Math Seminar Notes - Cara Zhu

9.7 Composition of quantum channels

We mentioned that quantum channels are combinations of

1. Adding a physical system in a fixed state (via tensoring),
2. Unitary transformations, and
3. Discarding a physical system (taking a partial trace).

Each of these operations admits an operator-sum decomposition. This might be obvious for the unitary transformations (second), but less so for the other two.

- **Adding a system**
  Any quantum system can be expanded by bringing in an auxiliary system in a fixed state $|a\rangle$. This transformation takes vectors in the Hilbert space associated with the original system and tensors them with a fixed vector $|a\rangle$ in the Hilbert space associated with the auxiliary system:

$$|\psi\rangle \longmapsto |a\rangle \otimes |\psi\rangle = (|a\rangle \otimes \mathbf{1})|\psi\rangle.$$

  In terms of density operators, we write this "expansion" transformation as

$$\begin{aligned} \rho \longmapsto \rho' &= |a\rangle\langle a| \otimes \rho \\ &= (|a\rangle \otimes \mathbf{1})\rho(\langle a| \otimes \mathbf{1}) \\ &= V\rho V^\dagger \end{aligned}$$

  Where $V = |a\rangle \otimes 1$
  We notes that $V^\dagger V = \langle a|a\rangle \otimes 1 = 1$ is the identity in the Hilbert space associated with the system, and so V is an isometry. Indeed, this transformation is an isometric embedding.

  Def: isometry: a mapping of a metric space onto another or onto itself so that the distance between any two points in the original space is the same as the distance between their images in the second space. (Necessary to keep?)

- **Discarding a system**

Conversely, given a composite system in state ρ, we can discard one of its subsystems. The partial trace over an auxiliary system can be written in the Kraus representation as

$$\begin{aligned}\rho \longmapsto \rho' &= \operatorname{tr}_{\mathcal{A}} \rho \\ &= (\operatorname{tr} \otimes \mathbf{1})\rho \\ &= \sum_i (\langle i| \otimes \mathbf{1})\rho(|i\rangle \otimes \mathbf{1}) \\ &= \sum_i E_i \rho E_i^\dagger\end{aligned}$$

where the vectors $|i\rangle$ form an orthonormal basis in the Hilbert space associated with the auxiliary system.

Any sequential composition of two quantum channels ε and F with Kraus operators {Ai}i∈I and {Bj}j∈J (respectively) is another quantum channel described by the Kraus operators {BjAi}i∈I & j∈J

representation: let

$$\mathcal{E} = \sum_i A_i \cdot A_i^\dagger$$
$$\mathcal{F} = \sum_j B_j \cdot B_j^\dagger$$

where $\sum_i A_i^\dagger A_i = \sum_j B_j^\dagger B_j = \mathbf{1}$; then the sequential composition of $\mathcal{E}$ followed by $\mathcal{F}$ can be written as

$$\mathcal{F} \circ \mathcal{E} = \sum_{i,j} (B_j A_i) \cdot (B_j A_i)^\dagger$$

so that the $B_j A_i$ are the Kraus operators associated with the new channel $\mathcal{F} \circ \mathcal{E}$, where the normalisation condition (or completeness relation) follows from

$$\begin{aligned}\sum_{i,j} (B_j A_i)^\dagger (B_j A_i) &= \sum_i A_i^\dagger \left(\sum_j B_j^\dagger B_j \right) A_i \\ &= \sum_i A_i^\dagger A_i \\ &= \mathbf{1}.\end{aligned}$$

So why did we call the above composition “sequential”? It’s how we always compose functions.

- There is another way of composition called **parallel composition.**
- If we have systems A and B with channels εA acting on A and εB acting on B, then the parallel composition is denoted by εA ⊗ εB, acting on the joint system of A ⊗ B, and with Kraus operators given by the Ai ⊗ Bj.

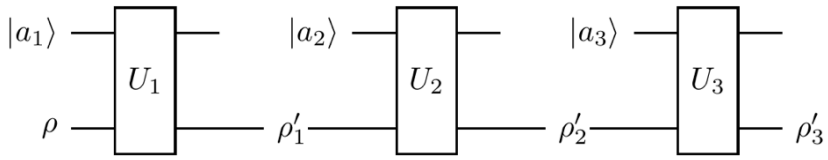
The normalization condition follows from a simple calculation:

$$\sum_{i,j}(A_i \otimes B_j)^\dagger(A_i \otimes B_j) = \sum_{i,j} A_i^\dagger A_i \otimes B_j^\dagger B_j$$
$$= \mathbf{1}_A \otimes \mathbf{1}_B.$$

Remark: When we compose quantum channels, each channel needs its own independent ancilla - do not share ancillas between different channels.

What are the ps?

For example, say we have three channels, $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$, with $\mathcal{E}_i$ defined by the unitary U_i and the state $|a_i\rangle$ of its ancilla. Then the (sequential) composition $\mathcal{E}_3 \circ \mathcal{E}_2 \circ \mathcal{E}_1$ is given by



where each $\mathcal{E}_i$ has its own associated ancilla $|a_i\rangle$. For more on this, see Section 9.12.2, where we talk about **Markov approximation**.

9.8 Completely positive trace-preserving map

Recall density operators ρ are positive (semi-definite) Hermitian operators that tr(ρ) = 1 where "positive" means ⟨v|ρ|v⟩≥0 for all |v⟩ (or, equivalently, that all its eigenvalues are non-negative real numbers).

It is easy to verify that quantum channels preserve positivity and trace, but the converse is not true! That is, there are linear maps that preserve positivity and the trace, but are not quantum channels, and thus which are not "physical operations".

- Ex: The matrix transpose operation ρ ↦ ρ T is a good example of such an unphysical operation: it preserves both trace and positivity, and if ρ is a density matrix then so too is ρT, but we will show that the transpose cannot be written in the Stinespring (or the Kraus) form; it is not induced by a unitary operation on some larger Hilbert space, and it cannot be physically implemented.
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- So, the question becomes what is the class of physically admissible maps? That is, how can we classify which maps are quantum channels and which are not?

1. We say that a linear operator f:H→H′between Hilbert spaces is bounded if there exists some real number B>0 such that $\| f(x) \|_{H'} \leqslant B \| x \|_{H}$ for every vector x∈H. Given a pair of Hilbert spaces H and H' , we denote the set of bounded linear operators from H to H' by B(H, H'). We write B(H) as a shorthand for B(H,H).??
2. Then, a quantum channel ε is a specific type of map that ε: B(H) → B(H')
3. Consider an ensemble of systems, with a fraction p1 of them in the state ρ1, and the remaining p2 of them in the state ρ2. The overall ensemble is described by
$$\rho = p1\rho1+p2\rho2.$$
If we apply ε to each member of the ensemble individually, then the overall ensemble will be described by the density operator ρ′ = ε(ρ), which should be given by
$$\rho' = p1\varepsilon(\rho1) + p2\varepsilon(\rho2)$$
We conclude that ε must be a linear map.
4. Next, since ε must map density operators to density operators, it has to be both **positive** (ε(ρ)⩾0 whenever ρ⩾0) and **trace preserving** (trε(ρ) =trρ for all ρ).
5. The final characteristic is **complete positivity**. Complete positivity of ε implies positivity, but the converse does not hold: there are maps which are positive but not completely positive. The matrix transpose operation ρ⟶ρ T is a classic example of such a map. (WHY?)

Matrix transpose example:

- Consider the transpose operation on a single qubit: T:$|i\rangle\langle j| \mapsto |j\rangle\langle i|$ (for i,j∈{0,1}). It preserves both trace and positivity.
- However, if the input qubit is part of a two qubit system, initially in the entangled state $|\Omega\rangle = 1/\mathrm{sqrt}(2)(|0\rangle|0\rangle+|1\rangle|1\rangle)$, and the transpose is applied to only one of the two qubits (say, the second one), then the density matrix of the two qubits evolves under the action of the partial transpose 1⊗T as

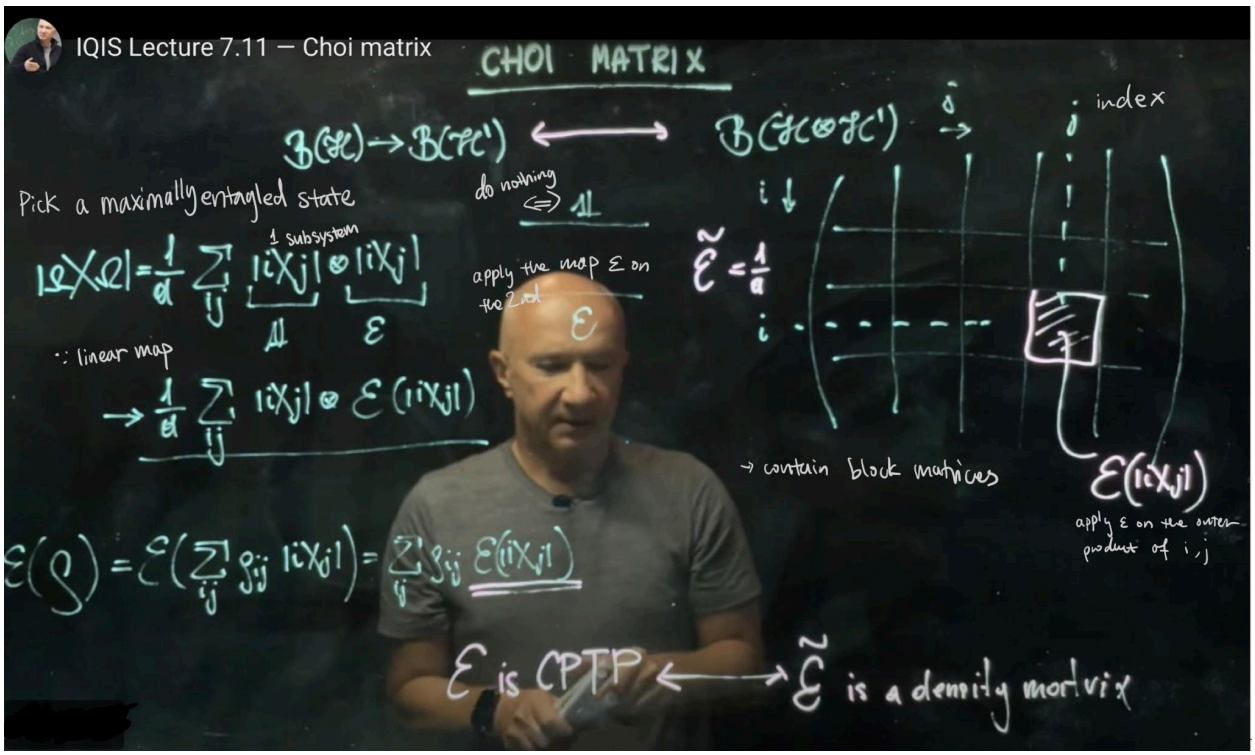
$$|\Omega\rangle\langle\Omega| = \frac{1}{2}\sum_{i,j}|i\rangle\langle j| \otimes |i\rangle\langle j| \overset{\mathbf{1}\otimes T}{\longmapsto} \frac{1}{2}\sum_{i,j}|i\rangle\langle j| \otimes T(|i\rangle\langle j|)$$
$$= \frac{1}{2}\sum_{i,j}|i\rangle\langle j| \otimes |j\rangle\langle i|.$$

- The output is known as the SWAP matrix, since it describes the SWAP operation:$|i\rangle\langle j| \mapsto |j\rangle\langle i|$

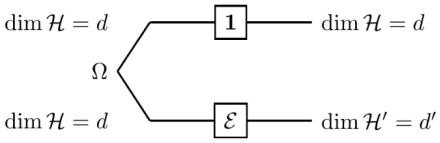
9.9 Chanel-state duality

Choi Matrix: denoted ε tilda

- Another way of representing the linear map: ε: B(H) → B(H')



Pictorially, we might represent this by something like



Let $\widetilde{\mathcal{E}}$ be the Choi matrix of a linear map $\mathcal{E}: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$. Then

1. $\mathcal{E}$ is completely positive if and only if $\widetilde{\mathcal{E}}$ is positive semi-definite.
2. $\mathcal{E}$ is trace preserving if and only if $(\mathbf{1} \otimes \mathrm{tr})\widetilde{\mathcal{E}} = \frac{1}{d}\mathbf{1}$.
3. $\mathcal{E}$ sends the identity operator to the identity operator if and only if $(\mathrm{tr} \otimes \mathbf{1})\widetilde{\mathcal{E}} = \frac{1}{d}\mathbf{1}$.
4. $\mathcal{E}$ sends Hermitian operators to Hermitian operators and only if $\widetilde{\mathcal{E}}$ is Hermitian.

9.10 The math of "can" and "cannot"

So we have the convex-sum of expression of a linear map

$$\mathcal{E}(|i\rangle\langle j|) = p|j\rangle\langle i| + (1-p)\delta_{ij}\frac{1}{2}|i\rangle\langle j|$$

Where $0 \leq p \leq 1$

So we take the input state ρ and either

- (i) apply the transpose, with probability p
- or (ii) replace it with the maximally mixed state, with probability (1−p).

But because the transpose operation is not completely positive it is not physically admissible.

But this does not mean that the map ε itself cannot be implemented

- So there are 2 cases
    1. P = 0, which correspond to (ii) - just replacing the input with the maximally mixed state
    2. P increases from 0 to 1: at some critical point the map switches from **completely positive** to **merely positive**. In order to find this critical value of p, we first calculate ε(|i⟩⟨j|) for i, j∈{0,1} as follows:

$$\begin{aligned}
|0\rangle\langle 0| &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \overset{\mathcal{E}}{\longmapsto} \begin{bmatrix} \frac{1+p}{2} & 0 \\ 0 & \frac{1-p}{2} \end{bmatrix} \\
|0\rangle\langle 1| &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \overset{\mathcal{E}}{\longmapsto} \begin{bmatrix} 0 & 0 \\ p & 0 \end{bmatrix}, \\
|1\rangle\langle 0| &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \overset{\mathcal{E}}{\longmapsto} \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} \\
|1\rangle\langle 1| &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \overset{\mathcal{E}}{\longmapsto} \begin{bmatrix} \frac{1-p}{2} & 0 \\ 0 & \frac{1+p}{2} \end{bmatrix},
\end{aligned}$$

We can then write down the Choi matrix:

$$\tilde{\mathcal{E}} = \frac{1}{2}\begin{bmatrix} \mathcal{E}(|0\rangle\langle 0|) & \mathcal{E}(|0\rangle\langle 1|) \\ \mathcal{E}(|1\rangle\langle 0|) & \mathcal{E}(|1\rangle\langle 1|) \end{bmatrix} = \frac{1}{2}\left[\begin{array}{cc|cc} \frac{1+p}{2} & 0 & 0 & 0 \\ 0 & \frac{1-p}{2} & p & 0 \\ \hline 0 & p & \frac{1-p}{2} & 0 \\ 0 & 0 & 0 & \frac{1+p}{2} \end{array}\right]$$

which lets us apply channel-state duality: ε is completely positive (and hence physically realizable) if and only if $\varepsilon\sim \geqslant 0$, and the latter is true only when $p \leqslant 1/3$(note that the eigenvalues of ε~ are ¼(1+p) and ¼ (1-3p)

9.11 Kraus operator

Channel-state duality gives us more than just a one-to-one correspondence between states ε~ and channels ε - it also gives a one-to-one correspondence between vectors in the statistical ensemble ε~ and the Kraus operators in the decomposition of ε.

We already know that if two mixtures[188] $(p_k, |\psi_k\rangle)$ and $(q_l, |\phi_l\rangle)$ are described by the same density operator

$$\sum_k |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k| = \tilde{\mathcal{E}} = \sum_l |\tilde{\phi}_l\rangle\langle\tilde{\phi}_l|$$

(where $|\tilde{\psi}_k\rangle = \sqrt{p_k}|\psi_k\rangle$ and $|\tilde{\phi}_l\rangle = \sqrt{q_l}|\phi_l\rangle$) then they are related to one another: there exists some unitary R such that

$$|\tilde{\psi}_k\rangle = \sum_l R_{kl}|\tilde{\phi}_l\rangle.$$

So how many Kraus operators do we really need?

- Channel-state duality tells us that the minimal number of Kraus operators needed to express ε:B(H)→B(H′) in the operator-sum form is given by the rank of its Choi matrix ε~
- In fact, this minimal set of Kraus operators corresponds to the spectral decomposition of E~.

Remark:

A linear map $\mathcal{E}: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ is completely positive if and only if it admits an operator–sum decomposition of the form

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger.$$

If this is the case, then this decomposition has the following properties:

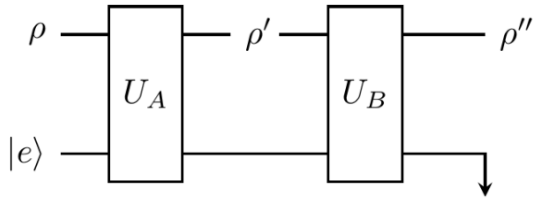
- $\mathcal{E}$ is trace preserving if and only if $\sum_k E_k^\dagger E_k = \mathbf{1}$.
- Two sets of Kraus operators $\{E_k\}$ and $\{F_l\}$ represent the same map $\mathcal{E}$ if and only if there exists a unitary R such that $E_k = \sum_l R_{kl} F_l$ (where the smaller set of the Kraus operators is padded with zeros, if necessary).

9.12 Remarks and exercises

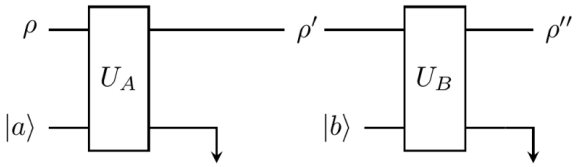
The Markov approximation

States: environment has essentially no memory.

- For example, suppose that our system is an atom, surrounded by the electromagnetic field (which serves as the environment). Let the field start in the vacuum state. If the atom emits a photon into the environment, then the photon quickly propagates away, and the immediate vicinity of the atom appears to be empty, i.e. resets to the vacuum state. In this approximate model, we assume that the environment quickly forgets about the state resulting from any previous evolution.



- Here the system, initially in state ρ, undergoes two stages of evolution, and the environment, initially in state $|e\rangle$, is not discarded after the first unitary evolution UA; the environment persists and participates in the second unitary evolution UB. In this case the evolutions ρ↦ρ′ and ρ↦ρ″are both well defined quantum channels, but the evolution ρ′↦ρ″ is not



- In this case, all three evolutions are well-defined: if εA describes the evolution from ρ↦ρ′, and εB from ρ↦ρ″, then the composition εA ∘ εB describes the evolution from ρ′↦ρ″