NATHAN RAGHAVAN

CHAPTER 2 - QUBITS

[Considering quantum bits and quantum circuits]

- Classical information theory → strings of 1s and 0s
  - ↳ in this case, when considering an individual bit, it doesn't provide a huge amount of information and is not the most interesting

- Quantum world → quantum bits ("qubits")
  - ↳ These provide a whole range of different cases and mathematical scenarios
  - ↳ e.g. Single-qubit interference is the fundamental building block for quantum computing

Now Let Us Investigate and Understand This In More Detail...

① COMPOSING QUANTUM OPERATIONS

- To understand this in detail, let us consider the most simple case

i.e. Quantum interference in the simplest possible computing machine

↳ Here, there are two computational states : $|0\rangle$ and $|1\rangle$

↳ The machine begins in state $|0\rangle$ before it "evolves" with a number of computational steps which transitions the machine between its states

↳ The output state, denoted as $|\psi\rangle$ can be seen as such :

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

i.e. two output states are reached according to the probability coefficients of α_0 and α_1

- In the more general case, each computational step can be called U and sends state $|k\rangle$ to state $|l\rangle$ where $k, l = 0, 1$ with some amplitude U_{lk} :

$$|k\rangle \rightarrow \sum_l U_{lk} |l\rangle$$

i.e. The state k evolves into state l with probability amplitude U_{lk} and probability $|U_{lk}|^2$
The whole event is the superposition of all these, hence the sum

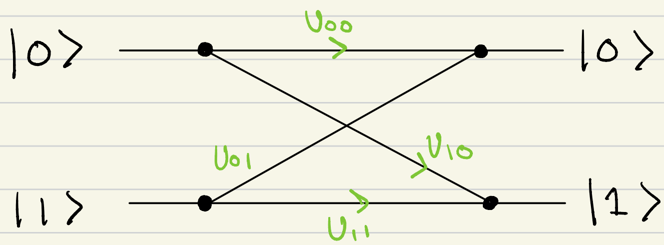
- Considering this, any computational step U can be seen by matrix:

$$U_{KL} = \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & O_{11} \end{bmatrix}$$

considers all transition amplitudes

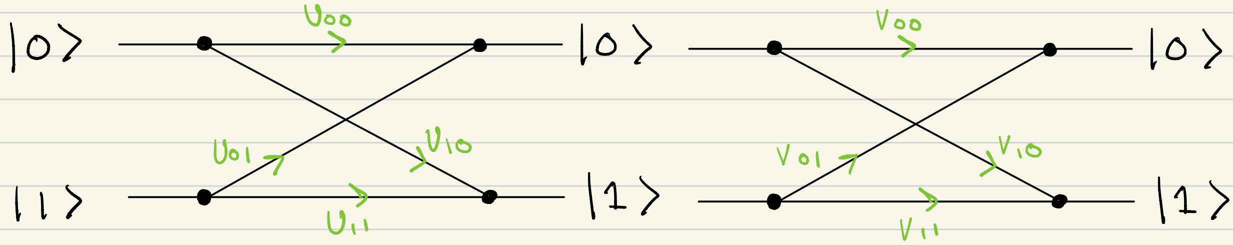
↳ Here, U_{lK} represents amplitude of transition from the respective states $|K\rangle$ and $|L\rangle$
↳ Each entry is unitary → since their moduli squared are probabilities
↳ $\boxed{U^{\dagger}U = UU^{\dagger} = I \text{, where } U^{\dagger} \text{ is adjoint matrix}}$

- This can also be represented by a diagram:



- **Now**: Let us consider quantum interference with two steps, U and V

What amplitude that input $|K\rangle$ will generate $|m\rangle$? We must consider every case, as on the diagram...



- This shows all the possible output examples with each combined computational path

e.g. $|0\rangle$ and $|1\rangle$: by route $|0\rangle \to |0\rangle \to |1\rangle$ $[V_{10} U_{00}]$
: by route $|0\rangle \to |1\rangle \to |1\rangle$ $[V_{11} U_{10}]$

∴ Total amplitude $= V_{10}U_{00} + V_{11}U_{10}$

where the interference term is $|V_{10}U_{00} + V_{11}U_{10}|^2$

- More generally therefor :

$$|k\rangle \to \sum_l U_{lk} |l\rangle \qquad |l\rangle \to \sum_m U_{ml} |m\rangle$$

- And composing the two, we apply U and then V :

$$\left.\begin{aligned} |k\rangle &\to \sum_l U_{lk} \left(\sum_m V_{ml} |m\rangle \right) \\ &= \sum_m \left(\sum_l V_{ml} U_{lk} \right) |m\rangle \\ &= \sum_m (VU)_{mk} |m\rangle \end{aligned}\right\}$$

in another way, we can multiply the matricies, which takes care of the multiplication and addition of amplitudes corresponding to each path

② QUANTUM BITS (QUBITS)

- A two state machine as described can also be called a qubit

  e.g. for an atom $|0\rangle$ is ground state and $|1\rangle$ the excited state

  ↳ Some pulses of light can take atoms between $|0\rangle$ and $|1\rangle$ states

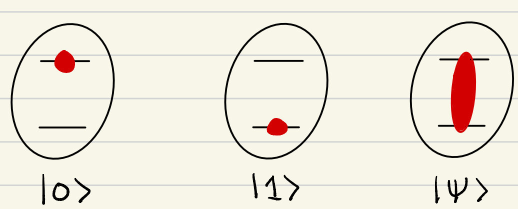
  ↳ Other pulses of light can take the atom into states with no classical analogue

    ↳ These are coherant superpositions of $|0\rangle$ and $|1\rangle$

    i.e. a qubit in $|0\rangle$ with some amplitude α_0 and in $|1\rangle$ with some amplitude in α_1

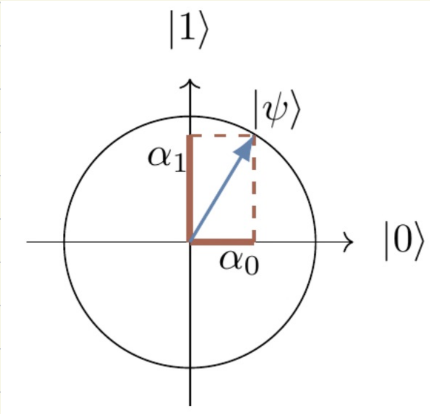
  This can be seen as $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \leftrightarrow \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$

Therefore, a concluding set of scenarios can be seen as :



- We know from Born's rule that α_0 and α_1 cannot be arbitrary complex numbers and must satisfy $|\alpha_0|^2 + |\alpha_1|^2 = 1$

  ↳ We can thus draw a state vector geometrically...



We must take this with a pinch of salt since amplitudes ARE complex numbers

Therefore α_0 and α_1 cannot be drawn as 1-dimensional vectors, although it provides a good general understanding

We can describe the new qubit computation U with our matrix representation:

The qubit state modification can be seen as:

$$|\psi\rangle \longrightarrow |\psi'\rangle = U|\psi\rangle$$

Which we can write as:

$$\begin{bmatrix} \alpha_0' \\ \alpha_1' \end{bmatrix} = \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

i.e. Operation U turns the state $|\psi\rangle$, with components α_k into $|\psi'\rangle = U|\psi\rangle$ with components $\alpha_l' = \sum_k U_{lk} \alpha_k$

AS A CONCLUDING THOUGHT:

- Qubit is a quantum system where boolean states 0 and 1 are represented by a prescribed pair of states $|0\rangle$ and $|1\rangle$
- The coherent superposition can be written as:

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

Such that $|\alpha_0|^2 + |\alpha_1|^2 = 1$

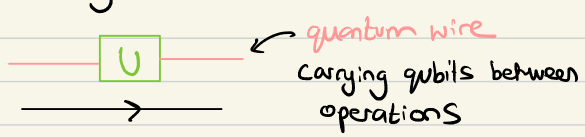
③ QUANTUM GATES AND CIRCUITS

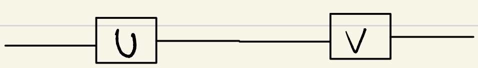
- Qubits (e.g atoms, trapped ions, molecules, nuclear spins) can be used to implement simple quantum interference and thus computation
- Any manipulations on qubits have to be performed by physically admissible operations which are represented by unary transformations

SOME DEFINITIONS:

- Quantum Logic Gate - Device which performs a fixed unitary operation on selected qubits in a fixed period of time
- Quantum Circuit - Device consisting of quantum logic gates whose computational steps are synchronised in time
- Circuit Size - The number of gates it contains
- Circuit Depth - Number of layers of gates in a circuit (gates can be layered where they operate simultaneously)

- Using these definitions, a unitary U acting on a single qubit can be represented diagrammatically as:





This shows two gates acting on the same qubit, U followed by V

↳ can be described by matrix product VU

④ SINGLE QUBIT INTERFERENCE

V. IMPORTANT

- Constructed as a sequence of three elementary operations :

  ① The Hadamard Gate

  ② A Phase-Shift Gate

  ③ The Hadamard Gate (again)

$$|0\rangle \;—\boxed{H}—\overset{\varphi}{\bullet}—\boxed{H}— \quad \cos\frac{\varphi}{2}|0\rangle - i\sin\frac{\varphi}{2}|1\rangle$$

- We have seen these before in the previous seminar :

other notation

Hadamard $\quad H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \begin{matrix} |0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \to |+\rangle \\ |1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \to |-\rangle \end{matrix}$

Phase-Shift $\quad P_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix} \quad \begin{matrix} |0\rangle \to |0\rangle \\ |1\rangle \to e^{i\varphi}|1\rangle \end{matrix}$

- These gates are FUNDAMENTAL → They will prop up over again

- Quantum vs Classical Computers

  Quantum computers are not necessarily quicker than classical computers
  ↓
  BUT They can implement quantum algorithms, some faster than classical ones

  They do not just "do all the computations at once"
  ↓
  BUT They rely on thoughtfully using interference, either destructive or constructive, to modify problems

i.e. The power of quantum computing comes from quantum interference

- When considering this sequence $H P_\varphi H$, we can express this as the product of the three matricies to show the transition amplitudes between states $|0\rangle$ and $|1\rangle$:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{i\varphi}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = e^{i\frac{\varphi}{2}}\begin{bmatrix}\cos \varphi/2 & -i\sin \varphi/2\\ -i\sin \varphi/2 & \cos \varphi/2\end{bmatrix}$$

- We can say that the majority of the time the input state is $|0\rangle$. Therefore we can follow the interference circuit step by step:

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$ Prepares equally weighted superposition of $|0\rangle$ and $|1\rangle$

$$\xmapsto{P_\phi} \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$$ controls entire evolution and determines output

$$\xmapsto{H} \cos\frac{\phi}{2}|0\rangle - i\sin\frac{\phi}{2}|1\rangle$$ closes interference by bringing paths together

Overall the probability of finding the qubit in either $|0\rangle$ or $|1\rangle$ are:

$$Pr(0) = \cos^2\frac{\phi}{2}$$

$$Pr(1) = \sin^2\frac{\phi}{2}$$

Thus, the Hadamard - Phase Shift - Hadamard can be seen as a fundamental quantum computation

↳ ① We prepare different computational paths [Hadamard]

② We evaluate function which introduces phase shifts [Phase-Shifts]

③ We bring together computational paths [Hadamard]

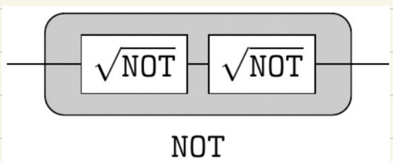
⑤ THE SQUARE ROOT OF NOT

- In this section, we must consider how quantum logic challenges conventional logic

  Let us pose the question :

  Design a gate that operates on a single bit which, when followed by an identical gate, the output is always the negation of the input.

  We can denote the resulting logic gate as $\sqrt{\text{NOT}}$ so that...

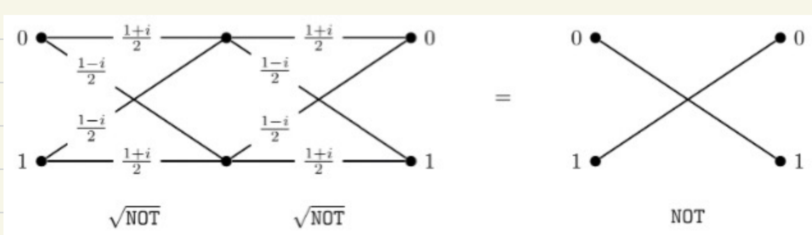


- In conventional logic, considering truth tables, this is impossible → no operation can exist
  ↳ **BUT** it does exist...

$$\sqrt{\text{NOT}} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \end{bmatrix}$$

This works since

$$\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



As with previously, we can denote this as the evolution of states

$$|0\rangle \xmapsto{\sqrt{\text{NOT}}} \frac{1}{\sqrt{2}}\left[e^{i\frac{\pi}{4}}|0\rangle + e^{i\frac{\pi}{4}}|1\rangle\right] \xmapsto{\sqrt{\text{NOT}}} |1\rangle$$

In any representation, quantum theory explains the behaviour of $\sqrt{\text{NOT}}$
↳ Therefore, $\sqrt{\text{NOT}}$ must exist as there exists a physical model for it in nature

⑥ PHASE GATES GALORE

- We have seen the phase gate already in qubit interference:

$$P_\psi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\psi} \end{bmatrix} \qquad \begin{aligned} |0\rangle &\longmapsto |0\rangle \\ |1\rangle &\longmapsto e^{i\psi}|1\rangle \end{aligned}$$

However, there are three specific examples of phase gates:

Phase-flip	$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\|0\rangle \longmapsto \|0\rangle$ $\|1\rangle \longmapsto -\|1\rangle$
$\frac{\pi}{4}$-phase	$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$	$\|0\rangle \longmapsto \|0\rangle$ $\|1\rangle \longmapsto i\|1\rangle$
$\frac{\pi}{8}$-phase	$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$	$\|0\rangle \longmapsto \|0\rangle$ $\|1\rangle \longmapsto e^{i\frac{\pi}{4}}\|1\rangle$

$\boxed{\text{global phase factor} = e^{i\frac{\psi}{2}}}$

Additionally, states which differ by a global phase are indistinguishable

Note: Phase gates can be written as either

$$P_\psi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\psi} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\psi}{2}} & 0 \\ 0 & e^{i\frac{\psi}{2}} \end{bmatrix}$$

since phase gate P_ψ is only defined up to a global phase factor

Where the second version is useful as it has determinant of 1 and belongs to a group which is called $SU(2)$

Phase flip Z is arguably the most important since it is one of the Pauli operators...

↳ Rimas will discuss this in the next talk