Quantum Information Theory Math Seminar

Chapter 3: Sections 3.4-3.6

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by

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Contents

1 Introduction

Before getting into the new material, let's review some of the basics again to refresh our memories.

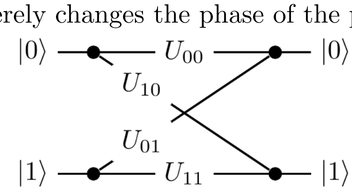
One of the key ideas we're meant to take away is that "computation is a physical process", which can be kind of strange to think about. Computers and any kind of information and data are all technically physical objects, though we tend to think of them as purely logical/theory-based processes.

Another key idea is that we're replacing the old view of probability theory with an updated version that accounts for *probability amplitudes*, which corrects for the failure of Kolmogorov's additivity axiom in cases such as the double slit experiment, where the probability that a particle goes through either one slit or the other is not simply $p_1 + p_2 = |z_1|^2 + |z_2|^2$, but instead also requires an interference term: $p = p_1 + p_2 + 2\sqrt{p_1 p_2}\cos(\varphi_2 - \varphi_1)$. This reliance of the probability on the *relative phase*, $\varphi_2 - \varphi_1$ conveys the physical superposition of the particle, as this relies on the particle physically going through both slits simultaneously.

We then looked at other interference experiments, particularly single-qubit interference experiments. As a reminder, a *qubit* is a quantum bit, or a quantum system which has two Boolean states, $|0\rangle$ and $|1\rangle$, which form a computational/standard basis. This allows a qubit to be written as a superposition: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ such that $|\alpha|^2 + |\beta|^2 = 1$.

We looked at how (unitary) matrices can represent the amplitude of transitioning from state to state, as shown below. These transitions happen due to *quantum (logic) gates* on a *quantum circuit*.

We've previously talked about the *Hadamard* and *phase-shift* gates, as these are essential to single-interference. The Hadamard gate is a resonant interaction, allowing the qubit to go from ground to the excited state or vice versa, while the phase gate is a dispersive interaction which merely changes the phase of the particle.



2 Section 3.4: Unitaries as rotations

While we previously discussed how single-qubit state vectors can be geometrically represented by points on the Bloch sphere, we can represent (2x2) unitary matrices through rotating the Bloch sphere.

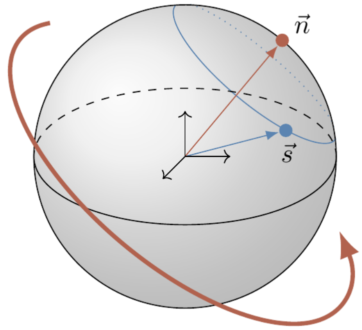
As a reminder, a matrix U is *unitary* if $U^\dagger U = UU^\dagger = 1$, where $U^\dagger$ is the Hermitian conjugate of the matrix U, meaning you first transpose U and then apply complex conjugation to each entry (e.g. the complex conjugate of $a + ib$ is $a - ib$ for any real numbers a and b).

First, we express these matrices as $U = u_0\mathbf{1} + i(u_x\sigma_x + u_y\sigma_y + u_z\sigma_z) = u_0\mathbf{1} + i\vec{u}\cdot\vec{\sigma}$, where $u_0^2 + |\vec{u}|^2 = 1$. Here, we assume/fix the determinant of the matrix is equal to one. This is because any two unitary matrices that are the same except for a global multiplicative phase factor still represent the same kind of physical operation. Remember that the sigma terms correspond to the Pauli operators we discussed last Monday, where σ_x is the bit-flip operator (which takes a qubit from $|0\rangle \to |1\rangle$ and vice-versa), σ_z is the phase-flip operator (which leaves $|0\rangle$ the same but shifts $|1\rangle \to -|1\rangle$), and σ_y is the bit-phase-flip operator (which transitions $|0\rangle \to i|1\rangle$ and $|1\rangle \to -i|0\rangle$). Matrices of this form the *special* (due to the determinant being equal to 1) unitary group, labeled SU(2).

The restriction on u_0 and $\vec{u}$ allows for the following parametrization of U: $U = (\cos\theta)\mathbf{1} + i(\sin\theta)\hat{n}\cdot\vec{\sigma}$ or $U = e^{i\theta\vec{n}\cdot\vec{\sigma}}$.

With this new expression of the unitary matrix, we can see that $U = e^{i\theta\vec{n}\cdot\vec{\sigma}}$ geometrically symbolizes a clockwise rotation about the axis defined by the unit vector $\hat{n}$ through the angle 2θ.

If you prefer the rotation angle to be through θ, you can also parametrise the unitary as $U = e^{\frac{-i\theta}{2}\hat{n}\cdot\vec{\sigma}}$, using the right-hand rule to determine the direction for the rotation of the Bloch sphere. Below is a representation of rotating the Bloch sphere according to the unitary, where we rotate the $\vec{s}$ about $\hat{n}$ by 2θ. After the rotation, we have $\vec{s}$ directed to a point on this circle, which is a unique circle whose center is marked by $\hat{n}$.



For example, a rotation of 2θ can be represented by the matrix $e^{i\theta\sigma_x} = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$ about the x-axis. Similarly, $e^{i\theta\sigma_y} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ represents a rotation by 2θ about the y-axis, and $e^{i\theta\sigma_z} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ about the z-axis.

Rotations about a *Pauli axis*, the x-, y-, or z-axes, are referred to as *Pauli rotations*, and can be written as $e^{i\theta\sigma_k} = (\cos\theta)\mathbf{1} + (i\sin\theta)\sigma_k$, for $k \in x, y, z$.

Therefore, we can represent the Hadamard gate by the following: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) = (-i)e^{i\frac{\pi}{2}\frac{1}{\sqrt{2}}(\sigma_x+\sigma_z)}$.

Note that quantum information science sometimes ignores the global multiplicative phase factors that allow the determinant to be fixed to 1, such as in how phase gates are usually represented as $P_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$ instead of the more technically correct $P_\alpha = \begin{bmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{bmatrix}$, as the Hadamard gate is effectively a rotation about the $(x+z)$–axis by an angle of π.

Physically, we can see the connection between unitaries and rotations by analyzing how the U(2) group acts on (2x2) Hermitian matrices with zero trace (meaning the sum of the terms along the diagonal is equal to 0), represented by V. All traceless matrices of this form can be written as $S = \vec{s} \cdot \vec{sigma}$, where $\vec{s}$ represents a Euclidean vector in R^3.

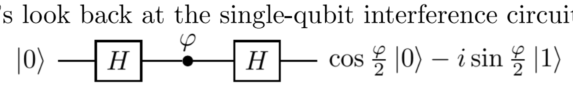
Now we can look directly at how U(2) acts on V: $S \mapsto S' = USU^\dagger$. Since this is a linear mapping

($R_U : R^3 \mapsto R^3$, this map is isometric (or is a distance preserving operation) and orthogonal, but this is only true in three-dimensional Euclidean space, allowing us to represent the rotations in the first place. This thus represents a group homomorphism (which is "a structure-preserving map between two algebraic structures of the same type" Wikipedia): $U(2) \mapsto SO(3)U \mapsto R_U$, where SO(3) is the three-dimensional special orthogonal group.

Physicists would usually represent this with a first order approximation such that $\vec{s'} = \vec{s} + 2\alpha(\hat{n} \cdot \vec{s}$, which geometrically represents an infinitesimal clockwise rotation of $\vec{s}$ through angle 2α about the axis angle $\hat{n}$.

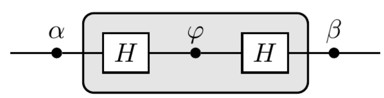
3 Section 3.5: Universality, again

Now that we've discussed unitaries and how we can represent them with Bloch sphere rotations, let's look back at the single-qubit interference circuit, shown below.



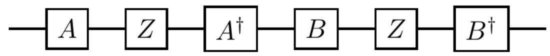
Using our newfound geometrical understanding of unitary operations, we can see that the circuit shows how a rotation about the z-axis, which is induced by the phase-gate P_ϕ, is turned into a rotation about the x-axis due to the two Hadamard gates. We can represent this with the following matrix: $H(e^{\frac{-i\varphi Z}{2}})H = e^{\frac{i\varphi X}{2}} \begin{bmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$.

Now, we can look at the following circuit:



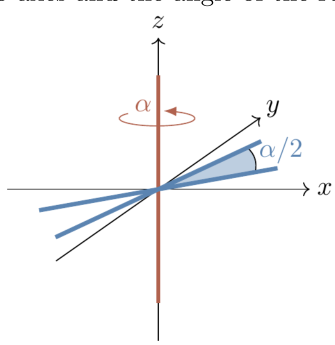
We can write this unitary as: $U(\alpha,\beta,\varphi) = e^{\frac{i\beta Z}{2}} e^{\frac{i\beta X}{2}} e^{\frac{i\beta Z}{2}} =$
$$\begin{bmatrix} e^{\frac{-i(\alpha+\beta)}{2}}\cos\frac{\varphi}{2} & ie^{\frac{i(\alpha-\beta)}{2}}\sin\frac{\varphi}{2} \\ ie^{\frac{-i(\alpha-\beta)}{2}}\sin\frac{\varphi}{2} & e^{\frac{i(\alpha+\beta)}{2}}\cos\frac{\varphi}{2} \end{bmatrix}$$
Therefore, with a pair of Hadamard gates and a number of phase gates, you can define an arbitrary unitary operation on a single qubit.

While this may seem like a special property of the z and x axes, the only important quality they have is their orthogonality, so the same will apply to any two orthogonal axes.

Now, let's look at this next circuit: 

, where A and B are unitary operations. As discussed above, we can represent *any* unitary matrix U in this form.

To prove this, we can see that this circuit represents two 180°rotations about the two axes found by rotation the z-axis using unitaries A and B. The final axis of rotation is perpendicular to the two axes and the angle of the rotation is twice of that between the two axes, as shown below.



4 Section 3.6: Some quantum dynamics

Now, we're going to switch gears and look at more fundamental quantum theory.

A *Hamiltonian* is a Hermitian operator which generates the time evolution of a quantum state, denoted by $\hat{H}$. This term specifies all possible interactions within the system and will generally change with time. However, in an isolated system, the state vector $|\psi(t)\rangle$ changes smoothly in time according the the *Schrödinger equation*: $\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}|\psi(t)\rangle$, where $\hbar$ is the Planck constant. Schrödinger's equation can be seen as an analog to Newton's second law, as while Newton describes the "certain future behavior of a classical system given some initial knowledge", Schrödinger describes "the future behavior of a quantum system given some initial knowledge".

In the case of *time-independent* Hamiltonians, the Schrödinger equation becomes $|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$. Therefore, this function is separable and we can see that the exponential term is the phase factor U(t) (which doesn't affect the resulting probabilities) and the ψ term is constant. This means that $||\psi\rangle|^2$ is constant and thus represents a stationary or *standing wave*.

5 Section 3.7: Remarks and exercises

Exercise 3.7.7: Phase as rotation

1. Show that the phase gate $P_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$ represents an anticlockwise rotation about the z-axis through the angle φ.