate on the evolution of system B.

An orthonormal basis $|i\rangle$ for the ancilla's Hilbert space $\mathcal{H}_A$ can be chosen. With this, any unitary operation on the combined system $AB$ can be described by:

$$U = \sum_{i,j} |i\rangle\langle j| \otimes B_{ij}$$

Here, the $B_{ij}$ terms are operators acting on $\mathcal{H}_B$, the Hilbert space of system B. These operators need not be unitary, but to ensure $U$ is unitary, they must satisfy:

$$\sum_i B_{ki}^\dagger B_{il} = \delta_{kl} I_{AB} \sum_i B_{ik} B_{ll}^\dagger = \delta_{kl} I_B$$

where $I_{AB}$ and $I_B$ are the identity operators on $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{H}_B$, respectively. These conditions ensure that the matrix columns and rows formed by the operators are orthonormal, analogous to the elements in a unitary matrix.

The evolution of system B not only depends on the unitary operation $U$ but also on the initial state of the ancilla $A$. Assuming the ancilla is in a pure state, which can be represented by a basis state $|k\rangle$, the unitary transformation of system B for any state $|\psi\rangle$ is:

$$U : |k\rangle \otimes |\psi\rangle \to \sum_i |i\rangle \otimes B_{ik}|\psi\rangle$$

To find the resulting density operator for system B, we consider the density operator of the joint system $AB$ after transformation and then trace out the ancilla:

$$tr_A \left( \sum_{i,j} |i\rangle \, \langle j \, | \otimes B_{ik} | \, \psi\rangle \, \langle\psi| B_{jk}^\dagger \right) = \sum_i B_{ik}|\psi\rangle\langle\psi|B_{ik}^\dagger$$

This operation gives the final state of system B after considering the effects of the unitary operation and the initial state of the ancilla.

Generalizing quantum state evolution, for any input density operator $\rho$, subsystem B undergoes the transformation:

$$\rho \to \rho' = \sum_i B_{ik} \rho B_{ik}^\dagger$$

We've omitted the index $k$ in the last term as it was initially included to track the ancilla's state. Since $U$ is a unitary operator, the operators $B_i$ fulfill the normalization condition:

$$\sum_i B_i^\dagger B_i = I$$

This ensures that the trace, or the total probability, remains constant.

To conceptualize this process, consider a sequence of three operations: 1. Attach an ancilla in a fixed state to the system ($\rho \to |k\rangle\langle k| \otimes \rho$). 2. Evolve the combined system and ancilla unitarily $\left(U(|k\rangle \otimes \rho)U^\dagger\right)$. 3. Discard the ancilla, leaving the subsystem in state $\rho'$.

Describing the process in steps: - Begin with a quantum system of interest, typically in a mixed state $\rho$, possibly entangled with other degrees of freedom which remain passive. - Introduce an ancilla to encompass all potential interactions, ensuring it's large enough for the combined system to be considered isolated and then apply a unitary evolution $U$. - Post-evolution, the ancilla is discarded, focusing on the subsystem alone. Notably, discarding can be partial, affecting only a portion of the ancilla or any other part of the expanded system.

It's the addition and subsequent removal of the ancilla that enables the non-unitary appearance of the system's evolution.

Moving forward, we apply our understanding of isometries, which are like unitaries but can change dimensionality, to combine the first two operations (adding an ancilla and applying unitary evolution) into one. This leads to Stinespring dilation theorem and its ancilla-free counterpart, the Kraus decomnosition

**Why this matters:**

- Bridging Theory and Real-world Quantum Systems: The study of open systems' evolution provides a critical link between ideal quantum mechanics and the realities of quantum systems interacting with their environments. This understanding is essential for designing quantum channels that can operate effectively in real-world conditions, where noise and decoherence are unavoidable.

- Foundation for Quantum Thermodynamics: Exploring the dynamics of open quantum systems lays the groundwork for quantum thermodynamics, offering insights into energy transfer, entropy, and the fundamental limits of quantum computing and sensing technologies. It shapes the design of quantum channels that can harness these processes for energy-efficient quantum computations and thermal management.
- Enabling Quantum Control and Feedback: The evolution of open systems underpins the development of quantum control techniques and quantum feedback mechanisms. By understanding how quantum systems interact with their surroundings, engineers can create quantum channels that dynamically adjust to external disturbances, enhancing the stability and performance of quantum computers, sensors, and communication networks.

# 6 Stinespring's Dilation and Kraus's Ambiguity

In the realm of quantum mechanics, as we begin to consider more complex systems with higherdimensional Hilbert spaces, it becomes practical to transition our focus from unitary transformations to isometries. This shift is primarily for the sake of mathematical convenience, and while it may not add to our physical understanding, simplifying our equations is generally beneficial.

Remember that any unitary transformation $U$ for a combined system $AB$ can be expressed as a sum involving an orthonormal basis $|i\rangle$ of system $A$ and operators $B_{ij}$ acting on the Hilbert space of system $B$ :

$$U=\sum_{i,j}|i\rangle\langle j|\otimes B_{ij}$$

These operators $B_{ij}$ are not required to be unitary. However, to ensure that $U$ is unitary, they must satisfy the following conditions:

$$\sum_i B_{ki}^{\dagger}B_{il}=\delta_{kl}I_{AB}\sum_i B_{ik}B_{ll}^{\dagger}=\delta_{kl}I_B$$

Additionally, if we consider the initial state of system $A$ to be $|k\rangle$, the action of $U$ can be represented by:

$$U : |k\rangle \otimes |\psi\rangle \to \sum_i |i\rangle \otimes B_{ik}|\psi\rangle$$

for an arbitrary state $|\psi\rangle$ of system $B$.

From here, we define an isometry $V$ that maps from $\mathcal{H}_B$ to $\mathcal{H}_A \otimes \mathcal{H}_B$ as follows:

$$V : |\psi\rangle \mapsto \sum_i |i\rangle \otimes E_i|\psi\rangle$$

where $E_i$ corresponds to $B_{ik}$ and satisfies the condition:

$$\sum_i E_l^\dagger E_i = I_B$$

The matrix representation of an isometry is a rectangular matrix given by selecting only a few of the columns from a unitary matrix; here, with $|k\rangle$ fixed, it is only the $k$-th column of the block matrix $U$ that determines the evolution of $\mathcal{B}$, as shown in Figure 9.1.

$$U = \begin{bmatrix} B_{11} & B_{12} & B_{13} & \cdots \\ B_{21} & B_{22} & B_{23} & \cdots \\ B_{31} & B_{32} & B_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \longmapsto V = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \end{bmatrix}$$

Figure 9.1: For $k = 2$, the second block column is selected. The matrix representation of the isometry $V$ on the right-hand side look like a column vector, but remember that the entries $E_i := B_{ik}$ are matrices.

We can streamline our understanding of the evolution of a quantum system B by utilizing isometries. An isometry V, depicted in a figure not shown here, acts by transforming a state $|\psi\rangle$ according to:

$$V|\psi\rangle V^\dagger = \sum_{i,j} |i\rangle \otimes E_i|\psi\rangle\langle j|E_j^\dagger$$

By tracing out the ancilla, we capture the evolution of system B, transforming an input state $\rho$ into an output state $\rho'$, described by:

$$\rho \to \rho' = trAV\rho V^\dagger = \sum_i E_i\rho E_i^\dagger$$

Here, the Kraus operators $E_i$ meet the normalization condition $\sum_i E_i^\dagger E_i = 1$, allowing for two interpretations of quantum evolutions: the Stinespring dilation and the Kraus representation.

In the Stinespring dilation, a quantum channel $\mathcal{E}$ arises from unitary evolution on a dilated system, encapsulating both the adding of an ancilla and its removal after evolution. This is synonymous with the 'Church of the Larger Hilbert Space' in quantum information science.

The Kraus representation, or operator-sum decomposition, offers a more direct approach by dealing with operators that map directly from the input to the output Hilbert space, without invoking an ancilla. This yields:

$$\rho \to \rho' = \sum_i E_i \rho E_i^\dagger$$

where the $E_i$ 's are the Kraus operators satisfying the completeness relation.

These two methods are equivalent, offering two perspectives of the same quantum process. Transitioning from a unitary evolution $U$ to an isometry $V$, we 'select' a set of Kraus operators $E_i$. Conversely, starting with a Kraus representation, we can 'build up' an isometry and then extend it to a full unitary transformation $U$, although the latter step involves some arbitrary choices as long as the end result is unitary.

It's important to note that the set of Kraus operators is not unique-different sets related by a unitary transformation describe the same physical process. This is due to the fact that the choice of basis in the ancilla's Hilbert space influences the form of the Kraus operators. To illustrate, let $V = \sum_i |e_i\rangle \otimes E_i$ and let $|e_i\rangle$ and $|f_j\rangle$ be two orthonormal bases. Then:

$$V = \sum_j |f_j\rangle \otimes F_j$$

with $F_j = \sum_i R_{ji} E_i$, where $R$ is the unitary matrix relating the two bases. This shows the unitary equivalence of different Kraus operator sets. Channels described by different Kraus operators are identical if the operators are related by a unitary matrix $R$.

In summary, two sets of Kraus operators describe the same quantum channel if they can be related by a unitary transformation. For instance, an identity channel can only have Kraus operators that are proportional to the identity operator.

**Why this matters:**

- Versatility in Quantum Channel Representation: This section illuminates how the Stinespring dilation and Kraus representations provide flexible and powerful frameworks for describing quantum channels. These mathematical tools allow for a deeper understanding of quantum processes, offering multiple perspectives for modeling quantum noise and interactions, which is fundamental for quantum algorithm optimization and error correction.

- Clarifying Quantum Operations: The exploration of ambiguity in Kraus representations highlights the nuanced nature of quantum operations, emphasizing that different sets of Kraus operators can describe the same quantum channel. This understanding is crucial for quantum information theory, revealing the complexity of quantum state transformations and guiding the development of more efficient quantum communication protocols.

- Implications for Quantum Computing and Simulation: Understanding the interplay between Stinespring dilation and Kraus operators aids in the simulation of quantum systems and the design of quantum computing architectures. By effectively capturing the effects of environmental interactions on quantum states, these concepts enable more accurate simulations and potentially more powerful quantum computational models, pushing the boundaries of what's computationally feasible.

# 7 Single-Qubit Channels

Now let's delve into single-qubit channels, which allow us to visualize their effects as changes to the Bloch sphere.

Any single qubit's state can be represented by a density matrix:

$$\rho = \frac{1}{2}(1 + \vec{s} \cdot \vec{\sigma}) = \frac{1}{2}\left(1 + s_x X + s_y Y + s_z Z\right)$$

Here, $\vec{s}$ is the Bloch vector, and $X, Y, Z$ are the Pauli matrices. While unitary operations rotate the Bloch sphere, general quantum channels can deform the sphere into spheroids or displace it.

Examples include: 1. Bit-flip Channel: It flips the qubit's state with a probability $p$. The transformation is:

$$\rho \to (1 - p)\rho + pX\rho X$$

The corresponding Kraus operators are $\sqrt{1-p}I$ and $\sqrt{p}X$, leading to a prolate spheroid aligned with the x-axis. At $p = \frac{1}{2}$, the Bloch sphere collapses to a line segment along the x-axis. Phase-flip Channel: It flips the phase of the qubit's state with a probability $p$. The transformation is:

$$\rho \to (1-p)\rho + pZ\rho Z$$

The Kraus operators for this channel are $\sqrt{1-p}I$ and $\sqrt{p}Z$, resulting in a prolate spheroid along the z-axis. At $p = \frac{1}{2}$, it also collapses to a line segment, but along the z-axis. Depolarizing Channel: It maintains the qubit's state with probability $1-p$ and applies any of the three Pauli operations with equal probability, leading to:

$$\rho \to (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

This channel uniformly contracts the Bloch sphere when $p < \frac{3}{4}$, and inverts the Bloch vector for $p > \frac{3}{4}$. At $p = \frac{3}{4}$, the sphere shrinks to a point at the center, and for $p = 1$, it flips to the opposite direction, increasing the magnitude up to $\frac{1}{3}$.

In quantum mechanics, the way we describe the action of quantum channels with Kraus operators can offer different narratives for the same physical process. For instance, consider the phase-flip channel with a probability $p = \frac{1}{2}$. We can represent the channel with two different sets of Kraus operators:

$$\left\{E_1 = \frac{1}{\sqrt{2}}I, E_2 = \frac{1}{\sqrt{2}}Z\right\} \quad or \quad \left\{F_1 = \frac{1}{\sqrt{2}}\left(E_1 + E_2\right), F_2 = \frac{1}{\sqrt{2}}\left(E_1 - E_2\right)\right\}$$

These translate to:

$$\{F_1 = |0\rangle \langle 0|, F_2 = | 1\rangle \langle 1|\}$$

Both sets of operators describe the same phase-flip channel, but the story they tell is different. The first set suggests a random, probabilistic choice between letting the qubit be or applying a phase flip. The second set suggests the channel is akin to a measurement in the standard basis without revealing the outcome.

This exemplifies that describing quantum channels solely in terms of Kraus operators can sometimes lead to ambiguity.

Another point is the limitation on how quantum channels transform the Bloch sphere. Not all conceivable shape deformations are possible; for example, you cannot flatten the Bloch sphere into an oblate spheroid, or 'pancake' shape. Such restrictions are due to the requirement of complete positivity in quantum channels, a stronger condition than mere positivity. This concept ensures that quantum evolutions remain physically viable when applied to part of a larger, entangled system.

**Why this matters:**

- Core of Quantum Error Processes: This section demystifies how single-qubit channels model basic quantum errors (bit-flip, phase-flip, and depolarizing), foundational for understanding quantum error correction. By grasping these error mechanisms, researchers and engineers can design quantum channels that mitigate such errors, enhancing the reliability and performance of quantum computing systems.

- Visualizing Quantum State Transformations: The discussion on single-qubit channels provides a vivid picture of how quantum states evolve, using the Bloch sphere representation. This visualization is crucial for intuitively understanding the effects of quantum operations and for designing quantum algorithms and protocols that are robust against noise and decoherence.

- Basis for Advanced Quantum Technologies: Single-qubit channels serve as building blocks for more complex multi-qubit systems and channels, laying the groundwork for advancements in quantum computing, secure quantum communication, and quantum sensing. Understanding these fundamental channels is essential for pushing the envelope in quantum technology development, aiming for practical quantum applications in cybersecurity, drug discovery, and beyond.computationally feasible.

# 8 Properties

The different characterizations we have obtained for quantum channels allow to derive some basic properties.

Proposition 6.8. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{L}_1, \mathcal{L}_2$ be two quantum channels on $\mathcal{L}_1(\mathcal{H})$ with Krauss decompositions

$$\mathcal{L}_1(\mathrm{~T}) = \sum_{i \in N} \mathrm{A}_i \mathrm{T} \mathrm{A}_i^* \quad and \quad \mathcal{L}_2(\rho) = \sum_{i \in N} \mathrm{B}_i \mathrm{~T~} \mathrm{B}_i^*,$$

respectively. 1) The composition $\mathcal{L}_2 \circ \mathcal{L}_1$ is a quantum channel on $\mathcal{L}_1(\mathcal{H})$. It admits a Krauss decomposition given by

$$\mathcal{L}_2 \circ \mathcal{L}_1(\mathrm{~T}) = \sum_{i,j \in N} \mathrm{B}_j \ \mathrm{A}_i \ \mathrm{T} \ \mathrm{A}_i^* \ \mathrm{B}_j^*.$$

2) Any convex combination $\lambda\mathcal{L}_1 + (1-\lambda)\mathcal{L}_2$ (with $0 \leq \lambda \leq 1$) is a quantum channel on $\mathcal{L}_1(\mathcal{H})$ with Krauss decomposition

$$(\lambda\mathcal{L}_1 + \ (1-\lambda)\mathcal{L}_2)\,(\mathrm{T}) = \quad = \sum_{i\in N} \left(\sqrt{\lambda}\mathrm{A}_i\right) \mathrm{T} \left(\sqrt{\lambda}\mathrm{A}_i\right)^* + \sum_{i\in N} \left(\sqrt{1-\lambda}\mathrm{B}_i\right) \mathrm{T} \left(\sqrt{1-\lambda}\mathrm{B}_i\right)^*.$$

Proof. 1) By definition, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are quantum channels on $\mathcal{L}_1(\mathcal{H})$, this means that for $i = 1, 2$ there exists Hilbert spaces $\mathcal{K}_i$, states $\omega_i$ on $\mathcal{K}_i$ and unitary operators $\mathrm{U}_i$ on $\mathcal{K}_i$ such that

$$\mathcal{L}_i(\mathrm{~T}) = Tr_{\mathcal{K}_i}\left(\mathrm{U}_i\left(\rho \otimes \omega_i\right)\mathrm{U}_i^*\right),$$

for all $\mathrm{T} \in \mathcal{L}_1(\mathcal{H})$. We shall now consider the Hilbert space $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$, the quantum state $\omega = \omega_1 \otimes \omega_2$. We consider the natural ampliations $\widehat{\mathrm{U}}_i$ of $\mathrm{U}_i$ to $\mathcal{H} \otimes \mathcal{K}$ by tensorizing $\mathrm{U}_i$ with the identity operator on the space $\mathcal{K}_j$ where it is not initially defined. Finally, put $\mathrm{U} = \widehat{\mathrm{U}}_2\widehat{\mathrm{U}}_1$, it is obviously a unitary operator on $\mathcal{H} \otimes \mathcal{K}$. Now we have, using basic properties of the partial traces

$$\mathrm{Tr}_{\mathcal{K}}\left(\mathrm{U}(\mathrm{T} \otimes \omega)\mathrm{U}^*\right) = Tr_{\mathcal{K}_1 \otimes \mathcal{K}_2}\left(\widehat{\mathrm{U}}_2\widehat{\mathrm{U}}_1(\mathrm{T} \otimes \omega_1 \otimes \omega_2)\widehat{\mathrm{U}}_1^*\widehat{\mathrm{U}}_2^*\right)$$
$$= Tr_{\mathcal{K}_2}\left(Tr_{\mathcal{K}_1}\left(\widehat{\mathrm{U}}_2(\mathrm{U}_1(\mathrm{T} \otimes \omega_1)\mathrm{U}_1^* \otimes \omega_2)\widehat{\mathrm{U}}_2^*\right)\right)$$
$$= Tr_{\mathcal{K}_2}\left(\mathrm{U}_2\left(Tr_{\mathcal{K}_1}\left(\mathrm{U}_1(\mathrm{T} \otimes \omega_1)\mathrm{U}_1^*\right) \otimes \omega_2\right)\mathrm{U}_2^*\right)$$
$$= Tr_{\mathcal{K}_2}\left(\mathrm{U}_2(\mathcal{L}_1(\mathrm{T}) \otimes \omega_2)\mathrm{U}_2^*\right)$$
$$= \mathcal{L}_2\left(\mathcal{L}_1(\mathrm{T})\right).$$

We have obtained $\mathcal{L}_2 \circ \mathcal{L}_1$ as the partial trace $\mathrm{T} \mapsto Tr_{\mathcal{K}}\left(\mathrm{U}(\mathrm{T} \otimes \omega)\mathrm{U}^*\right)$. By definition, it is a quantum channel.

The unitary dilations of $\mathcal{L}_1$ and $\mathcal{L}_2$ can be chosen in such a way that $\omega_1$ and $\omega_2$ are pure states $|\psi_i\rangle\langle\psi_i|$ (Theorem 6.7). We are given orthonormal bases $(e_i)_{i\in N}$ and $(f_j)_{j\in N}$ of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. In the proof of Theorem 6.5 it is shown that the coefficients of a Krauss representation of $\mathcal{L}_1$ can be obtained as

$$\mathrm{A}_i = {}_{\mathcal{K}_1}\langle e_i \,|\mathrm{U}_1|\, \psi_1\rangle_{\mathcal{K}_1}$$

and, in the same way, those of $\mathcal{L}_2$ are obtained as

$$\mathrm{B}_i = {}_{\mathcal{K}_2}\langle f_j \,|\mathrm{U}_2|\, \psi_2\rangle_{\mathcal{K}_2}.$$

Now if we compute those of $\mathcal{L}_2 \circ \mathcal{L}_1$ we get $\mathrm{M}_{ij} = {}_{\mathcal{K}_1\otimes\mathcal{K}_2}\langle e_i \otimes f_j | \mathrm{U} | \psi_1 \otimes \psi_2 \rangle_{\mathcal{K}_1\otimes\mathcal{K}_2}$
$= {}_{\mathcal{K}_2}\left\langle f_j|_{\mathcal{K}_1} \left\langle e_i \left| \widehat{U}_2 \widehat{U}_1 \right| \psi_1 \right\rangle_{\mathcal{K}_1} | \psi_2 \right\rangle_{\mathcal{K}_2}$
$= {}_{\mathcal{K}_2}\left\langle f_j \left| \mathrm{U}_2 \left\langle e_{\mathcal{K}_1} \left| \widehat{U}_1 \right| \psi_1 \right\rangle_{\mathcal{K}_1} \right| \psi_2 \right\rangle_{\mathcal{K}_2}$
$= {}_{\mathcal{K}_2}\langle f_j | \mathrm{U}_2 | \psi_2 \rangle_{\mathcal{K}_2 \mathcal{K}_1} \langle e_i | \mathrm{U}_1 | \psi_1 \rangle_{\mathcal{K}_1}$
$= \mathrm{B}_j \ \mathrm{A}_i$.

This gives the announced Krauss representation, together with a proof of the strong convergence of $\sum_{i,j\in N} \mathrm{A}_i^* \ \mathrm{B}_j^* \ \mathrm{B}_j \ \mathrm{A}_i$. We have proved 1).

The property 2) is very easy to prove, using directly their Krauss decompositions.

# 9 Examples of Quantum Channels

Here are some concrete physical examples of Quantum Channels

**Spontaneous Emission** Here is the amplitude-damping channel or spontaneous emission. Here the environment is 2 dimensional and the unitary evolution is given by

$$U(|0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle U(|1\rangle \otimes |0\rangle) = \sqrt{1-p}|1\rangle \otimes |0\rangle + \sqrt{p}|0\rangle \otimes |1\rangle.$$

In other words, if the small system is in the ground state $|0\rangle$ then nothing happens, if it is in the excited state $|1\rangle$ then it may emit this energy into the environment with probability $p$. This is the simplest model of spontaneous emission of an excited particle: the excited particle goes down to the ground state, emiting a photon into the environment.

In this model there are only two Krauss operators for the associated completely positive map:

$$\mathrm{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad \mathrm{M}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

Successive applications of the associated completely positive map $\mathcal{L}$ make any initial state $\rho_0$ converge exponentially fast to the ground state

$$\rho_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|.$$

That is, as we explained above, the system $\mathcal{H}_A$ ends up emitting all its energy into the environment and hence converges to the ground state.

**The Depolarizing Channel**

The quantum channel that we describe here is part of the so-called noisy channels in Quantum Information Theory. The noisy channels describe what occurs to a qubit which is transmitted to someone else and which is affected by the fact that the transmission is not perfect: the communication channel has to undergo some perturbations (some noise) coming from the environment. Hence, the noisy channel tries to describe the typical defects that the quantum bit may undergo during its transmission.

The noisy channel that we shall describe is the depolarizing channel. It describes the fact that the qubit may be left unchanged with probability $q = 1 - p \in [0, 1]$, or may undergo, with probability $p/3$, one the three following transformations: - bit flip: - phase flip: - both:

This channel can be represented through a unitary evolution $U$ staking a four dimensional environment $\mathcal{H}_E$ with orthonormal basis denoted by $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. More precisely, the small system is $\mathcal{H}_A = C^2$, which we identify to the subspace $\mathcal{H}_A \otimes |0\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_E$. The operator U acts on $\mathcal{H}_A \otimes \mathcal{H}_E$ by

$$\mathrm{U}(|\psi\rangle \otimes |0\rangle) = \sqrt{1-p}|\psi\rangle \otimes |0\rangle + \sqrt{\frac{p}{3}}\left[\sigma_x|\psi\rangle \otimes |1\rangle + \sigma_y|\psi\rangle \otimes |2\rangle + + \sigma_z|\psi\rangle \otimes |3\rangle\right]$$

and U is completed in any way as a unitary operator on $\mathcal{H}_A \otimes \mathcal{H}_E$. The effect of the transform U when seen only from the small system $\mathcal{H}_A$ is then

$$\mathcal{L}(\rho) = Tr_{\mathcal{K}}\left(\mathrm{U}(\rho \otimes |0\rangle\langle 0|)\mathrm{U}^*\right).$$

An easy computation gives the Krauss representation

$$\mathcal{L}(\rho) = \sum_{i=0}^{3} \mathrm{M}_i \rho \mathrm{M}_i^*$$

with

$$\mathrm{M}_0 = \sqrt{1-p}\mathrm{I}, \mathrm{M}_1 = \sqrt{\frac{p}{3}}\sigma_x, \mathrm{M}_2 = \sqrt{\frac{p}{3}}\sigma_y, \mathrm{M}_3 = \sqrt{\frac{p}{3}}\sigma_z.$$

Another interesting way to see this map acting on density matrices is to see it acting on the Bloch sphere. Recall that any qubit state can be represented as a point $(x, y, z)$ in the 3 dimensional ball:

$$\rho = \frac{1}{2}\left(\mathrm{I} + x\sigma_x + y\sigma_y + z\sigma_z\right)$$

with $x^2 + y^2 + z^2 \leq 1$. An easy computation shows that

$$\mathcal{L}(\rho) = \frac{1}{2}\left(\mathrm{I} + \left(1 - \frac{4p}{3}\right)(x\sigma_x + y\sigma_y + z\sigma_z)\right).$$

As a mapping of the ball, the depolarizing channel acts simply as an homothetic transformation with rate $1 - 4p/3$.