5.1-5.7 Quantum Entanglement

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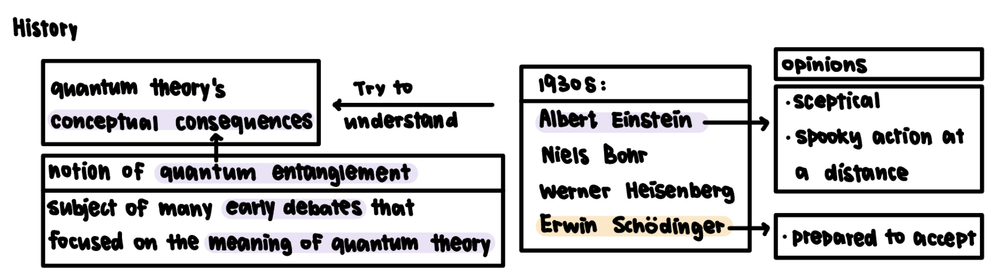
0 Introduction to Quantum Entanglement

Quantum entanglement is a fundamental principle of quantum mechanics that describes a phenomenon where particles become interconnected and the state of one cannot be described without the state of the other, regardless of the distance separating them.

1 Historical Background



2 From One Qubit to Two: Entanglement

2.1 Single Qubit States

For a single qubit, the general state can be written as:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

where c_0 and c_1 are complex coefficients. Since the state must be normalized ($|c_0|^2 + |c_1|^2 = 1$), and considering the global phase factor (which does not affect the physical state), we are left with two real parameters to fully describe a single qubit state.

2.2 Two Qubit States

For two qubits, the most general state is a superposition of the four basis states:

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

Here, c_{00}, c_{01}, c_{10}, and c_{11} are complex coefficients, leading initially to eight real parameters. Normalization imposes one constraint, reducing the count to seven. The global phase factor further reduces this to six real parameters.

2.3 Separable vs. Entangled States

2.3.1 Separable States

A separable state is one that can be decomposed into the tensor product of two individual qubit states. The first state:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

can be viewed as a pair of state vectors, where each one pertains to one of the two qubits. This can be simplifies to:

$$= \frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle)$$

This shows that the state is separable because it can be written as a product of the states of individual qubits:

- Qubit 1: $|0\rangle$.
- Qubit 2: $(|0\rangle + |1\rangle)$.

2.3.2 Entangled States

An entangled state, on the other hand, cannot be decomposed into a tensor product of two individual qubit states. For example,

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

does not admit such a decomposition. There do not exist states $|\psi_1\rangle$ and $|\psi_2\rangle$ such that

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\psi_1\rangle|\psi_2\rangle$$

This means that the state cannot be represented as a product of the states of individual qubits, making it an entangled state. The entanglement here means that the state of one qubit cannot be described independently of the state of the other qubit.

2.3.3 Conclusion

1. Separable State: Can be decomposed into a tensor product like $|\psi\rangle = |a\rangle \otimes |b\rangle$ of the states of individual qubits. Each qubit's state can be described independently.

2. Entangled State: Cannot be decomposed into a tensor product of the states of individual qubits. The state of one qubit is dependent on the state of the other qubit, indicating a quantum correlation between them.

3 Tensor products

1 Consider two systems, A and B, described by vectors in n-dimensional and m-dimensional Hilbert spaces, $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The combined system is described by vectors in the nm-dimensional tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$.

3.1 Basis of Tensor Product Space

Given orthonormal bases $\{|a_i\rangle\}_{i=1}^{n}$ of $\mathcal{H}_A$ and $\{|b_j\rangle\}_{j=1}^{m}$ of $\mathcal{H}_B$, we form a basis of the tensor product space by taking ordered pairs $|a_i\rangle \otimes |b_j\rangle$, for $i = 1, \ldots, n$

and $j = 1, \ldots, m$. These can be abbreviated as $|a_i\rangle\ |b_j\rangle$ or $|a_i b_j\rangle$. The space $\mathcal{H}_A \otimes \mathcal{H}_B$ consists of all linear combinations of such tensor product basis vectors:

$$|\psi\rangle = \sum_{i,j} c_{ij} |a_i\rangle \otimes |b_j\rangle.$$

3.2 Properties of Tensor Products

- **Distributivity**

$$|a\rangle \otimes (\beta_1|b_1\rangle + \beta_2|b_2\rangle) = \beta_1|a\rangle \otimes |b_1\rangle + \beta_2|a\rangle \otimes |b_2\rangle$$

$$(\alpha_1|a_1\rangle + \alpha_2|a_2\rangle) \otimes |b\rangle = \alpha_1|a_1\rangle \otimes |b\rangle + \alpha_2|a_2\rangle \otimes |b\rangle.$$

- **Hilbert Space:** The tensor product of Hilbert spaces is itself a Hilbert space, equipped with a natural inner product defined for any two product vectors $|a\rangle \otimes |b\rangle$ and $|a'\rangle \otimes |b'\rangle$ by

$$(\langle a'| \otimes \langle b'|)(|a\rangle \otimes |b\rangle) = \langle a'|a\rangle\langle b'|b\rangle,$$

  and extended by linearity to sums of tensor products of vectors, and, by associativity to any number of subsystems.

Note: the bra corresponding to the tensor product state $|a\rangle \otimes |b\rangle$ is written as $(|a\rangle \otimes |b\rangle)^\dagger = \langle a| \otimes \langle b|$ where the order of the factors on either side of $\otimes$ does not change when the dagger operation is applied.

3.3 Tensor Product of Operators

A useful fact about tensor products is that

$$\lambda\mathbf{a} \otimes \mathbf{b} = \mathbf{a} \otimes \lambda\mathbf{b}$$

(where $\mathbf{a}$ and $\mathbf{b}$ are vectors, and λ is a scalar). This means that we don't need to worry about where exactly we put λ, and can write something like $(\mathbf{a} \otimes \mathbf{b})\lambda$.

We will also need the concept of the tensor product of two operators. If A is an operator on $\mathcal{H}_A$ and B an operator on $\mathcal{H}_B$, then the tensor product operator $A \otimes B$ is an operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ defined by its action on product vectors via

$$(A \otimes B)(|a\rangle \otimes |b\rangle) = (A|a\rangle) \otimes (B|b\rangle)$$

and with its action on all other vectors determined by linearity:

$$A \otimes B \left(\sum_{i,j} c_{ij} |a_i\rangle \otimes |b_j\rangle \right) = \sum_{i,j} c_{ij} A|a_i\rangle \otimes B|b_j\rangle.$$

4 More Qubits, and Binary Representation

4.1 Quantum vs. Classical Registers

A classical register, consisting of three bits, can store only one of these binary strings at any time. A quantum register composed of three qubits can store both strings in a superposition, highlighting the difference between classical and quantum computing capabilities.

4.2 Application of the Hadamard Gate

Applying the Hadamard gate to the first qubit of the state $|011\rangle$, represented as $H \otimes 1 \otimes 1$, given that linear combinations distribute over tensor products, we obtain:

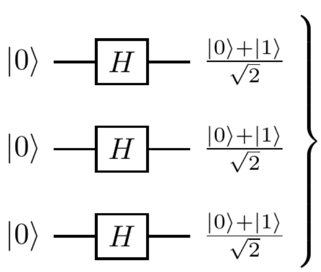
$$|011\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}(|011\rangle + |111\rangle)$$

4.3 Superposition of All Binary Strings

By applying the Hadamard transform $(H \otimes H \otimes H)$ to

$$|0\rangle \otimes |0\rangle \otimes |0\rangle = |000\rangle$$

we get



$$= \frac{1}{2^{3/2}} \left\{ \begin{array}{r} |000\rangle + |001\rangle + |010\rangle + |011\rangle \\ + |100\rangle + |101\rangle + |110\rangle + |111\rangle \end{array} \right\}.$$

This state is an equally weighted superposition of all binary strings of length 3, and can be written as

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

4.4 Hadamard Transform

The Hadamard transform, applying the Hadamard gate to each of the n qubits ($H^{\otimes n}$), maps product states to product states. It is fundamental in creating multi-qubit interference.

4.5 Notation and Decimal Representation

Instead of representing states as binary strings, we can also consider the decimal number each string represents, switching from binary strings of length n to considering all natural numbers from 0 to $2^n - 1$. For $n = 3$ qubits, we could either write

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

or instead switch to the decimal approach with $N = 2^n = 8$ and write

$$\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

so that we are writing $|7\rangle$ to mean $|111\rangle$, and $|3\rangle$ to mean $|011\rangle$, and $|0\rangle$ to mean $|000\rangle$, and so on.

5 Separable or Entangled?

Most vectors in the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ are entangled, meaning they cannot be represented as product states $|a\rangle \otimes |b\rangle$ where $|a\rangle \in \mathcal{H}_A$ and $|b\rangle \in \mathcal{H}_B$.

5.1 Understanding Joint States

Consider a joint state $|\psi\rangle$ of systems A and B. We can express $|\psi\rangle$ in a product basis as follows:

$$|\psi\rangle = \sum_{i,j} c_{ij}|a_i\rangle \otimes |b_j\rangle = \sum_i |a_i\rangle \otimes \left(\sum_j c_{ij}|b_j\rangle\right) = \sum_i |a_i\rangle \otimes |\phi_i\rangle \qquad (1)$$

Here, $|\phi_i\rangle = \sum_j c_{ij}|b_j\rangle$ are vectors in $\mathcal{H}_B$ that need not be normalized.

5.2 Identifying Product States

For any product state $|\psi\rangle = |a\rangle \otimes |b\rangle$, expanding the first state in the $|a_i\rangle$ basis yields:

$$|\psi\rangle = \sum_i |a_i\rangle \otimes \left(\sum_i \alpha_i |b\rangle\right). \qquad (2)$$

This expression aligns with the previous equation with $|\phi_i\rangle = \alpha_i|b\rangle$, indicating each $|\phi_i\rangle$ vector is a multiple of the same vector $|b\rangle$.

Conversely, if $|\phi_i\rangle = \alpha_i|b\rangle$ for all i in equation(1), then $|\psi\rangle$ must be a product state. To determine whether joint states are product states or not, we write the joint state according to the equation and check if all the vectors $|\phi_i\rangle$ are multiples of a single vector. Given that choosing states randomly makes it unlikely for this condition to be satisfied, we almost certainly pick an entangled state.

5.3 Describing States with Parameters

For n qubits, we need $2(2^n - 1)$ real parameters to describe their state vector, but only 2^n to describe separable states. As n grows, 2^n becomes much smaller than $2(2^n - 1)$.

6 Controlled-NOT

The Controlled-NOT gate is a two-qubit entangling gate in quantum computing. It affects two qubits: a control qubit and a target qubit. The C-NOT gate flips the state of the target qubit if the control qubit is in state $|1\rangle$, and does nothing if the control qubit is in state $|0\rangle$. In the standard basis

$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the Controlled-NOT (CNOT) gate is represented by the following unitary matrix:

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We represent the C-NOT gate in circuit notation as shown in Figure 5.1.

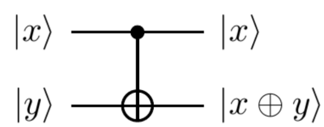


Figure 5.1: Where $x, y \in \{0, 1\}$, and $\oplus$ denotes `XOR`, or addition modulo 2.

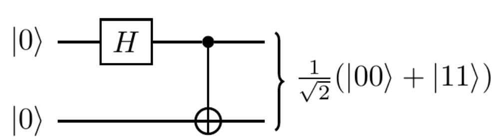
Note that this gate does not admit any tensor-product decomposition, but can be written as a sum of tensor products(where X is the Pauli bit-flip operation).:

$$C - NOT = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

7 Bell States

Bell states illustrate the phenomenon of quantum entanglement and are generated using the Controlled-NOT (C-NOT) gate.

Here is a simple circuit that demonstrates the entangling power of C-NOT:



In this circuit, the separable input $|0\rangle|0\rangle$ evolves as

$$|00\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{\text{C-NOT}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The result is one of the Bell states, specifically $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. By applying similar logic, we can generate the four Bell states from the standard computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$:

$$
\begin{aligned}
|00\rangle &\longmapsto |\psi_{00}\rangle \coloneqq \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\\
|01\rangle &\longmapsto |\psi_{01}\rangle \coloneqq \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\\
|10\rangle &\longmapsto |\psi_{10}\rangle \coloneqq \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),\\
|11\rangle &\longmapsto |\psi_{11}\rangle \coloneqq \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned}
$$

The more standard notation for these states, is the following:

$$
\begin{aligned}
\Phi^+ &\coloneqq |\psi_{00}\rangle\\
\Psi^+ &\coloneqq |\psi_{01}\rangle\\
\Phi^- &\coloneqq |\psi_{10}\rangle\\
\Psi^- &\coloneqq |\psi_{11}\rangle
\end{aligned}
$$

These states form an orthonormal basis in the Hilbert space of two qubits. Measuring in the Bell basis involves "rotating" the Bell states back to the standard basis using the reverse of the entangling circuit, followed by standard basis measurements.

The Bell states are said to be maximally entangled, since their reduced density operators are maximally mixed. Roughly, this means that the outcomes of any measurement performed on them are completely random.