1/29 Chapter 1 Notes:

Tags

Preliminary Info

Notation

$x =: y$: x is defined to be y

$x \equiv y$: x is another name for y

$|0>, |1>$: Notates a quantum bit holding either 0 or 1

$|\text{up}>, |\text{down}>$: Notates an electron of either spin up or spin down

Complex Numbers

A complex number is a $x + yi$ where $x, y \in R$ and $i = \sqrt{-1}$

Euclidean Vectors + Vector Spaces

- **Basis**: Collection of vectors such that every vector in V can be written in exactly one way as a linear combination of the basis vectors
  - ie). Identity matrix
- **Complex Vector Space**: set V such that any two vectors a, b in V and any two complex numbers α, β can form the linear combination $\alpha a + \beta b$
  - Must be communicative, associative, have identity, and inverse
- **Dimension**: Number of elements in a basis

Bras & Kets

- **Ket**: Column vector notated by $|a>$
- **Bra**: Row vector notated by $< a|$

$< u|v >$: Result of matrix multiplication of row and column vectors

Daggers

- **Hermitian Conjugation**: aka dagger operation †
  - Anti-linear operation that maps a ket to a bra & vice versa.
  - H represents kets while H^* represents bras.
  - Same thing as the conjugate transpose; interchanging rows and columns of A & then taking complex conjugates of each entry.

Geometry Background

- Length of a vector given by $||v|| = \sqrt{< v|v >}$
- Two vectors are orthogonal if $< u|v >= 0$
- Vector can be expressed as a linear combination of basis vectors: $|v >= \sum_i v_i|e_i >$

Operators

- **Linear Map**: Function A that maps from vector spaces $H \mapsto K$ represents linear combinations $A(c_1|v_1 > +c_2|v_2 >) = c_1A|v_1 > +c_2A|v_2 >$
- **Hermitian Conjugate**: denoted by A^t, $< i|A^t|j >=< j|A|i >^*$ where t turns (nxm) matrices into (mxn) matrices
  - Operator A is said to be:
    - Normal: if $AA^t = A^tA$
    - Unitary: if $A^t = A^{-1}$
      - If unitary, then it must be normal
    - Hermitian/self-adjoint: if $A^t = A$

Eigenvalues + Eigenvectors

- **Eigenvector**: A non-zero vector $|v >$ such that $A|v >= \lambda|v >$.
  - Assumed length is 1, must be orthogonal, so can always be scaled.

- **Trace**: Equal to the sum of eigenvalues
- **Determinant**: Equal to the product of the eigenvalues

Identities

Dagger for bras and kets:

- $|a\rangle^\dagger = \langle a|$
- $\langle a|^\dagger = |a\rangle$
- $(|a\rangle\langle b|)^\dagger = |b\rangle\langle a|$
- $(\alpha|a\rangle + \beta|b\rangle)^\dagger = \alpha^\star\langle a| + \beta^\star\langle b|$

Dagger for operators:

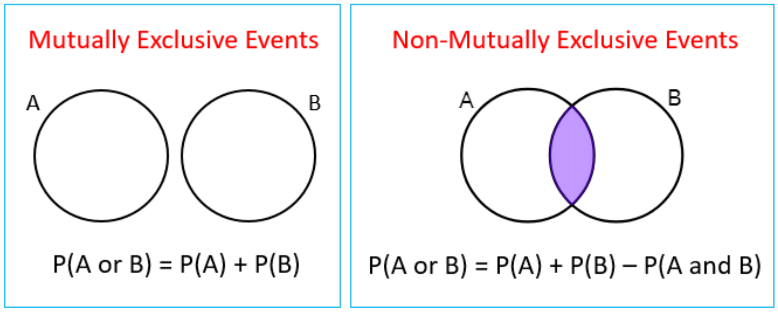
- $(AB)^\dagger = B^\dagger A^\dagger$
- $(A^\dagger)^\dagger = A$
- $(\alpha A + \beta B)^\dagger = \alpha^\star A^\dagger + \beta^\star B^\dagger$

Trace:

- $\mathrm{tr}(\alpha A + \beta B) = \alpha\,\mathrm{tr}(A) + \beta\,\mathrm{tr}(B)$
- $\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$
- $\mathrm{tr}\,|a\rangle\langle b| = \langle b|a\rangle$
- $\mathrm{tr}(A|a\rangle\langle b|) = \langle b|A|a\rangle = \mathrm{tr}(|a\rangle\langle b|A)$

Probabilities

- **Sample Space**: Ω, set of all possible outcome values
  - Ex. rolling an odd number on a six sided die has $\Omega = \{1, 3, 5\}$
- **Mutually Exclusive**: If $P(A \wedge B) = 0$
  - Cannot happen at same time
- **Independent**: $P(A \wedge B) = P(A) * P(B)$
  - Events do not depend on each other at all



Chapter 1

Introduction

> As quoted by the textbook, “information is physical”
> “Any information/computation is a physical process”
> Thus in order to understand this computation, we must look at the underlying physics
> “The laws of physics are written in the language of quantum physics”

1.1 Probability

> Q: What is the difference between a probability and a probability amplitude?
> A: “Positive real number probabilities replaced with complex numbers
> z such that $|z|^2$ are the probabilities in quantum theory”

Probability	**Probability Amplitude**
In classical probability theory, probability is a measure of the likelihood of an event occurring.	In quantum mechanics, probability amplitudes are complex numbers associated with the quantum state of a system. The square of the magnitude of a

	probability amplitude gives the probability of finding a particle in a particular state.
It is a real number between 0 and 1, where 0 represents impossibility, 1 represents certainty, and values in between represent degrees of likelihood.	Unlike classical probabilities, quantum probability amplitudes can be added or subtracted with phases, which leads to interference phenomena.
Classical probabilities are additive, meaning the probability of either of two mutually exclusive events occurring is the sum of their individual probabilities.	Probability amplitudes allow for the description of wave-particle duality and phenomena such as superposition.

Quantum physics is essentially a new probability theory that can be summarized in 3 basic rules:

1. **Born's Rule**: For a complex number alpha representing the probability amplitude, $p = |\alpha|^2$
2. **Product Rule**: Totally probability amplitude is the product of two consecutive probability amplitudes, $\alpha = \alpha_1 * \alpha_2$
3. **Addition Rule**: For a mutually exclusive event where two configurations lead to the same end state (alternatives), probability of the end state is the sum of the probability amplitudes, $\alpha = \alpha_1 + \alpha_2$

1.2 Kolmogorov Axioms

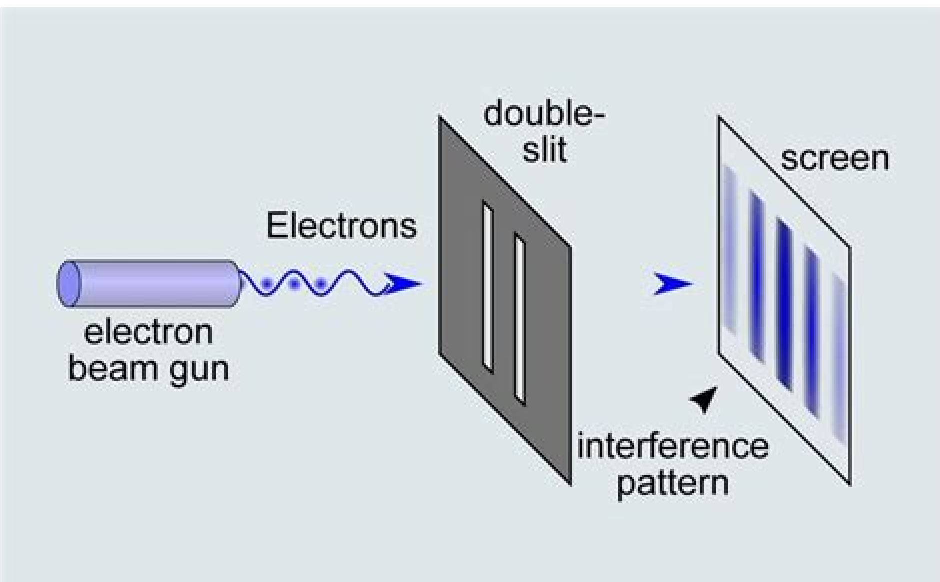
> *Andrey Kolmogorov (1903-1987)***:** Soviet mathematician mainly involved in probability theory + information theory. Proposed the relativity axiom in *"Foundations of Probability Theory"*.

1. Once you identify all elementary outcomes/events, you may assign probabilities to them where:
2. A probability is a number between 0 and 1, an event which is certain has probability.
3. **Kolmogorov Relativity Axiom**: If something can happen in two mutually exclusive ways, you add up the probabilities associated with each wave. $P = p_1 + p_2$

Though probability theory is ubiquitous in today's world, it fails to describe many quantum phenomena.

1.3 Double Slit Experiment + Interference

> There is no fundamental reason why nature should conform to the additivity axiom.
> Instead, we can "find out how nature works" through experimentation.



Particle (in this example, it is a photon) emitted from source S and can reach the detector D through 2 different paths. The results are inconsistent with the predictions of probability theory.

- Upper slit is taken with probability $p_1 = |z_1|^2$, lower slit is taken with probability $p_2 = |z_2|^2$
  - These two events are mutually exclusive.
- Probability theory states that particle should reach detector with probability $p_1 + p_2 = |z_1|^2 + |z_2|^2$, but this doesn't happen in experimentation

- Classical probability sum $p_1 + p_2$ is modified by the interference term $2\sqrt{p_1 p_2} cos(phi_2 - phi_1)$
  - Relative phase $phi_2 - phi_1$ determines if interference is positive or negative
    - Positive: constructive interference
    - Negative: destructive interference
    - Can either suppress or enhance total probability p

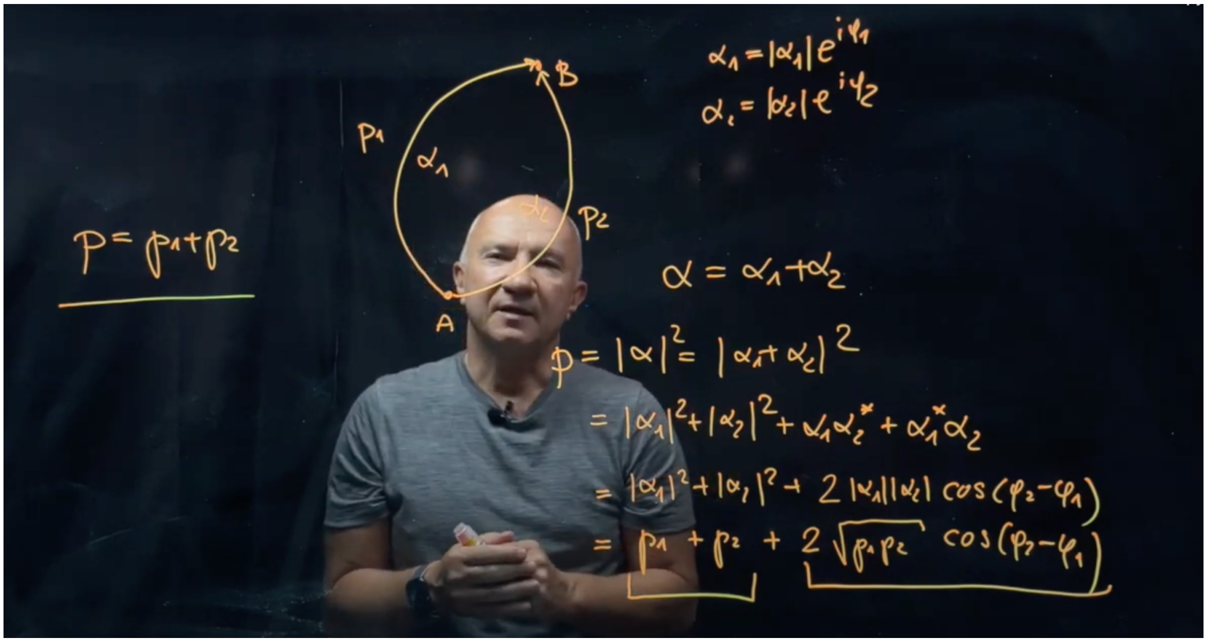
Proof to obtain quantum interference term:

$$z_1 = |z_1|e^{i\phi_1}$$
$$z_2 = |z_2|e^{i\phi_2}$$
$$p = |z|^2 = |z_1 + z_2|^2$$
$$= |z_1|^2 + |z_2|^2 + z_1^* z_2 + z_1 z_2^*$$
$$= p_1 + p_2 + |z_1||z_2|(e^{i(\phi_2 - \phi_1)} + e^{-i(\phi_2 - \phi_1)})$$
$$= p_1 + p_2 + 2\sqrt{p_1 p_2} cos(\phi_2 - \phi_1)$$

Where $p_1 + p_2$ are the classical terms and $2\sqrt{p_1 p_2} cos(\phi_2 - \phi_1)$ are the quantum inference terms and * represents the complex conjugate.

Example in the classical case:

ie. this will work with probabilities, not probability amplitudes



Key:

Complex Conjugate: $*$ (ie. $a + bi \mapsto a - bi$)

Phase Factor: ϕ

The first part of the equation ($p_1 + p_2$) is the classical part of the equation

The second part of the equation is the quantum interference (can be positive or negative depending on the cosine result being $+-1$, & thus can increase or decrease probability).

Interpretation by most physicists: the system enters a state where it follows both paths at the same time, this challenges Kolmogorov Relativity Axiom.

1.4 Superpositions

> "According to quantum theory, a particle that goes through the upper and the lower slit with certain amplitudes does explore *both* of the two paths, not just one of them."

- **Basis States**: particle is either in the upper slit or the lower slit
- **Superposition States:** $|\psi >= \alpha|\text{upper slit} > +\beta|\text{lower slit} >$

- aka. particle goes thru upper slit with amplitude alpha & lower slit with amplitude beta

- **Dirac Notation**: Using $| >$ to specify whatever the vector represents
  - ie. spin up/down via $|\uparrow>, |\downarrow>$, or $|0>, |1>$ for the quantum bit holding $0/1$

1.5 Interferometers

- **Ramsey Interferometry**: Generic name for an interference experiment in which atoms are sent through two separate Ramsey Zones
  - **Ramsey Zones**: “resonant interaction” zones separated by an intermediate “dispersive interaction” zone.
  - The first of these experiments were executed in Paris in the 90s. These experiments showed that the clear dependence of the atom’s outcome of either 0 or 1 was dependent on the intermediary phase shift, not the Ramsey Zones.

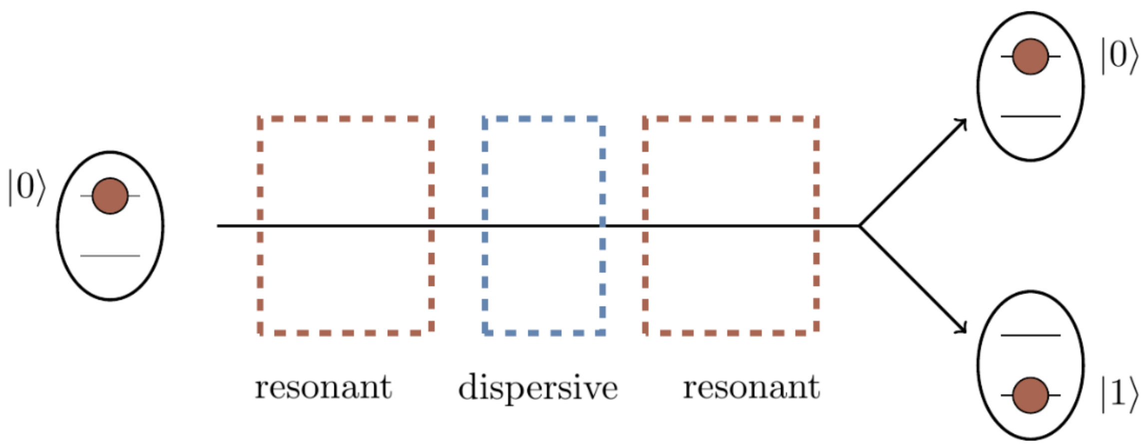
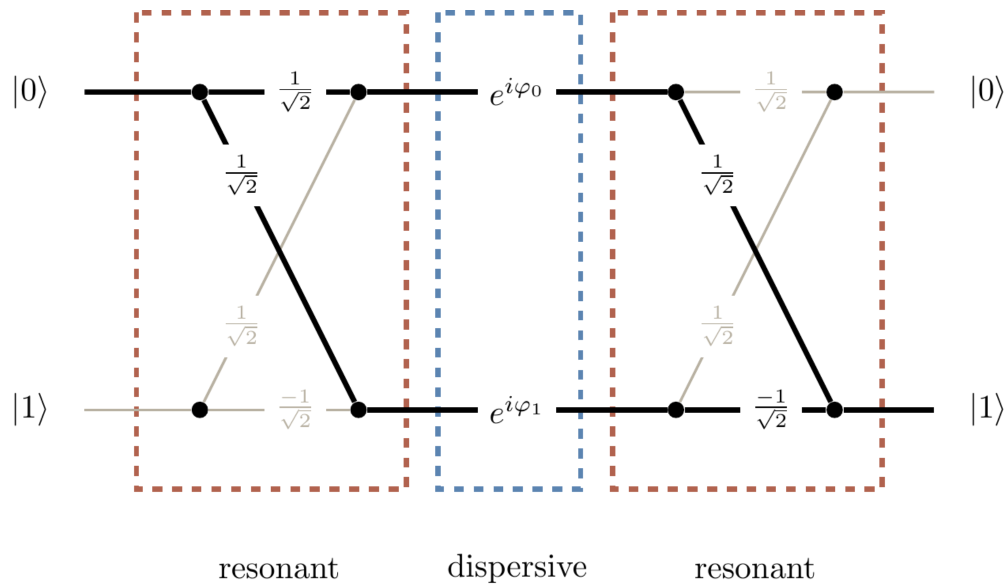


Figure 1.1: A schematic diagram of a Ramsey interference experiment.

Circle represents rubidium atoms traveling with fixed velocity through three areas with pre-selected states 0 or 1.

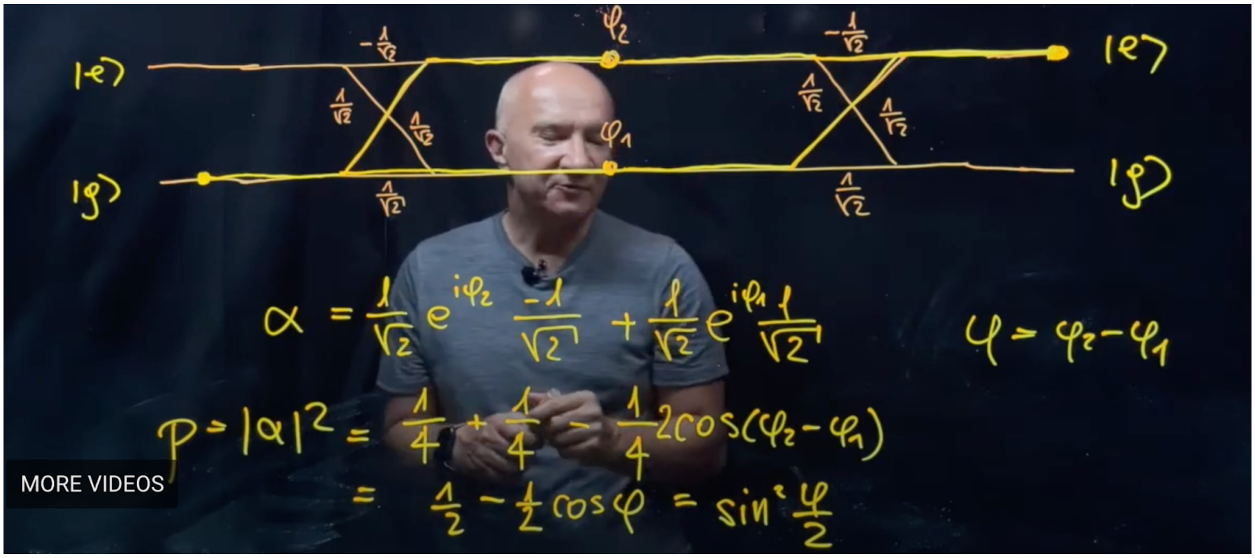
Rectangles are cavities containing arrangements of mirrors where you can trap an electromagnetic field.

- **Resonant Interaction**: Two external cavities are tuned to the relative frequencies of the atom, so the atom exchanges energy with the cavities going back and forth between 0 and 1.
- **Dispersive Interaction**: Central cavity does not induce transition, but instead phase shifts. The magnetic field is too off-resonance to exchange energy with the atom, but can still acquire a phase shift



1. Atom starts in the ground state & goes through the first cavity where it interacts with light there, moving it towards the excited state.
2. In the second cavity the atom undergoes phase shifts to alter the energy levels.
3. The third cavity is the same as the first.

First gate opens quantum inferences, 2nd gate controls action, third gate closes quantum interference



Calculating the probability amplitude of the Ramsey Interferometer:

$$\alpha = \frac{1}{\sqrt{2}}e^{i\phi_2}\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}e^{i\phi_2}\frac{1}{\sqrt{2}}$$

Where: α is the probability amplitude and ϕ is the phase factor for the cavity

This is the overall probability that the atom enters excited state in cavity 1 and stays there for cavity 3 or that the atom remains grounded in cavity 1 and enters excited state in cavity 3.

We can then square this magnitude to calculate the probability:

$$p = |\alpha|^2 = \frac{1}{4} + \frac{1}{4} - \frac{1}{4}(cos(\phi_2 - \phi_1)) = \frac{1}{2} - \frac{1}{2}(cos(\phi))$$
$$\text{where } \phi \text{ is } \phi_2 - \phi_1$$
$$= \frac{sin^2(\phi)}{2}$$

For more detail into how we obtained the probability:

We can see from the diagram that

$$\begin{aligned} U_{10} &= \frac{1}{\sqrt{2}} e^{i\varphi_0} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} e^{i\varphi_1} \frac{-1}{\sqrt{2}} \\ &= \frac{1}{2}\left(e^{i\varphi_0} - e^{i\varphi_1}\right). \end{aligned}$$

Then, using the trick of writing $x = \frac{x+y}{2} + \frac{x-y}{2}$ and $y = \frac{x+y}{2} - \frac{x-y}{2}$, followed by Euler's formula ($e^{i\alpha} = \cos\alpha + i\sin\alpha$), we see that

$$\begin{aligned} U_{10} &= \frac{1}{2}\left(e^{i\varphi_0} - e^{i\varphi_1}\right) \\ &= \frac{1}{2}\left(e^{i\frac{\varphi_0+\varphi_1}{2}} e^{i\frac{\varphi_0-\varphi_1}{2}} - e^{i\frac{\varphi_0+\varphi_1}{2}} e^{-i\frac{\varphi_0-\varphi_1}{2}}\right) \\ &= \frac{1}{2} e^{i\frac{\varphi_0+\varphi_1}{2}} \left(e^{i\frac{\varphi_0-\varphi_1}{2}} - e^{-i\frac{\varphi_0-\varphi_1}{2}}\right) \\ &= \frac{1}{2} e^{i\frac{\varphi_0+\varphi_1}{2}} \left(2i\sin\left(\frac{\varphi_0-\varphi_1}{2}\right)\right) \\ &= -ie^{i\frac{\varphi_0+\varphi_1}{2}} \sin\frac{\varphi_1-\varphi_0}{2} \end{aligned}$$

$$\begin{aligned} P_{10} &= |U_{10}|^2 \\ &= \left|-ie^{i\frac{\varphi_0+\varphi_1}{2}} \sin\frac{\varphi_1-\varphi_0}{2}\right|^2 \\ &= \left|\sin\frac{\varphi_1-\varphi_0}{2}\right|^2 \\ &= \frac{1}{2}(1-\cos\varphi) \end{aligned}$$

We can turn these probability amplitudes into a matrix:

described by the product of these matrices. We compile a

$$\begin{aligned} U &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{i\varphi_0} & 0 \\ 0 & e^{i\varphi_1} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \\ &= e^{i\frac{\varphi_0+\varphi_1}{2}} \begin{bmatrix} \cos\frac{\varphi}{2} & -i\sin\frac{\varphi}{2} \\ -i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix} \\ &= \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} \end{aligned}$$

where $\varphi = \varphi_1 - \varphi_0$, as before.

Optional Exercises:

1.1:

An old-fashioned incandescent lamp in the attic is controlled by one of three on–off
switches downstairs labelled A, B and C. But which one? Your mission is to do something with the switches, then determine after one trip to the attic which switch
is connected to the attic lamp.

Solution:

1. Turn switch A on and leave it on for a few minutes.
2. After some time, turn switch A off and turn switch B on.
3. Quickly go up to the attic.

Now, there are three possible scenarios:

- If the lamp is on, then switch B controls the lamp.
- If the lamp is off but warm to the touch, then switch A controls the lamp.
- If the lamp is off and cool, then switch C controls the lamp.

Here's the explanation:

- If switch B is connected, it will be on since it was turned on by the switch
- If switch A is connected, it will be on for a bit still since you left it on for a while before switching to B. Therefore, the residual heat in the filament will keep the lamp warm for a short period.
- If switch C is connected, it will be off and cool in the attic.

This solution relies on the physical properties of the incandescent lamp (it retains heat even after being turned off) and cleverly uses the switches to determine the connection without directly observing the wiring.

1.2:

Mathematicians view computation as an operation on abstract symbols. Any finite set of symbols is called an alphabet. A string over an alphabet is a finite sequence of symbols from that alphabet. Here, without any loss of generality, we will use the binary alphabet $\{0,1\}$. The set of all possible binary strings of length n is written as $\{0,1\}^n$. For example, $\{0,1\}^2$ contains the strings $00, 01, 10$ and 11. **How many elements does the set $\{0,1\}^n$ contain?**

Solution:

2^n, because there are 2 possible choices for n possible digits.

Show that the Hamming distance is a proper metric on the set
$\{0,1\}^n$

Solution:
To be a proper metric, we need to demonstrate that it satisfies three properties: non-negativity, identity of indiscernibles, and the triangle inequality.
1. Non negativity: Hamming distance is non-negative
2. Identity of Indiscernibles: Hamming distance is zero iff the strings are identical

3. Triangle Inequality: Hamming distance between direct route is always less than the indirect route

0.9.A **Dirac notation.** In this course we will use vectors called kets $|v\rangle$, linear functionals called bras $\langle u|$, inner products $\langle u|v\rangle$ which are complex numbers, outer products $|u\rangle\langle v|$ which are operators, and tensor products $|a\rangle \otimes |b\rangle$ which describe composite systems. Kets can be identified with column vectors, e.g.

$$|a\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}, \ |b\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \ |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The adjoint $|v\rangle^\dagger$, denoted by a dagger, is called a bra vector, or just a bra, and is written using a mirror image bracket $\langle v|$. Recall that for any complex matrix A (row and column vectors can be viewed as matrices) the adjoint $A^\dagger$ (also known as conjugate transpose or Hermitian conjugate) is formed by interchanging rows and columns and taking complex conjugate of each entry. Thus, in components bras are always written as row vectors.

(1) Express $\langle a|$, $\langle b|$, $\langle 0|$ and $\langle 1|$ in components as row vectors.

(2) Express the inner product $\langle a|b\rangle$ in terms of components of $|a\rangle$ and $|b\rangle$.

1.3:

A quantum computer starts calculations in some initial state, then follows n different computational paths which lead to the final output. The computational paths are followed with probability amplitudes $\frac{1}{n}e^{ik\varphi}$, where φ is a fixed angle $0 < \varphi < 2\pi$ and $k = 0, 1, ...n-1$. Using the fact that $1 + z + z^2 + \ldots + z^n = \frac{1-z^{n+1}}{1-z}$, show that the probability P of generating the output is given by

$$P = \frac{1}{n^2}\left|\frac{1-e^{in\varphi}}{1-e^{i\varphi}}\right|^2 = \frac{1}{n^2}\frac{\sin^2(n\frac{\varphi}{2})}{\sin^2(\frac{\varphi}{2})}.$$

for $0 < \varphi < 2\pi$, and that $P = 1$ when $\varphi = 0$. Plot the probability as a function of φ.

1.4:

Imagine two distant stars, A and B, that emit *identical* photons. If you point a single detector towards them you will register a click every now and then, but you never know which star the photon came from. Now prepare two detectors and point them towards the stars. Assume the photons arrive with the probability amplitudes specified in Figure 1.8. Every now and then you will register a coincidence: the two detectors will both click.

a. Calculate the probability of such a coincidence.
b. Now assume that $z \approx \frac{1}{r}e^{i2r\pi/\lambda}$, where r is the distance between detectors and the stars and λ is some fixed constant. How can we use this to measure r?

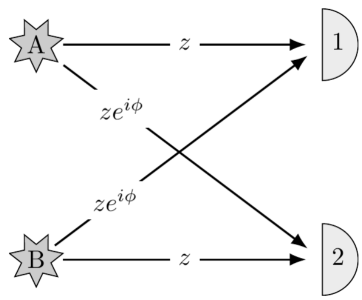


Figure 1.8: Two photon detectors pointing at two stars, with the probabilities of detection labelling the arrows.