8.5 Subsystems of Entangled Systems

Given a quantum state of the composite system AB described by ρ_{AB}, how can we obtain the reduced density operators ρ_A, ρ_B?

→ Partial trace

$$\rho_{AB} \mapsto \rho_A = \mathrm{tr}_B(\rho_{AB}) \quad \text{"partial trace over B"}$$

$$\rho_{AB} \mapsto \rho_B = \mathrm{tr}_A(\rho_{AB}) \quad \text{" \quad " \quad A"}$$

$\mathrm{tr}_B(\cdot)$ denotes "tracing out" the B subsystem
kinda like taking marginal distribution
over joint prob. dist.

Def. (Partial trace) $\mathrm{tr}_B(A \otimes B) = A \cdot (\mathrm{tr}\, B)$

$$\mathrm{tr}_A(A \otimes B) = (\mathrm{tr}\, A) \cdot B$$

Ex. $R = \sum_i M_i \otimes N_i$ on AB, $\mathrm{tr}_B(R) = \sum_i M_i \cdot \mathrm{tr}(N_i)$

Ex. $\mathrm{tr}_B(|00\rangle\langle 11|) = \mathrm{tr}_B(|0\rangle\langle 1| \otimes |0\rangle\langle 1|) = |0\rangle\langle 1| \cdot \mathrm{tr}(|0\rangle\langle 1|) = |0\rangle\langle 1| \cdot \langle 0|1\rangle = 0$

Ex. Composite system AB in a pure entangled state $|\Psi_{AB}\rangle$.

$$|\Psi_{AB}\rangle = \sum_i c_i |a_i\rangle \otimes |b_i\rangle$$

orthonormal bases

$$\sum_i |c_i|^2 = 1$$

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}| = \sum_{i,j} c_i c_j^* |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

$$\begin{aligned}
\rho_A &= \mathrm{tr}_B \rho_{AB} \\
&= \mathrm{tr}_B |\Psi_{AB}\rangle\langle\Psi_{AB}| \\
&= \mathrm{tr}_B \sum_{i,j} c_i c_j^* |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j| \\
&= \sum_{i,j} c_i c_j^* |a_i\rangle\langle a_j| (\mathrm{tr}\, |b_i\rangle\langle b_j|) \\
&= \sum_{i,j} c_i c_j^* |a_i\rangle\langle a_j| \underbrace{\langle b_i | b_j\rangle}_{\delta_{ij}} = \sum_i |c_i|^2 |a_i\rangle\langle a_i|
\end{aligned}$$

→ ρ_A is diagonal w/ $p_i = |c_i|^2$

Ex. (Maximally mixed state)

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{d}} \sum_{i}^{d} |a_i\rangle|b_i\rangle \qquad (\dim \mathcal{H}_A = \dim \mathcal{H}_B = d)$$

then $\rho_A = \rho_B = \frac{1}{d} I = \begin{pmatrix} 1/d & & 0 \\ & 1/d & \\ 0 & & \ddots \; 1/d \end{pmatrix}$

$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ has $\rho_A = \rho_B = \frac{1}{2} I = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

$$\rho = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$

$$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

$$= \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

$$\rho_A = \rho_B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

8.6 Mixtures and Subsystems

Consider the joint state of a bipartite system AB

$$|\psi_{AB}\rangle = \sum_{i,j} c_{ij} |a_i\rangle \otimes |b_j\rangle$$
$$= \sum_j |\tilde{\psi}_j\rangle |b_j\rangle \quad \text{where } |\tilde{\psi}_j\rangle = \sum_i c_{ij} |a_i\rangle$$
$$= \sum_j \sqrt{p_j}\, \underbrace{|\psi_j\rangle}_{\text{normalized version of } |\tilde{\psi}_j\rangle} |b_j\rangle$$
$$p_j = \langle \tilde{\psi}_j | \tilde{\psi}_j \rangle$$

Then the partial trace over B gives the reduced density operator of subsystem A.

$$\rho_A = tr_B\Big(\sum_{i,j} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_j| \otimes |b_i\rangle\langle b_j|\Big)$$
$$= \sum_{i,j} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_j| \big(tr_B(|b_i\rangle\langle b_j|)\big)$$
$$= \sum_{i,j} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_j| \langle b_i | b_j\rangle$$
$$= \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

ρ_A describes subsystem A; is it ever affected by what happens to subsystem B?

Say A, B are in separate labs.
Bob measures in basis $|b_j\rangle$ and get k w/ probability p_k.

$$\sum_{i=1} \sqrt{p_i}|\psi_i\rangle|b_i\rangle \mapsto |\psi_k\rangle|b_k\rangle$$

Subsystem A is then in state $|\psi_k\rangle$
Bob doesn't communicate his result
Alice's perspective: Bob prepares a mixture of $|\psi_1\rangle, \ldots, |\psi_m\rangle$ w/ prob. $p_1, \ldots, p_m$
Alice's density matrix: $\rho_A = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Now suppose Bob measures in some other basis $|d_i\rangle$
We can write $|b_i\rangle = U|d_i\rangle$

$$|\psi_{AB}\rangle = \sum_i |\tilde{\psi}_i\rangle|b_i\rangle$$
$$= \sum_i |\tilde{\psi}_i\rangle\Big(\sum_j U_{ij}|d_j\rangle\Big)$$
$$= \sum_j \underbrace{\Big(\sum_i U_{ij}|\tilde{\psi}_i\rangle\Big)}_{|\tilde{\phi}_j\rangle}|d_j\rangle = \sum_j |\tilde{\phi}_j\rangle|d_j\rangle$$

Alice's perspective: mixture of states $|\phi_1\rangle, \dots, |\phi_m\rangle$ w/ $p_k = \langle\tilde{\phi}_k|\tilde{\phi}_k\rangle$
But this mixture has the same density operator as the previous one.

$$\sum_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j| = \sum_{i,j,l} U_{ij} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| U^*_{lj}$$

$$= \sum_{i,l} \underbrace{\left(\sum_j U_{ij} U^*_{lj}\right)}_{\delta_{il}} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_l|$$

$$\sum_k U_{ik} U^*_{jk} = \delta_{ij}$$

if U unitary

$$= \sum_j |\tilde{\psi}_j\rangle\langle\tilde{\psi}_j| = \rho_A$$

8.7 Partial trace, revisited

Given a density matrix of two qubits in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, how can we easily find the reduced density matrices?

ρ_{AB} is a 4×4 matrix, which we'll write as $\rho_{AB} = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right]$

$$\rho_A = tr_B(\rho_{AB}) = \left[\begin{array}{c|c} trP & trQ \\ \hline trR & trS \end{array}\right]$$

$$\rho_B = tr_A(\rho_{AB}) = P + S$$

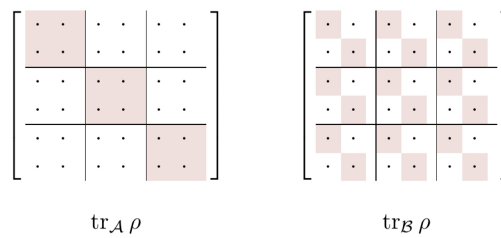

Figure 8.1: Visualising the two partial traces of a matrix *written in the tensor product basis*.

Def. (Partial trace) The partial trace over B is the unique map $\rho_{AB} \mapsto \rho_A$ s.t.

$$tr[X\rho_A] = tr[(X\otimes I)\rho_{AB}]$$

where X is any observable acting on X and I acts on B

Sanity check $|\psi_{AB}\rangle = \sum_{i,j} c_{ij}|a_i\rangle \otimes |b_j\rangle$

$$= \sum_j |\tilde{\psi}_j\rangle|b_j\rangle = \sum_j \sqrt{p_j}\,|\psi_j\rangle|b_j\rangle$$

Now Alice measures some observable X on her system.
We can think of this as $(X\otimes I)$ acting on the entire system
The expected value of this obs. in $|\psi_{AB}\rangle$ is

$$tr(X\otimes I)|\psi_{AB}\rangle\langle\psi_{AB}|$$

$$tr[(X\otimes I)\rho_{AB}] = tr\Big[(X\otimes I)\Big(\sum_{i,j} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_j| \otimes |b_i\rangle\langle b_j|\Big)\Big]$$

$$= \sum_{i,j}\big[tr(X|\tilde{\psi}_i\rangle\langle\tilde{\psi}_j|)\big]\underbrace{\big[tr(|b_i\rangle\langle b_j|)\big]}_{\delta_{ij}}$$

$$= \sum_i tr[X|\tilde{\psi}_i\rangle\langle\tilde{\psi}_j|]$$

$$= tr\Big[X\sum_i p_i|\psi_i\rangle\langle\psi_j|\Big]$$

$$= tr\Big[X\sum_i p_i|\psi_i\rangle\langle\psi_j|\Big] = tr[X\rho_A]$$

WTS uniqueness.

Suppose $\exists$ arbitrary map T s.t. $\mathrm{tr}[X T(\rho_{AB})] = \mathrm{tr}[(X \otimes I)\rho_{AB}]$ —— $(*)$

$\forall$ density matrices ρ_{AB}

$\forall$ observables X acting on A.

Now take some orthonormal basis $\{M_i\}$ of the space of Hermitian matrices

M_i Hermitian $\Rightarrow (M_i | T(\rho_{AB})) = \frac{1}{2}\mathrm{tr}[M_i T(\rho_{AB})]$ ↳ norm is defined by Hilbert-Schmidt inner product $(A|B) = \frac{1}{2}\mathrm{tr} A^\dagger B$

Expand $T(\rho_{AB})$ in this basis:

* any vector v in an inner product sp. w/ orthonormal basis $\{e_i\}$ can be expanded as $v = \sum_i (e_i|v) e_i$

$$
\begin{aligned}
T(\rho_{AB}) &= \frac{1}{2}\sum_i (M_i | T(\rho_{AB})) M_i \\
&= \frac{1}{2}\sum_i \mathrm{tr}[M_i T(\rho_{AB})] M_i \\
&\overset{(*)}{=} \frac{1}{2}\sum_i \mathrm{tr}[(M_i \otimes I)\rho_{AB}] M_i
\end{aligned}
$$

↳ indep. of choice of T, so we're done.