Quantum Gates

Review:

A gate is a feature of a computer that performs some operation on a bit (either 1 or 0). A not gate in particular flips the bit: if it is a 1, it becomes a 0. If it is a 0, it becomes a 1. Quantum gates are devices that perform some **unitary** operation on a qubit over a certain period. This means that there is some input and some output. Specifically, we are concerned with the math *behind* the output. What exactly is possible?

We represent these gates with 2-by-2 matrices acting on the vector representation of a qubit.

Pauli operators represent special gates with specific nice properties:

Identity	$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{array}{rcr} \lvert 0\rangle & \longmapsto & \lvert 0\rangle \\ \lvert 1\rangle & \longmapsto & \lvert 1\rangle \end{array}$
Bit-flip	$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{array}{rcr} \lvert 0\rangle & \longmapsto & \lvert 1\rangle \\ \lvert 1\rangle & \longmapsto & \lvert 0\rangle \end{array}$
Bit-phase-flip	$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	$\begin{array}{rcr} \lvert 0\rangle & \longmapsto & i\lvert 1\rangle \\ \lvert 1\rangle & \longmapsto & -i\lvert 0\rangle \end{array}$
Phase-flip	$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{array}{rcr} \lvert 0\rangle & \longmapsto & \lvert 0\rangle \\ \lvert 1\rangle & \longmapsto & -\lvert 1\rangle \end{array}$

Bit-flip is analogous to a NOT gate in classical computing. The rest have no analogs, but this gives us an idea of what the gates should represent.

Beam Splitters:

A symmetric beam splitter is a cube that splits light into two beams: one that reflects, acting as a sort of mirror, and one that absorbs, allowing the light to pass through.

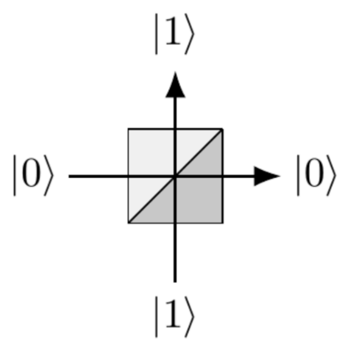


Figure 3.1: A symmetric beam-splitter, with input ports on the bottom and the left sides, and output ports on the top and the right sides.

Great. But what's happening at the photon level?

Individual photons don't split into two. Each photon hits the exits at the same probability: 0.5. This makes sense based on classic physics and probability, but this general trend doesn't really carry on.

Recall the double slit experiment: the moral of the story is that qubits don't generally follow "basic" rules of physics and probability.

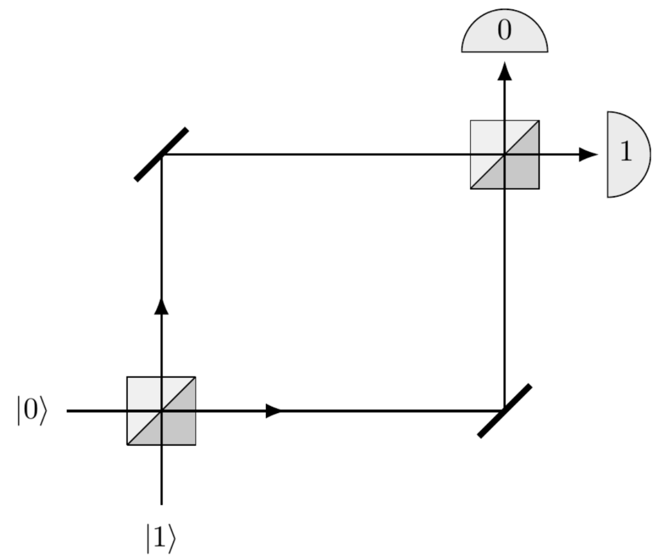


Figure 3.2: Two beam-splitters with mirrors, arranged so that the photon travels through both, along with two detectors. We label the detectors in such a way that, if a photon enters input $|j\rangle$ and is *transmitted* (not reflected) through both beam-splitters, then it is detected by detector j.

Consider the above beam splitter contraption. If we send a photon through, we'd expect there to be an equal probability of hitting both the 0 and 1 outputs. The first splitter splits the probability but the mirrors recombine it until it's split again at the end, right? The Kolmogrov additivity axiom says that we can add mutually exclusive events.

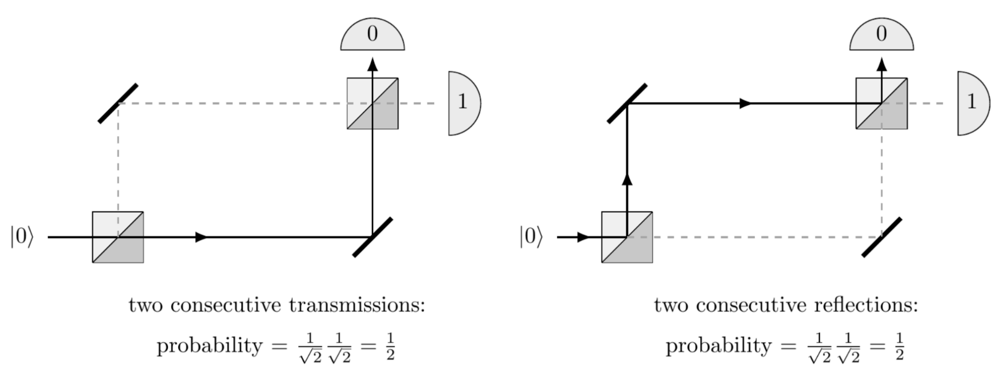


Figure 3.3: The two possible classical scenarios. Note that this is *not* what actually happens in the real physical world!

Take for example the above diagram: there are two ways for us to hit the 0 output. There are 4 possible routes for the photon to take. Each reflection happens with a probability of ½, as does each transmission. We would think that if we fire a photon from the 0 input and check the probability of it hititng the 0 output, we could add the probability of double transmissions (½)(½) to the probability of double reflections (½)(½) and end up with a total probability of ½. We would expect both the 0 and the 1 to be hit at ½ probability. However, this is not what happens in reality!!! The input from 0 always hits 1 and the input from 1 always hits 0. Thus, the beam splitter is essentially a $\sqrt{NOT}$ gate (there are many different ways to implement such a gate). We can describe the action of the beam splitter using a matrix. Consider the following:

$$B = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Each B_{lk} represents a different path for the particle to take (the second number path first followed by the first number path). Reflections happen with amplitude $i/\sqrt{2}$, while transmissions happen with amplitude $1/\sqrt{2}$. Let's try to then figure out the amplitude of a particle from the input port 0 hitting the output port 0. We notice that multiplying the terms with i results in -½ while the non i terms results in ½. Indeed, there is a 0 probability that the photon goes along this path. It is the sum of both possible paths (two consecutive reflections and two consecutive transmissions).

The textbook seems to think that it's important to note that **amplitudes can cancel each other out, unlike probabilities**. We actually have a *negative* amplitude because of the i terms, thus resulting in that zero chance of going from input port 0 to output port 0. If we find the path from 0 to 1, we find that there is a total amplitude of i, giving a probability of 1 (as |i|^2=1, as discussed in the first lecture).

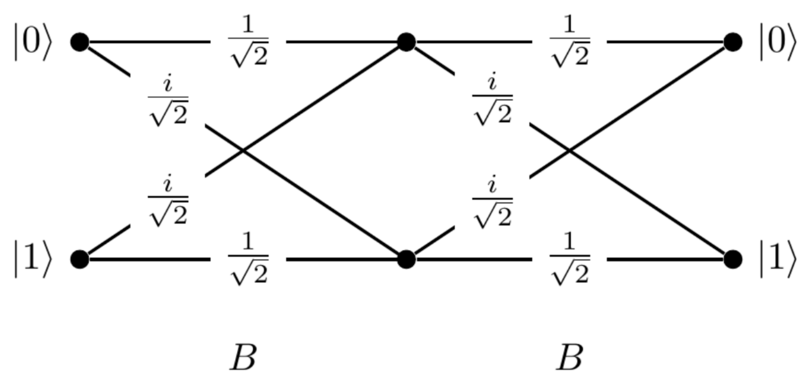


Figure 3.4: All possible transitions and their amplitudes when we compose two beam-splitters, as described by the matrix B above.

To finish showing that this is a $\sqrt{NOT}$ gate, we do the following calculations

$$BB = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = iX$$

where

$$X = \mathtt{NOT} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note what has happened! Two beam splitters together function essentially as a NOT gate does. In other words, the beam splitter action squared is equal to the NOT gate, and we have a proper $\sqrt{NOT}$ gate in mathematical terms. To summarize, a $\sqrt{NOT}$ gate can be represented with the matrix B.

Quantum Interference:

A fun little beam-splitter system is **Mach-Zehnder Interferometer**, which is essentially the double beam-splitter we looked at earlier with phase shifters (this could be like glass or something).

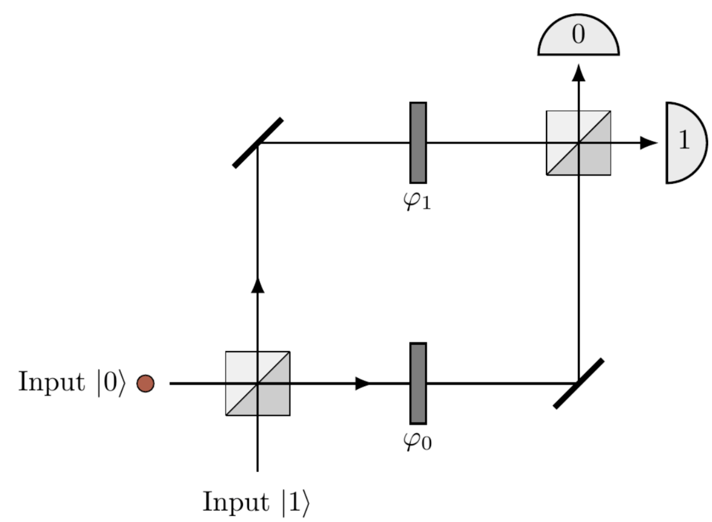


Figure 3.5: The Mach–Zehnder interferometer, with the input photon represented by the coloured dot. This experimental set-up is named after the physicists Ludwig Mach and Ludwig Zehnder, who proposed it back in 1890s.

Here, the ϕ_0 and ϕ_1 represent the thickness of the phase shifters and are measured in the units of the of the photon's wavelength multiplied by 2π . The phase shifters shift the probability amplitude by a factor of $e^{i\phi_0}$ and $e^{i\phi_1}$ respectively. The other difference between a Mach-Zehnder Interferometer and the beam splitter that we looked at earlier is the probability split: these splitters can be **non-symmetric**, meaning that they don't split at an equal probability. They now reflect with some fixed probability of $i\sqrt{R}$ and transmit at probability $\sqrt{T}$, and we can check that the absolute values of these numbers squared equals 1.

Let's now look at an example of the following interferometer. Say we call a probability amplitude U_{ij} where j represents the value of the input port and i represents the value of the output port. For U_{00}. We note that as before, there are two different paths for which a photon can begin at the 0 input and end at the 0 output. There is the double reflection and ϕ_1 path and the double transmission and ϕ_0 path. So when we multiply by the correct phase shifters, we end up with:

$$U_{00} = \sqrt{T}e^{i\varphi_0}\sqrt{T} + i\sqrt{R}e^{i\varphi_1}i\sqrt{R}$$

we square the absolute value of the amplitude to find the overall probability:

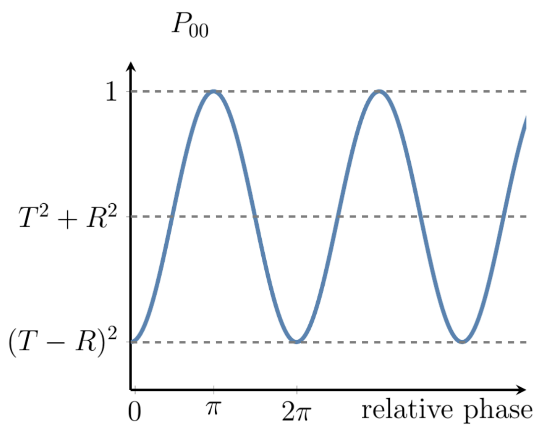
$$\begin{aligned} P_{00} &= \left|\sqrt{T}e^{i\varphi_0}\sqrt{T} + i\sqrt{R}e^{i\varphi_1}i\sqrt{R}\right|^2 \\ &= \left|Te^{i\varphi_0} - Re^{i\varphi_1}\right|^2 \\ &= T^2 + R^2 - 2TR\cos(\varphi_1 - \varphi_0). \end{aligned}$$

Let's take a closer look at this equation: the $T^2 + R^2$ part is exactly what we would expect from classic physics. There are either 2 consecutive transmissions T^2 or 2 consecutive reflections R^2. However, this is not what happens in reality: the

$2TRcos(\phi_1 - \phi_2)$ term is very very important. We define the **relative phase** as

$\phi = \phi_1 - \phi_2$, and depending on the value of this, our probability can range from

$(T - R)^2$ to 1, as this term affects the cosine value:



Let's consider what this interferometer looks like using a familiar diagram:

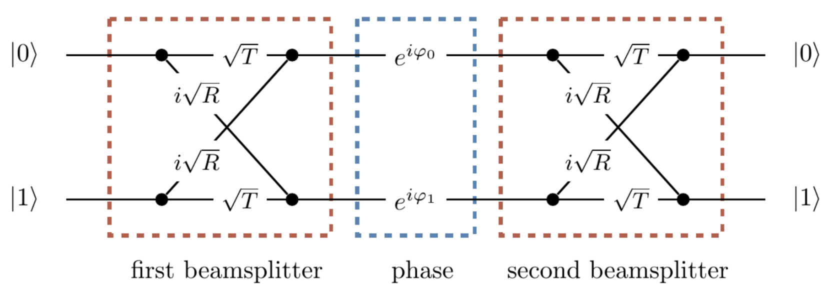


Figure 3.6: The Mach–Zehnder interferometer represented as an abstract diagram.

Let's try to caclulate the probability amplitude and corresponding probability of U_{10}.

$$U_{10} = \sqrt{T}e^{i\varphi_0}i\sqrt{R} + i\sqrt{R}e^{i\varphi_1}\sqrt{T}$$

(reminder, we start at 0 and end up at 1 and thus calculate the two paths that get us there, multiplying all the terms along the way). We find that the probability ends up being:

$$
\begin{aligned}
P_{10} &= \left| \sqrt{T} e^{i\varphi_0} i\sqrt{R} + i\sqrt{R} e^{i\varphi_1} \sqrt{T} \right|^2 \\
&= 2RT + 2RT\cos(\varphi_1 - \varphi_0).
\end{aligned}
$$

Again, the term without the cosine is what we'd expect in classic physics, while the second term makes things weird, We can add up both of our probabilities to find that

$$
P_{00} + P_{10} = R^2 + 2RT + T^2 = (T + R)^2 = 1.
$$

as a little sanity check. The probability does equal 1 as expected. Oftentimes, Mach-Zehnder interferometers are **symmetric**, meaning that reflects and transmissions both happen at a probability of ½ . We can represent them as follows:

$$
U = \begin{bmatrix} -\sin\varphi/2 & \cos\varphi/2 \\ \cos\varphi/2 & \sin\varphi/2 \end{bmatrix}
$$

where $\varphi = \varphi_1 - \varphi_0$.

Pauli Matrices:

In mathematics, we use matrices to represent **vector spaces**. If we have a 2x2 vector space, its basis can be represented with four matrices, each having a 1 in one of the four spots. Using linear combinations of these four matrices, we can represent any 2x2 matrix. However, we can also use the Pauli matrices;

Identity	$\mathbf{1} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\|0\rangle \longmapsto \|0\rangle$ $\|1\rangle \longmapsto \|1\rangle$
Bit-flip	$X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$	$\|0\rangle \longmapsto \|1\rangle$ $\|1\rangle \longmapsto \|0\rangle$
Bit-phase-flip	$Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$	$\|0\rangle \longmapsto i\|1\rangle$ $\|1\rangle \longmapsto -i\|0\rangle$
Phase-flip	$Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$	$\|0\rangle \longmapsto \|0\rangle$ $\|1\rangle \longmapsto -\|1\rangle$

These matrices are **Hermitian,** meaning that the transposes of the conjugates are equal to the matrices. They are also unitary (quantum operations are reversible), square to the identity matrix, and **anti-commute** (AB = -BA).

Cool. What does this look like? If we add linear combinations of the matrices, we end up with:

$$\begin{aligned} A &= \begin{bmatrix} a_0 + a_z & a_x - ia_y \\ a_x + ia_y & a_0 - a_z \end{bmatrix} \\ &= a_0\mathbf{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z. \end{aligned}$$

for some complex numbers a_0, a_x, a_y, and a_z.

This looks like a dot product! And indeed, we can simplify the above matrix representation to:

$= a_0 1 + \vec{a} * \vec{\sigma}$

The **multiplication rule:**

$$(\vec{a}\cdot\vec{\sigma})\,(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\,\mathbf{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}.$$

Next, we have to review the **trace** of a matrix, i.e, the sum of the diagonals. We find that tr(ab +cd) = atr(b) + ctr(d). , and that trace is invariant under cyclic permutation, meaning that tr(ABC) = tr(BCA) = tr(CAB).

Finally, we define an **inner product** (generalization of the dot product) that will become important later on.

The **Hilbert–Schmidt product** of A and B is given by

$$(A|B) = \frac{1}{2} \operatorname{tr} A^{\dagger} B.$$

This weird cross thing refers to the **Hermitian.**