Quantum Information Theory Seminar – Chapter 6

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Format

1. INTRODUCTION – Bell's theorem

a. <u>General explanation</u>

In **classical physics**, there is an assumption that the properties of physical systems are determined by measurable variables. These variables describe the system's state at any given time and predict the outcome of any measurement of the system ("expected value"). The notion of "**locality**", that defines classical physics, posits that a particle is only influenced by its immediate environment so that the interactions via physical fields do not exceed the speed of light (consistent with the theory of relativity which states that no information can travel faster than the speed of light).

However, **quantum mechanics** introduces phenomena that challenge these classical intuitions. For instance, **quantum entanglement** describes a situation where particles become interconnected in such a way that the state of one (no matter how distant) cannot be fully described without considering the state of the other. This leads to **correlations** between the properties of entangled particles that are stronger than what can be explained by classical physics (see 6.2).

Albert Einstein and his colleagues, Boris Podolsky and Nathan Rosen, in their EPR paper in 1935, introduced the concept of "**local hidden variables**" to reconcile these quantum phenomena with classical physics. The theory suggests that quantum systems possess predetermined properties (the "hidden variables") that determine the outcomes of quantum measurements. These hidden variables are "local" in the sense that they obey the principle of locality.

The most significant challenge to the local hidden variable's theory came from physicist John Bell in 1964, who formulated **Bell's theorem**. This theorem shows that if local hidden variables were responsible for quantum correlations then the results of certain types of quantum experiments would be subject to statistical limitations known as **Bell inequalities.** However, numerous experiments have demonstrated violations of Bell inequalities which support the non-local nature of quantum entanglement and challenge the viability of local hidden variables as an explanation for quantum phenomena (see 6.3 and 6.4). This suggests that the quantum mechanical description of nature - its inherent randomness and non-locality - cannot be reproduced by any theory based on local hidden variables.

b. Physics explanation – polarization of light paradox

Before getting into the mathematics of Bell's theorem, it is important to understand the physical interpretation, and how things work in the real world. To do this, let's begin with the **polarization of light paradox**. A simplified explanation of light polarization begins with the understanding that light consists of **transverse waves**[1]. Transverse waves possess a unique characteristic referred to as "polarization," which dictates their "direction." A polarizing filter is designed to allow only waves of a specific polarization to be transmitted. When two polarizing filters are positioned at right angles (90 degrees) to one another, virtually no light is transmitted, as the light that manages to pass through the first filter is obstructed by the second, owing to their mutually perpendicular polarizations.

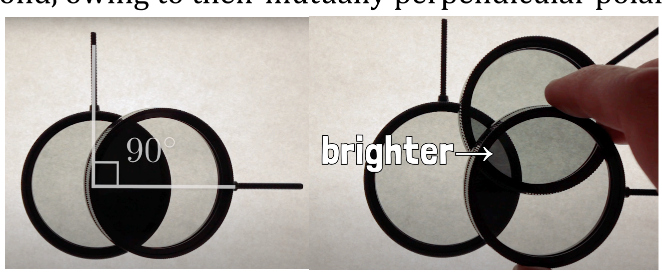

Figure 1: polarizing filters at different angles

What you are observing in figure 1 is that the photons that allow them to pass through a filter along one axis have a much lower probability of passing through a second filter along a perpendicular axis – in other words, a 0 probability. However, introducing a third filter between these two, angled at 45 degrees, surprisingly allows more light to pass than if this middle filter were absent. Somehow introducing another filter allows more light through.

[1] Transverse wave: wave that oscillates perpendicularly to the direction of the wave's advance

More specifically, photons are waves in the electromagnetic field passing through polarization filters (90-degree bend). Photons are quantum objects so they either pass fully or not at all through the filters.

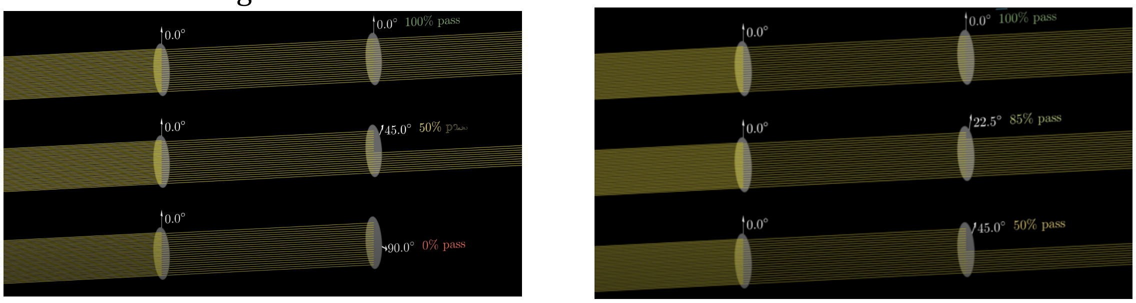

Figure 2: politizing filters at different angles and the probabilities of the photons passing

This is only considering the angles of the filters and no other hidden variables for instance what the photon did before (like passing another filter). What is interesting also is that when you have an angle smaller than 45 degrees then it becomes increasingly higher probability (figure 2). This does not make sense as you would expect for a 22.5-degree filter to have a 75% pass not 85%. Looking at it deeper to show how this does not make sense, another example is if the second filter B is absent, and C is at 45 degrees then there is a 50% block. If there is B, there is 85% that the photons pass through B and 85% that they pass through C so 15% times 2 that are blocked, how does that = 50% (figure 3).

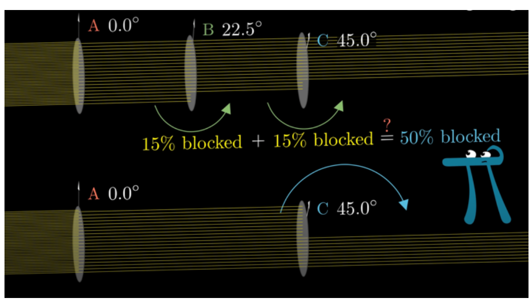

Figure 3: differences between having 2 or 3 polarizing filters at different angles.

To study this phenomenon further, **Bell experimentation** comes into play as it is an experiment where the interactions cannot affect each other without faster-than-light communication. The key is not to make photons pass through different filters at different times but at different points in space at the same time and for this, we need **entanglement**[2].

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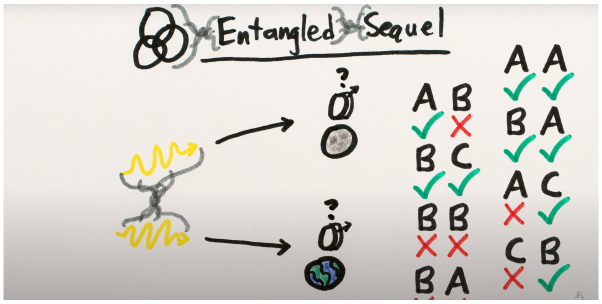
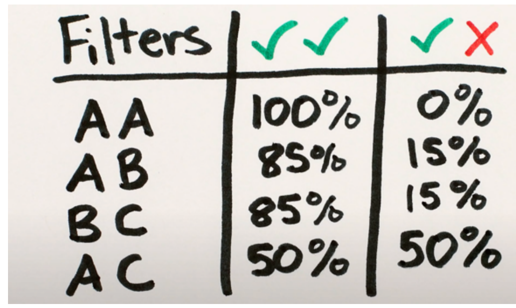
[2] **Entanglement** is a phenomenon where the quantum states of two or more particles become interconnected in such a way that the state of each particle cannot be described independently of the state of the others, even when the particles are separated by large distances. This interconnectedness leads to correlations in their observable properties (such as polarization or spin) that are stronger than what can be explained by classical physics or by any theory based on local hidden variables.
Entangled means here that if you were to pass individual photons through filters **in the same orientation**, either both pass or both get blocked, they behave the same way when measured along the same axis, they **have a correlated behavior**.

The idea here is to have two **entangled photon pairs** pass through two different filters randomly at the same time. Doing this many times will enable you to collect data on how often both photons in an entangled pair pass through the different combinations of filters (Figure 4). You see the same results as above which confirms the paradox and seems to contradict the idea of "hidden variable" (Figure 5)[3].





Filters	✓✓	✓✗
A A	100%	0%
A B	85%	15%
B C	85%	15%
A C	50%	50%

Figure 4,5: Entangled sequel diagram and results

This begs the final question of how does the active passing of entangled photon pairs through multiple filter combinations (same orientation for the pairs) simultaneously correlate with each other. Is this effect due to local hidden variables (like Einstein says) or nonlocal randomness (like Bell says by defying classical physics?). What this experiment shows is that either realism is not how the universe works or locality or a combination - It just cannot be locally real[4].

Can this be confirmed mathematically?

Bell's theorem provided a way to test the predictions of quantum mechanics against those of local hidden variables theories. The experiments that followed, many of which used entangled photons and polarization filters, have repeatedly confirmed the predictions of quantum mechanics which demonstrates the non-local correlations that exist between entangled particles and thus support the inherently quantum nature of these correlations.

[3] You use filter A at one site and B at the other, among all photons that pass filter A, among 15% have an entangled partner that gets blocked at B. Likewise if they are set to be in C, about 15 % that do pass through B have an entangled partner that get blocks at C. For setting A and C, half of those that get through A get blocked at C.

[4] Two assumptions in science:

1. Realism: the assumption that there is an underlying state even if its not being probed is called realism
2. Locality: the assumption that faster than light influence is not possible is called locality

2. 6.1 Hidden variables

Applying this notion to quantum information theory, when measuring observables (quantifiable properties) of a qubit, such as a spin along different axes denoted (σ_x for the x-axis and σ_z for the z-axis), **quantum mechanics states that these properties do not have definite values until they are measured.** Due to the principle of quantum superposition, a qubit can be in a state where it has a probability of being in multiple states simultaneously. To resolve this and explain the contradiction explained above, the concept of "hidden variables" was developed by **Einstein due to his concerns regarding quantum theory's** completeness. A hidden-variable theory proposes a deterministic framework aiming to account for the probabilistic nature of quantum mechanics through the introduction of additional variables that may not be directly observable.

Take a single qubit as an instance. Looking back to the earlier discussion about **compatible operators** (referenced in Section 4.6 by Raunak) also known as commutative operators. For recall, consider an eigenbasis of A (an operator), where each vector $|e_k\rangle$ is an eigenvector. If A and B commute (AB = BA), any vector B $|e\rangle$ is also an eigenvector of A, meaning A and B share an eigenbasis. Conversely, if A and B share an eigenbasis, they commute (AB = BA). We call A and B compatible if they commute, and incompatible otherwise. Overall, compatible operators refer to operators that can be **measured simultaneously without interference**, meaning the order of their measurement does not affect the outcome,

It is understood that a qubit's quantum state cannot simultaneously be an eigenstate of two non-commuting (incompatible) operators, such as σ_x and σ_z. In practical terms, this implies that if a qubit possesses a definitive value of σ_x, then its σ_z value remains undefined, and the reverse is also true.

Accepting quantum theory as a comprehensive explanation of reality necessitates the conclusion that **it's impossible for both σ_x and σ_z values to be definite for the same qubit concurrently.** Einstein was notably uneasy with this notion, suggesting that quantum theory might not be all-encompassing and that observables σ_x and σ_z, can have concurrent definitive values, albeit we're only aware of one at any given time, thus **introducing the concept of hidden variables.**

In his EPR paper, Einstein presented several persuasive arguments in favor of "hidden variables" and it remained a significant critique of quantum theory's completeness for nearly three decades. From this perspective, **the uncertainty observed in quantum theory is simply due to our lack of knowledge about these "hidden variables" that exist in nature but are not acknowledged by the theory**. However, as mentioned above, in 1964, John Bell demonstrated that the hypothesis of local hidden variables could be experimentally tested and disproven. It is crucial to clarify that this discussion pertains specifically to what are termed local hidden variable theories, further elaborated in Section 6.7.

3. 6.2 Quantum correlations

A quantum correlation is a **mathematical description of how correlations between measurements of physical properties in quantum systems can be quantified, particularly in the context of entangled states**. It is the expected value of the product of the possible outcomes, i.e. the expected change in physical characteristics when a quantum system passes through an interaction site. We will now evaluate the expected value for a specific quantum state known as the **singlet state**[5], under the measurement of observables[6] A and B (by two parties Alice and Bob).

Consider two entangled qubits in the **singlet state**:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

and note that the **projector**[7] $|\psi\rangle\langle\psi|$ can be written as:

$$|\psi\rangle\langle\psi| = \frac{1}{4}(1\otimes 1 - \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y - \sigma_z\otimes\sigma_z)$$

where σ_x, σ_Y and σ_z are the previously discussed Pauli matrices.

Also recall that any **single-qubit observable** with values ±1 can be represented by the operator:

$$\vec{a}\cdot\vec{\sigma} = \alpha_x\sigma_x + \alpha_y\sigma_y + \alpha_z\sigma_z$$

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[5] **Singlet state**: denoted $|\psi\rangle$, is a particular entangled state of two particles where the measurements of certain properties (like spin along any axis) are perfectly anti-correlated. The expectation value $\langle\psi|\ A\otimes B\ |\psi\rangle$ is calculated to determine the average outcome of measuring observables A and B on this entangled state.

[6] [6] An **observable** A represents a measurable physical property, such as spin, position, momentum, or energy, with a numerical value. It extends to any basic measurement where each outcome has an associated numerical value. If λ_k is the numerical value associated with outcome $|e_k\rangle$, then the observable A is **represented** by the operator $A = \sum_k \lambda_k\,|e_k\rangle\langle e_k| = \sum_k \lambda_k P_k$,

[7] A **projector** is a Hermitian operator which is **idempotent** ($P^2 = P$). Reminder: Hermitian: ($P = P^\dagger$) The **rank** of P is given by tr(P).

where $\vec{a}$ is a unit vector in the three-dimensional Euclidean space.

a. Tensor product

Let's say that Alice and Bob both choose observables that we can characterize by vectors $\vec{a}$ and $\vec{b}$, respectively. If Alice measures the first qubit in our singlet state $|\psi\rangle$, and Bob the second, then the corresponding observable is described by the tensor product:

$$A \otimes B = (\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma})$$

The eigenvalues of A⊗B are the products of eigenvalues of A and B. Thus, A⊗B has two eigenvalues: +1, corresponding to the instances when Alice and Bob registered identical outcomes, i.e. (+1,+1) or (−1,−1); and −1, corresponding to the instances when Alice and Bob registered different outcomes, i.e. (+1,−1) or (−1,+1).

This means that the expected value of A⊗B, in any state, has a simple interpretation:

$$\langle A \otimes B \rangle = Pr(outcomes\ are\ the\ same) - Pr\ (outcomes\ are\ different)$$

This expression can take any real value in the interval [−1,1], where −1 means we have **perfect anti-correlations**, 0 means no **correlations**, and +1 means **perfect correlation.**

b. Trace[8]

We can evaluate the expectation value in the singlet state:

$$\begin{aligned}\langle\psi|\, A \otimes B\, |\psi\rangle &= \mathrm{tr}\,[\,(\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma})\ |\psi\rangle\langle\psi|] \\ &= -\frac{1}{4}\,\mathrm{tr}\,[(\vec{a} \cdot \vec{\sigma})\sigma_x \otimes (\vec{b} \cdot \vec{\sigma})\sigma_x + (\vec{a} \cdot \vec{\sigma})\sigma_y \otimes (\vec{b} \cdot \vec{\sigma})\sigma_y + (\vec{a} \cdot \vec{\sigma})\sigma_z \otimes (\vec{b} \cdot \vec{\sigma})\sigma_z]\ ^{9} \\ &= -\frac{1}{4}\,\mathrm{tr}[4(a_x b_x + a_y b_y + a_z b_z)\mathbf{1} \otimes \mathbf{1}] \\ &= -\,\vec{a} \cdot \vec{b}\end{aligned}$$

where we have used the fact that $\mathrm{tr}\,(\vec{a} \cdot \vec{\sigma})\sigma_k = 2a_k$ (for k=x, y, z). So, if Alice and Bob choose the same observable $\vec{a} = \vec{b}$ (meaning they measure along the same direction), then the

[8] The trace (tr) operation sums up the diagonal elements of a matrix, which is a step in calculating the expectation value.

[9] Simplification using Pauli Matrices: The Pauli matrices properties are used to simplify the expression which leads to a formula that depends on the dot product of the vectors $\vec{a}$ and $\vec{b}$ which represent the directions along which Alice and Bob measure spin.

expected value $\langle A \otimes B \rangle$ will be equal to -1, and their outcomes will *always* be opposite: whenever Alice registers +1 (resp. −1) Bob is bound to register −1 (resp. +1).

This mathematical description shows how quantum mechanics allows for the prediction of measurement outcomes in entangled systems as it highlights the non-classical quantum correlations (entanglement) that defy classical intuition. **The singlet state's perfect anti-correlation is a hallmark of quantum entanglement** as it leads to phenomena like the violation of Bell's inequalities, which challenge local realism[10] and classical notions of correlation.

4. 6.3 The CHSH inequality

So, Bell's theorem asserts that no theory of nature that relies on local realism can reproduce all the predictions of quantum mechanics (= Bell claims that the world must be nonlocal[11] and challenges local realism). To formalize his theorem, Bell derived inequalities that any local realistic theory must satisfy. Situations, specifically measurements on entangled quantum states, are predicted by quantum mechanics and he proves that these inequalities are violated. Introduced in 1969 by John Clauser, Michael Horne, Abner Shimony and Richard Holt (CHSH), we will now describe the most popular version of Bell's argument.

Firstly, we begin by making two very important assumptions:

1. **Hidden variables.** We assume that the outcomes of any measurement on any individual system are predetermined. This suggests that any probabilities we assign to describe the system simply indicate our lack of knowledge about these hidden variables. In other words, that observables have definite values.

2. **Locality.** Alice's choice of measurements (choosing between A_1 and A_2) does not affect the outcomes of Bob's measurement, and vice versa.

Secondly, we imagine a scenario where our two characters, Alice and Bob, have precise measuring devices and are stationed at two distant locations. They each have the option to measure one of two specific observables, which can result in either +1 and -1 outcome. Alice can choose between observables A_1 and A_2 and Bob between B_1 and B_2. There is a source that emits pairs of qubits that fly apart, one towards Alice and the other towards Bob. They indecently and randomly decide which observable to measure. This scenario allows us to

[10] **Local realism:** The idea that information about a system is localized to the system and that distant events cannot have instantaneous (faster than the speed of light) effects on it

[11] **Nonlocal:** This means that particles can instantaneously affect each other's state, no matter the distance separating them in ways that cannot be explained by signals traveling at or below the speed of light.

treat observables as random variables A_k and B_k (where k = 1,2) that can take on the value ±1. We can thus introduce a new random variable, the CHSH quantity:

$$\mathrm{S} = A_1\ (B_1 - B_2) + A_2(B_1 + B_2)$$

There are four potential outcomes for the pair $(B_1\ , B_2)$ and after examination, it becomes apparent that either $(B_1 \pm B_2)$ will result in 0 or ±2 (depending on if $B_1 = B_2$ or not). This means that looking at the four possible outcomes for the pair $(A_1\ , A_2)$, it is determined that $S = \pm 2$. But the average value of S must lie in between these two possible outcomes, i.e. $-2 \leqslant \langle S \rangle \leqslant 2$.

> There is absolutely *no quantum theory involved* in the CHSH inequality
>
> $$-2 \leqslant \langle S \rangle \leqslant 2$$
>
> because the CHSH inequality is not specific to quantum theory: it does not really matter what kind of physical process is behind the appearance of binary values of A_1, A_2, B_1, and B_2; it is merely a statement about correlations, and for all classical correlations we must have
>
> $$|\langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle| \leqslant 2$$
>
> (which is just another way of phrasing the CHSH inequality).

The assumptions are important here.

While the specifics of the locality principle will be further discussed only in 6.7 by Tasmim, a quick observation is important. In a universe governed by hidden variables, a prediction like "Bob will observe a +1 if he measures B_1 must hold a definitive truth value (true or false) before the actual measurement by Bob. Absent to the principle of locality, such predictions become murky because the outcome for B_1 might be influenced by Alice's choice to measure A_1 or A_2. This scenario is undesirable as it suggests the possibility of instantaneous signal exchange, implying that Alice's decision could instantly affect Bob's measurement outcomes which allows Bob to "see" Alice's actions immediately.

5. 6.4 Bell's theorem via CHSH

In the realm of quantum mechanics, the story of Alice and Bob continues as they explore the behavior of qubits through observables. To understand their experiments in the context of quantum mechanics, we translate their observations into a more formal language. The observables in question, denoted as $A_1\ A_{,2}, B_1, B_2$ are represented as 2x2 Hermitian matrices. Each of these matrices possesses two eigenvalues, +1 and -1. The expected value of the outcome, denoted ⟨S⟩, is calculated using 4x4 **CHSH matrix**, defined as:

$$\langle S \rangle = A_1 \otimes \ (B_1 - \ B_2) + A_2 \otimes (B_1 + \ B_2).$$

This equation sets the stage for evaluating ⟨S⟩ using quantum theory, leading to what is known as a CHSH test or Bell test.

a. CHSH Test and Quantum Measurements

The CHSH test involves measurements on a pair of qubits described by S. If the qubits are in the singlet state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, then as previously seen, the measurement outcome $\langle A \otimes B \rangle$ equals $-\vec{a} \cdot \vec{b}$. By choosing specific vectors $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$ (as shown in the figure below) the corresponding matrices satisfy

$$\langle A_1 \otimes B_1 \rangle = \langle A_2 \otimes B_1 \rangle = \langle A_2 \otimes B_2 \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle A_1 \otimes B_2 \rangle = \frac{1}{\sqrt{2}}$$

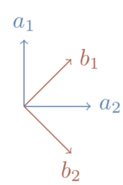

Figure 6.1: The relative angle between the two perpendicular pairs is 45°.

b. CHSH test Outcomes

Through this method, it's discovered that the outcome $\langle S \rangle = \langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle = -2\sqrt{2}$, a result that **starkly violates the CHSH inequality as it is strictly less than -2**. This inequality suggests that no local hidden variable theories can explain the observed behavior of entangled quantum system and indicates that the outcomes in quantum mechanics are inherently random and not merely a product of our ignorance.

c. Experimental Verification and Significance

The violation of the CHSH inequality is not just a theoretical prediction but has been confirmed through complex experiments. Initially, these experiments were monumental undertakings, but they have since become routine. Such experimental verifications underscore Bell's theorem: entangled quantum systems exhibit behaviors that defy explanation by local hidden variables, thereby reinforcing the randomness inherent in quantum mechanics.

d. Ensuring Quantum Security through CHSH Tests

In practice, conducting a CHSH test with careful control over the locality conditions (e.g., ensuring Alice and Bob are sufficiently distant to prevent any causal interaction between their measurements) provides a powerful method to verify the non-classical and unpredictable nature of quantum systems. This verification is crucial not only for understanding the fundamental aspects of quantum mechanics but also for securing quantum protocols against potential eavesdroppers. Thus, the CHSH test plays a pivotal role in both quantum theory and the practical implementation of quantum technologies, as further discussed in subsequent sections.

6. CONCLUSION

Bell's theorem posits that the behavior of entangled quantum systems cannot be explained by local hidden variables as we saw thanks to the CHSH inequality that was violated. In other words, outcomes in quantum mechanics really are random and it is not simply our lack of knowledge about some background process.
In the words of the physicist John Stewart Bell, for whom this family of results is named, "**If [a hidden-variable theory] is local it will not agree with quantum mechanics, and if it agrees with quantum mechanics it will not be local.”**

Definitions

- **Physician language vs mathematician language**
  - Observable vs Hermitian operator
  - Value vs eigenvalue
  - Measurable physical quantity vs observable

- **Operator**
  - In quantum theory, an operator is a mathematical entity that acts on elements of a Hilbert space, which represents the state space of a quantum system, to produce other elements within that space
  - Operators as observables
    - When an operator that represents an observable acts on a quantum state (if the state is an eigenvector of that operator), the outcome is an eigenvalue multiplied by the original state. These eigenvalues are the possible results of measuring the corresponding physical quantity. For instance, measuring the energy of a system involves the Hamiltonian operator, whose eigenvalues represent possible energy levels

- **Compatible operators**
  - Consider an eigenbasis of A (an operator), where each vector $|e_k\rangle$ is an eigenvector. If A and B commute (AB = BA), any vector B $|e\rangle$ is also an eigenvector of A, meaning A and B share an eigenbasis. Conversely, if A and B share an eigenbasis, they commute (AB = BA). We call A and B compatible if they commute, and incompatible otherwise
- **Eigenbasis**
  - if a basis $\{|e1\rangle,\ldots,|en\rangle\}\{|e1\rangle,\ldots,|en\rangle\}$ is such that each $|ek\rangle|ek\rangle$ is an eigenvector of an operator AA, then we call it an **eigenbasis of** AA
  -
- An **observable** A represents a measurable physical property, such as spin, position, momentum, or energy, with a numerical value. It extends to any basic measurement where each outcome has an associated numerical value. If λ_k is the numerical value associated with outcome $|e_k\rangle$, then the observable A is **represented** by the operator

$$A = \sum_k \lambda_k \, |e_k\rangle\langle e_k| = \sum_k \lambda_k P_k,$$

where λ_k now corresponds to the eigenvalue of the eigenvector $|e_k\rangle$ or the projector P_k.

- A **projector** is a Hermitian operator which is **idempotent** ($P^2 = P$). Reminder: Hermitian: ($P = P^\dagger$) The **rank** of P is given by tr(P).