Enumerative invariants and birational geometry
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Lectures by Various

Columbia University

Disclaimer

These notes were taken during the lectures using `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. Also, notation may differe between lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

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Seminar Website: `https://math.columbia.edu/~plei/s24-birat.html`

References used

We used the following references in the seminar:

1. The main references for Section 1.1 are [Giv04; Dub96; Coa03].
2. The reference for Section 1.2 is [CG07].
3. The reference for Section 1.3 is [Iri17].
4. The main references for Section 1.4 are [CIJ18; AGV08; Tse10]
5. The reference for Section 1.5 is [CIJ18, Section 3].
6. The reference for Section 2.1 and Section 2.2 is [CIJ18].
7. The reference for Section 2.3 is [PSW24]. Also see the very similar [LSW24].
8. The reference for Chapter 3 is [IK24].
9. The reference for Chapter 4 is [Iri23].

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Preliminaries

1.1 Givental formalism (Patrick, Feb 01)

1.1.1 Introduction Let X be a smooth projective variety. Then for any $g, n \in \mathbb{Z}_{\geq 0}, \beta \in H_2(X, \mathbb{Z})$, there exists a moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ (Givental's notation is $X_{g,n,\beta}$) of *stable maps* $f\colon C \to X$ from genus-g, n-marked prestable curves to X with $f_*[C] = \beta$. It is well-known that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \in A_\delta(\overline{\mathcal{M}}_{g,n}(X,\beta)), \qquad \delta = \int_\beta c_1(X) + (\dim X - 3)(1-g) + 3.$$

In addition, there is a universal curve and sections

$$\mathcal{C} \underset{\sigma_i}{\overset{\pi}{\rightleftarrows}} \overline{\mathcal{M}}_{g,n}(X,\beta).$$

In this setup, there are tautological classes

$$\psi_i := c_1(\sigma_i^* \omega_\pi) \in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta)).$$

This allows us to define individual Gromov-Witten invariants by

$$\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \phi_i \cdot \psi_i^{a_i}.$$

These invariants satisfy various relations. The first is the *string equation:*

$$\langle \tau_0(1) \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n+1,\beta} = \sum_{i=1}^n \langle \tau_{a_i-1}(\phi_i) \prod_{j \neq i} \tau_{a_j}(\phi_j) \rangle^X_{g,n,\beta}.$$

The next is the *dilaton equation:*

$$\langle \tau_1(1) \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n+1,\beta} = (2g-2+n) \langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n,\beta}.$$

Finally, we have the *divisor equation* when one insertion is a divisor $D \in H^2(X)$:

$$\begin{aligned} \langle \tau_0(D) \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n+1,\beta} = & \left(\int_\beta D \right) \cdot \langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n,\beta} \\ & + \sum_{i=1}^n \langle \tau_{a_i-1}(\phi_i \cdot D) \prod_{j \neq i} \tau_{a_j}(\phi_j) \rangle^X_{g,n,\beta}. \end{aligned}$$

It is often useful to package Gromov-Witten invariants into various generating series.

Definition 1.1.1. The *quantum cohomology* $QH^*(X)$ of X is defined by the formula

$$(a \star_t b, c) := \sum_{\beta,n} \frac{Q^\beta}{n!} \langle a, b, c, t, \ldots, t \rangle^X_{0,3+n,\beta}$$

for any $t \in H^*(X)$. This is a commutative and associative product.

The *small quantum cohomology* is obtained by setting $t = 0$ and the ordinary cohomology is obtained by further setting $Q = 0$.

Remark 1.1.2. Convergence of the formula does not hold in general, so quantum cohomology needs to be treated as a formal object.

Definition 1.1.3. Let ϕ_i be a basis of $H^*(X)$ and ϕ^i be the dual basis. Then the *J-function* of X is the cohomology-valued function

$$J_X(t,z) := z + t + \sum_i \sum_{n,\beta} \frac{Q^\beta}{n!} \langle \frac{\phi_i}{z - \psi}, t, \ldots, t \rangle^X_{0,n+1,\beta} \phi^i.$$

Definition 1.1.4. The *genus-*0 *GW potential* of X is the (formal) function

$$\mathcal{F}^X(t(z)) = \sum_{\beta,n} \frac{Q^\beta}{n!} \langle t(\psi), \ldots, t(\psi) \rangle^X_{0,n,\beta}.$$

The associativity of the quantum product is equivalent to the PDE

$$\sum_{e,f} \mathcal{F}^X_{abe} \eta^{ef} \mathcal{F}_{cdf} = \sum_{e,f} \mathcal{F}^X_{ade} \eta^{ef} \mathcal{F}^X_{bcf}$$

for any a, b, c, d, which are known as the *WDVV equations*. Here, we choose coordinates on $H^*(X)$ and set $z = 0$ (only consider primary insertions). In addition, set η_{ef} to be the components of the Poincaré pairing and let η^{ef} be the inverse matrix.

1.1.2 Frobenius manifolds

A Frobenius manifold can be thought of as a formalization of the WDVV equations.

Definition 1.1.5. A *Frobenius manifold* is a complex manifold M with a flat symmetric bilinear form $\langle -, - \rangle$ (meaning that the Levi-Civita connection has zero curvature) on TM and a holomorphic system of (commutative, associative) products $\star_t$ on T_tM satisfying:

1. The unit vector field $\mathbf{1}$ is flat: $\nabla \mathbf{1} = 0$;
2. For any t and $a, b, c \in T_tM$, $\langle a \star_t b, c \rangle = \langle a, b \star_t c \rangle$;
3. If $c(u, v, w) := \langle u \star_t v, w \rangle$, then the tensor $(\nabla_z c)(u, v, w)$ is symmetric in $u, v, w, z \in T_tM$.

If there exists a vector field E such that $\nabla\nabla E = 0$ and complex number d such that:

1. $\nabla\nabla E = 0$;
2. $\mathcal{L}_E(u \star v) - \mathcal{L}_E u \star v - u \star \mathcal{L}_E v = u \star v$ for all vector fields u, v;
3. $\mathcal{L}_E \langle u, v \rangle - \langle \mathcal{L}_E u, v \rangle - \langle u, \mathcal{L}_E v \rangle = (2 - d) \langle u, v \rangle$ for all vector fields u, v,

then E is called an *Euler vector field* and the Frobenius manifold M is called *conformal.*

Example 1.1.6. Let X be a smooth projective variety. Then we can give $H^*(X)$ the structure of a Frobenius algebra with the Poincaré pairing and the quantum product. Note that the quantum product does not converge in general, so we must treat this as a formal object. The Euler vector field is given by

$$E_X = c_1(X) + \sum_i \left(1 - \frac{\deg \phi_i}{2}\right) t^i \phi_i,$$

where a general element of $H^*(X)$ is given by $t = \sum_i t^i \phi_i$. We will also impose that $\phi_1 = \mathbf{1}$. There is another very important structure, the *quantum connection*, which is given by the formula

$$\begin{aligned} \nabla_{t^i} &:= \partial_{t^i} + \frac{1}{z}\phi_i \star_t \\ \nabla_{z\frac{\mathrm{d}}{\mathrm{d}z}} &:= z\frac{\mathrm{d}}{\mathrm{d}z} - \frac{1}{z} E_X \star_t + \mu_X. \end{aligned}$$

Here, μ_X is the *grading operator*, defined for pure degree classes $\phi \in H^*(X)$ by

$$\mu_X(\phi) = \frac{\deg \phi - \dim X}{2}\phi.$$

Finally, in the direction of the Novikov variables, we have

$$\nabla_{\xi Q \partial_Q} = \xi Q \partial_Q + \frac{1}{z}\xi \star_t.$$

Remark 1.1.7. For a general conformal Frobenius manifold $(H, (-,-), \star, E)$, there is still a *deformed flat connection* or *Dubrovin connection* given by

$$\begin{aligned} \nabla_{t^i} &:= \frac{\partial}{\partial t^i} + \frac{1}{z}\phi_i \star \\ \nabla_{z\frac{\mathrm{d}}{\mathrm{d}z}} &:= z\frac{\mathrm{d}}{\mathrm{d}z} - \frac{1}{z} E \star. \end{aligned}$$

Definition 1.1.8. The *quantum D-module* of X is the module $H^*(X)[z][\![Q, t]\!]$ with the quantum connection defined above.

Remark 1.1.9. It is important to note that the quantum connection has a fundamental solution matrix $S^X(t,z)$ given by

$$S_X(t,z)\phi = \phi + \sum_i \sum_{n,\beta} \frac{Q^\beta}{n!} \phi^i \langle \frac{\phi_i}{z-\psi}, \phi, t, \ldots, t\rangle^X_{0,n+2,\beta}.$$

It satisfies the important equation

$$S^*_X(t,-z)S(t,z) = 1.$$

Using this formalism, the J-function is given by $S^*_X(t,z)\mathbf{1} = z^{-1}J_X(t,z)$.

1.1.3 Givental formalism

The Givental formalism is a geometric way to package enumerative (CohFT) invariants cleanly. We begin by defining the symplectic space

$$\mathcal{H} := H^*(X, \Lambda)(\!(z^{-1})\!)$$

with the symplectic form

$$\Omega(f,g) := \mathrm{Res}_{z=0}(f(-z)g(z)).$$

This has a polarization by Lagrangian subspaces

$$\mathcal{H}_+ := H^*(X,\Lambda)[z], \qquad \mathcal{H}_- := z^{-1}H^*(X,\Lambda)[\![z^{-1}]\!]$$

giving $\mathcal{H} \cong T^*\mathcal{H}_+$ as symplectic vector spaces. Choose Darboux coordinates $\underline{p}, \underline{q}$ on $\mathcal{H}$. For example, there is a choice in Coates's thesis which gives a general element of $\mathcal{H}$ as

$$\sum_{k\geq 0}\sum_i q_k^i \phi_i z^k + \sum_{\ell \geq 0}\sum_j p_\ell^j \phi^j (-z)^{-\ell-1}.$$

Taking the *dilaton shift*

$$q(z) = t(z) - z = -z + t_0 + t_1 z + t_2 z^2 + \cdots,$$

we can now think of $\mathcal{F}^X$ has a formal function on $\mathcal{H}_+$ near $q = -z$. This convention is called the *dilaton shift*.

Before we continue, we need to recast the string and dilaton equations in terms of $\mathcal{F}^X$. Write $t_x = \sum t_k^i \phi_i$. Then the string equation becomes

$$\partial_0^1 \mathcal{F}(t) = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty}\sum_j t_{n+1}^j \partial_n^j \mathcal{F}(t)$$

and the dilaton equation becomes

$$\partial_1^1 \mathcal{F}(t) = \sum_{n=0}^{\infty} t_n^j \partial_n^j \mathcal{F}(t) - 2\mathcal{F}(t).$$

There are also an infinite series of topological recursion relations

$$\partial_{k+1}^i \partial_\ell^j \partial_m^k \mathcal{F}(t) = \sum_{a,b} \partial_k^i \partial_0^a \mathcal{F}(t) \eta^{ab} \partial_0^b \partial_\ell^j \partial_m^k \mathcal{F}(t).$$

We can make sense of these three relations for any (formal) function $\mathcal{F}$ on $\mathcal{H}_+$.

Now let

$$\mathcal{L} = \left\{ (\underline{p}, \underline{q}) \in \mathcal{H} \mid \underline{p} = \mathrm{d}_{\underline{q}}\mathcal{F} \right\}$$

be the graph of $\mathrm{d}\mathcal{F}$. This is a formal germ at $q = -z$ of a Lagrangian section of the cotangent bundle $T^*\mathcal{H}_+$ and is therefore a formal germ of a Lagrangian submanifold in $\mathcal{H}$.

Theorem 1.1.10. *The function $\mathcal{F}$ satisfies the string equation, dilaton equation, and topological recursion relations if and only if $\mathcal{L}$ is a Lagrangian cone with vertex at the origin $q = 0$ such that its tangent spaces L are tangent to $\mathcal{L}$ exactly along zL.*

Because of this theorem, $\mathcal{L}$ is known as the *Lagrangian cone*. It can be recovered from the J-function by the following procedure. First consider $\mathcal{L} \cap (-z + z\mathcal{H}_-)$. Via the projection to $-z + H$ along $\mathcal{H}_-$, this can be considered as the graph of the J-function. Next, we consider the derivatives $\frac{\partial J}{\partial t^i}$, which form a basis of $L \cap z\mathcal{H}_-$, which is a complement to zL in L. Then we know that

$$z\frac{\partial J}{\partial t^i} \in zL \subset \mathcal{L},$$

so

$$z\frac{\partial^2 J}{\partial t^i \partial t^j} \in L \cap z\mathcal{H}_-.$$

Writing these in terms of the first derivatives $\frac{\partial J}{\partial t^i}$ and using the fact that J is a solution of the quantum connection, so we recover the Frobenius structure of quantum cohomology.

We will now express some classical results in this formalism. Let X be a toric variety with toric divisors $D_1,\dots,D_N$ such that $D_1,\dots,D_k$ form a basis of $H^2(X)$ and Picard rank k. Then define the *I-function*

$$I_X = ze^{\sum_{j=1}^k t_i D_i}\sum_\beta Q^\beta \frac{\prod_{j=1}^N \prod_{m=-\infty}^0 (D_j+mz)}{\prod_{j=1}^N \prod_{m=-\infty}^{\langle D_j,\beta\rangle}(D_j+mz)}.$$

Theorem 1.1.11 ([Giv98])**.** *The formal functions I_X and J_X coincide up to some change of variables, which if $c_1(X)$ is semi-positive is given by components of the I-function.*

Theorem 1.1.12 (Mirror theorem in this formalism)**.** *For any t, we have*

$$I_X(t,z)\in\mathcal{L}.$$

Another direction in Gromov-Witten theory is the Virasoro constraints. In the original formulation, these involved very complicated explicit differential operators, but in the Givental formalism, there is a very compact formulation.

Define $\ell^{-1}=z^{-1}$ and

$$\ell_0 = z\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{2}+\mu+\frac{c_1(X)\cup -}{z}.$$

Then define

$$\ell_n=\ell_0(z\ell_0)^n.$$

Theorem 1.1.13 (Genus-0 Virasoro constraints)**.** *Suppose the vector field on $\mathcal{H}$ defined by ℓ_0 is tangent to $\mathcal{L}$. Then the same is true for the vector fields defined by ℓ_n for any $n\geq 1$.*

Proof. Let L be a tangent space to $\mathcal{L}$. Then if $f\in zL\subset\mathcal{L}$, the assumption gives us $\ell_0 f\in L$. But then $z\ell_0 f\in zL$, so $\ell_0 z\ell_0 f=\ell_1 f\in L$. Continuing, we obtain $\ell_n f\in L$ for all n. ☐

Later, we will learn that the Quantum Riemann-Roch theorem can be stated in this formalism. Let $\mathcal{L}^{\mathrm{tw}}$ be the twisted Lagrangian cone (where the twisted theory will be defined next week).

Theorem 1.1.14 (Quantum Riemann-Roch)**.** *For some explicit linear symplectic transformaiton Δ, we have $\mathcal{L}^{\mathrm{tw}}=\Delta\mathcal{L}$.*

1.1.4 Quantization

In the last part of the talk, we will briefly discuss the quantization formalism, which encodes the higher-genus theory. In Darboux coordinates p_a, q_b, we will quantize symplectic transformations by the standard rules

$$\widehat{q_aq_b}=\frac{q_aq_b}{\hbar},\qquad \widehat{q_ap_b}=q_a\frac{\partial}{\partial q_b},\qquad \widehat{p_ap_b}=\hbar\frac{\partial^2}{\partial q_a\partial q_b}.$$

This determines a differential operator acting on functions on $\mathcal{H}_+$.

We also need the genus-g potential

$$\mathcal{F}_g^X := \sum_{\beta,n}\frac{Q^\beta}{n!}\langle t(\psi),\dots,t(\psi)\rangle_{g,n,\beta}^X$$

and the *total descendent potential*

$$\mathcal{D}:=\exp\left(\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}_g^X\right).$$

In this formalism, the Virasoro conjecture can be expressed as follows. Let $L_n=\widehat{\ell}_n+c_n$, where c_n is a carefully chosen constant.

Conjecture 1.1.15 (Virasoro conjecture)**.** *If* $L_{-1}\mathcal{D} = L_0\mathcal{D} = 0$*, then* $L_n\mathcal{D} = 0$ *for all* $n \geq 1$*.*

In this formalism, the higher-genus version of the Quantum Riemann-Roch theorem takes the very simple form

Theorem 1.1.16 (Quantum Riemann-Roch)**.** *Let* $\mathcal{D}^{\mathrm{tw}}$ *be the twisted descendent potential. Then*

$$\mathcal{D}^{\mathrm{tw}} = \widehat{\Delta}\mathcal{D}.$$

1.2 Quantum Riemann-Roch (Shaoyun, Feb 08)

We will state and prove the Quantum Riemann-Roch theorem in genus 0, following Coates-Givental.

1.2.1 Twisted Gromov-Witten invariants

Again, let X be a smooth projective variety. Let E be a vector bundle on X. We should note that

$$\overline{\mathcal{M}}_{0,n+1}(X,\beta) \xrightarrow{\pi} \overline{\mathcal{M}}_{0,n}(X,\beta)$$

is the universal curve, and the universal morphism is simply ev_{n+1}. We will consider the sheaf

$$E_{0,n,\beta} \coloneqq R\pi_* \,\mathrm{ev}_{n+1}^* E \in K^0(\overline{\mathcal{M}}_{0,n}(X,\beta)).$$

We need to check that this is a well-defined K-theory class. Choose an ample line bundle $L \to X$. By definition, for $N \gg 1$, the cohomology

$$H^i(X, E \otimes L^N) = 0$$

whenever $i \geq 1$. This gives us an exact sequence

$$0 \to \ker(=: A) \to H^0(X, E \otimes L^N) \otimes L^{-N}(=: B) \to E \to 0.$$

For any stable map $f : \Sigma \to X$ of positive degree, we obtain a long exact sequence

$$0 \to H^0(\Sigma, f^*E) \to H^1(\Sigma, f^*A) \to H^1(\Sigma, f^*B) \to H^1(\Sigma, f^*E) \to 0,$$

so we obtain

$$R^0\pi_* \,\mathrm{ev}_{n+1}^* E - R^1\pi_* \,\mathrm{ev}_{n+1}^* E = R^1\pi_* \,\mathrm{ev}_{n+1}^* B - R^1\pi_* \,\mathrm{ev}_{n+1}^* A.$$

This expresses $E_{0,n,\beta}$ as a difference of vector bundles.

We will now introduce a *universal characteristic class*

$$\mathbf{c}(-) = \exp\left(\sum_{k=0}^{\infty} s_k \,\mathrm{ch}_k(-)\right),$$

where $s_0, s_1, s_2, \ldots$ are formal variables and ch_k is the k-th Chern character

$$\frac{x_1^k}{k!} + \cdots + \frac{x_r^k}{k!},$$

where x_i are the Chern roots.

Example 1.2.1. Let $E \to X$ be a vector bundle and equip it with the fiberwise $\mathbb{C}^*$-action by scaling. Let λ be the equivariant parameter and ρ_i be the Chern roots. Then

$$e(E) = \sum_i (\lambda + \rho_i).$$

We then rewrite

$$\begin{aligned}\prod(\lambda+\rho_i) &= \exp\left(\sum_i\left(\log\lambda-\sum_k\frac{(-\rho_i)^k}{k\lambda^k}\right)\right)\\ &= \exp\left(\mathrm{ch}_0(E)\log\lambda+\sum_{k>0}\frac{(-1)^{k-1}(k-1)!}{\lambda^k}\,\mathrm{ch}_k(E)\right),\end{aligned}$$

so for the (equivariant Euler class), we obtain

$$\begin{aligned}s_0 &= \log\lambda\\ s_k &= \frac{(-1)^{k-1}(k-1)!}{\lambda^k}, \qquad k>0.\end{aligned}$$

We are now ready to define the $(E,\mathbf{c})$-twisted Gromov-Witten invariants.

Definition 1.2.2. Define the *twisted Gromov-Witten invariants* by

$$\langle\alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n}\rangle_{0,n,\beta}^{X,(E,\mathbf{c})} := \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}}\prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i)\psi_i^{k_i}\cup\mathbf{c}(E_{0,n,\beta})$$

for $\alpha_i\in H^*(X)$ and $k_i\in\mathbb{Z}_{\geq 0}$.

We will now construct the Lagrangian cone for the twisted theory. Let R be the coefficient ring containing $s_0,s_1,\dots$ and define

$$\mathcal{H}_X^{\mathrm{tw}} := H^*(X)\otimes R(\!(z^{-1})\!)[\![Q]\!].$$

We also introduce the *twisted Poincaré pairing*

$$(a,b)_{(E,\mathbf{c})} = \int_X a\cup b\cup\mathbf{c}(E).$$

The symplectic structure is defined by

$$\Omega_{\mathrm{tw}}(f,g) = \mathrm{Res}_{z=0}(f(-z)g(z))_{(E,\mathbf{c})}.$$

There is a polarization

$$\mathcal{H}_X^{\mathrm{tw}} = \mathcal{H}_+^{\mathrm{tw}}\oplus\mathcal{H}_-^{\mathrm{tw}}$$

with

$$\begin{aligned}\mathcal{H}_+^{\mathrm{tw}} &:= H^*(X)\otimes R[z][\![Q]\!]\\ \mathcal{H}_-^{\mathrm{tw}} &:= H^*(X)\otimes R[\![z]\!][\![Q]\!].\end{aligned}$$

Finally, we have the *twisted genus*-0 *descendent potential*

$$\mathcal{F}_{X,\mathrm{tw}}^0(t) := \sum_{\beta,n}\frac{Q^\beta}{n!}\langle t,\dots,t\rangle_{0,n,\beta}^{X,(E,\mathbf{c})}.$$

Identifying $\mathcal{H}_X^{\mathrm{tw}}$ with $T^*\mathcal{H}_+^{\mathrm{tw}}$, we obtain the twisted Lagrangian cone $\mathcal{L}_X^{\mathrm{tw}}$ as the graph of $\mathrm{d}\mathcal{F}_{X,\mathrm{tw}}^0$. Denote the untwisted Lagrangian cone as $\mathcal{L}_X$.

Theorem 1.2.3. *We have*

$$\mathcal{L}_X^{\mathrm{tw}} = \Delta\mathcal{L}_X,$$

where

$$\Delta = \exp\left(\sum_{m\geq 0}\sum_{\ell\geq 0}s_{2m-1+\ell}\frac{B_{2m}}{(2m)!}\,\mathrm{ch}_\ell(E)z^{2m-1}\right).$$

Here, the Bernoulli numbers B_{2m} are defined by

$$\frac{t}{1-e^{-t}} = \frac{t}{2}+\sum_{m\geq 0}\frac{B_{2m}}{(2m!)}t^{2m}.$$

1.2.2 Proof of Theorem 1.2.3

The idea is to use the Grothendieck-Riemann-Roch theorem.

Proposition 1.2.4. *We can write*

$$[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}} \cap \mathrm{ch}_k(E_{0,n,\beta}) = \pi_*\left(\sum_{\substack{r+\ell=k+1\\ r,\ell\geq 0}} \frac{B_r}{r!}\,\mathrm{ch}_\ell(\mathrm{ev}_{n+1}^* E)\Psi(r)\right),$$

where

$$\begin{aligned}\Psi(r) = {} & \psi_{n+1}^r \cap [\overline{\mathcal{M}}_{0,n+1}(X,\beta)]^{\mathrm{vir}} \\ & - \sum_{i=1}^{n} (\sigma_i)_*(\psi_i^{n-1} \cap [\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}) \\ & + \frac{1}{2} j_*\left(\sum_{\substack{a+b=r-2\\ a,b\geq 0}} (-1)^a \psi_+^a \psi_i^b \cap [\widetilde{Z}_{0,n+1,\beta}]^{\mathrm{vir}}\right).\end{aligned}$$

Here, $Z_{0,n+1,\beta}$ is formed by the nodes of π, $\widetilde{Z}_{0,n+1,\beta}$ is a double cover of $Z_{0,n+1,\beta}$ formed by a choice of branch of the nodes, ψ_+ and ψ_- are the ψ-classes at the two branches of the nodes, and

$$j\colon \widetilde{Z}_{0,n+1,\beta} \to Z_{0,n+1,\beta} \to \overline{\mathcal{M}}_{0,n+1}(X,\beta)$$

is the "inclusion."

Proof. We will first assume that $\overline{M}_{0,n+1}(X,\beta)$, $\overline{\mathcal{M}}_{0,n}(X,\beta)$, and $Z_{0,n+1,\beta}$ are all smooth and that $\pi(Z_{0,n+1,\beta})$ is a normal crossings divisor. In general, we need a Cartesian diagram

$$\begin{tikzcd}
 & \mathrm{ev}_{n+1}^* E \arrow[ld] \arrow[rr] & & E \arrow[ld] \\
\overline{\mathcal{M}}_{0,n+1}(X,\beta) \arrow[rr] \arrow[dd] & & \mathcal{C} \arrow[dd] & \\
 & Z_{0,n+1,\beta} \arrow[lu, hook] & & Z \arrow[lu, hook] \\
\overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[rr] & & \mathcal{M}. &
\end{tikzcd}$$

Continuing in the ideal situation, we apply Grothendieck-Riemann-Roch[1] to obtain

$$\begin{aligned}\mathrm{ch}(E_{0,n,\beta}) &= \mathrm{ch}(R\pi_*\,\mathrm{ev}_{n+1}^* E) \\ &= \pi_*(\mathrm{ch}(\mathrm{ev}_{n+1}^* E)\cdot \mathrm{td}^\vee\,\Omega_\pi),\end{aligned}$$

where $\mathrm{td}^\vee$ is the dual Todd class, defined by $\frac{-x}{1-e^{tx}}$, and Ω_π is the sheaf of relative differentials.

We then have two short exact sequences

$$0 \to \Omega_\pi \to \omega_\pi \to \mathcal{O}_{Z_{0,n+1,\beta}} \to 0$$

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[1] We need to be careful about directly applying Grothendieck-Riemann-Roch in the stacky setting (and in general we are only quasi-smooth).

and

$$0 \to \omega_\pi \to L_{n+1} \to \bigoplus_{i=1}^{n} \mathcal{O}_{D_i} \to 0,$$

where D_i is the divisor where the marked points $i, n+1$ collide and their component has exactly three special points. Now we obtain

$$\Omega_\pi = L_{n+1} - \sum_{i=1}^{n} \mathcal{O}_{D_i} - \mathcal{O}_{Z_{0,n+1,\beta}}$$

in K-theory. Using the facts that $c_1(L_{n+1}) = \psi_{n+1}$, $D_i \cap D_j = \emptyset$ for $i \neq j$, and $D_i \cap Z_{0,n+1,\beta} = \emptyset$, we see that L_{n+1} is trivial when restricted to D_i and $Z_{0,n+1,\beta}$. Now we apply the dual Todd class.

Lemma 1.2.5. *If* $x_1 \cup x_2 = 0$, *then*

$$(\mathrm{td}^\vee(x_1) - 1)(\mathrm{td}^\vee(x_2) - 1) = 0.$$

Using the lemma, we obtain

$$\begin{aligned}\mathrm{td}^\vee(\Omega_\pi) &= \mathrm{td}^\vee(L_{n+1}) \prod_{i=1}^{n} \mathrm{td}^\vee(-\mathcal{O}_{D_i})\, \mathrm{td}^\vee(\mathcal{O}_{Z_{0,n+1,\beta}})^{-1} \\ &= 1 + (\mathrm{td}^\vee(L_{n+1}) - 1) + \sum_{i=1}^{n} \left(\frac{1}{\mathrm{td}^\vee(\mathcal{O}_{D_i})} - 1 \right) + \left(\frac{1}{\mathrm{td}^\vee(\mathcal{O}_{Z_{n+1,\beta}})} - 1 \right).\end{aligned}$$

The first term in the statement comes from the dual Todd class of L_{n+1}, the second comes from

$$0 \to \mathcal{O}(-D_i) \to \mathcal{O} \to \mathcal{O}_{D_i} \to 0$$

and the relation between $\mathcal{O}(-D_i)$ and L_i, and the last term can be found in Appendix A of Coates-Givental. □

To obtain the Quantum Riemann-Roch theorem, we use the previous proposition and manipulate the generating function. If E is convex and $Y \subset X$ is a complete intersection defined by E, then $\mathcal{L}_X^{\mathrm{tw}}$ is closely related to $\mathcal{L}_Y$, so we are able to study the Gromov-Witten theory of Y using this.

1.3 Shift operators (Melissa, Feb 15)

Let X be a semiprojective smooth variety. This means that X is projective over its affinization. Also assume that X has an action by $T = (\mathbb{C}^\times)^m$ such that all T-weights in $H^0(X, \mathcal{O})$ are contained in a strictly convex cone in $\mathrm{Hom}(T, \mathbb{C}^\times)_\mathbb{R}$ and $H^0(X, \mathcal{O})^T = \mathbb{C}$. All such X imply that

(a) The fixed locus X^T is projective;

(b) The T-variety X is equivariantly formal. This means that $H_T^*(X)$ is a free module over $H_T^*(\mathrm{pt}) = \mathbb{Q}[\lambda] := \mathbb{Q}[\lambda_1, \ldots, \lambda_m]$ and there is a non-canonical isomorphism

$$H_T^*(X) \cong H^*(X) \otimes H_T^*(\mathrm{pt})$$

as $H_T^*(\mathrm{pt})$-modules.

(c) The evaluation maps $\mathrm{ev}_i \colon X_{0,n,d} \to X$ are proper.

Using (b), we may choose a basis $\{\phi_i\}_{i=0}^N$ of $H_T^*(X)$ over $H_T^*(\mathrm{pt})$. Let τ^i be the dual coordinates.

1.3.1 Equivariant big quantum cohomology

Let $(-,-)$ be the T-equivariant Poincaré pairing, which in general takes values in $\mathbb{Q}(\lambda)$. Then the T-equivariant big quantum product is defined by

$$\begin{aligned}(\phi_i \star_\tau \phi_j, \phi_k) &= \langle\!\langle \phi_i, \phi_j, \phi_k \rangle\!\rangle_{0,3}^{X,T} \\ &= \sum_{d,n} \frac{Q^d}{n!} \langle \phi_i, \phi_b, \phi_j, \tau, \dots, \tau \rangle_{0,n+3,d}^{X,T}.\end{aligned}$$

This can also be defined using the evaluation maps

$$(\mathrm{ev}_i)_* \colon H_T^*(X_{0,n+3,d}) \to H_T^{*-2(c_1(X)\cdot d+n)}(X)$$

as

$$\phi_i \star_\tau \phi_j = \sum_{d,n} \frac{Q^d}{n!} (\mathrm{ev}_3)_* \left(\mathrm{ev}_1^*(\phi_i)\, \mathrm{ev}_2^*(\phi_j) \prod_{i=4}^{n+3} \mathrm{ev}_i^*(\tau) \cap [X_{0,n+3,d}]^{\mathrm{vir}} \right) \in H_T^*(X)[\![Q]\!][\![\tau_0, \dots, \tau_n]\!].$$

1.3.2 Quantum connection

We will define

$$\nabla_i \colon H_T^*(X)[z][\![Q]\!][\![\tau]\!] \to z^{-1} H_T^*(X)[z][\![Q]\!][\![\tau^0, \dots, \tau^N]\!]$$

by setting

$$\nabla_i = \frac{\partial}{\partial \tau^i} + \frac{1}{z}(\phi_i \star).$$

We can view z as the loop variable by setting $\widehat{T} = T \times \mathbb{C}^\times$. If the extra copy of $\mathbb{C}^\times$ acts trivially on X, then

$$H_{\widehat{T}}^*(X) = H_T^*(X)[z].$$

This has a fundamental solution

$$M(\tau) \colon H_{\widehat{T}}^*(X)[\![Q,\tau]\!] \to H_{\widehat{T}}^*(X)_{\mathrm{loc}}[\![Q,\tau]\!]$$

where

$$H_{\widehat{T}}^*(X)_{\mathrm{loc}} := H_{\widehat{T}}^*(X) \otimes_{\mathbb{Q}[\lambda,z]} \mathbb{Q}(\lambda(z)).$$

This satisfies the differential equation

$$z \frac{\partial}{\partial \tau^i} M(\tau) = M(\tau)(\phi_i \star),$$

which is equivalent to

$$\frac{\partial}{\partial \tau^i} \circ M(\tau) = M(\tau) \circ \nabla_i.$$

The solution has the form

$$(M(\tau)\phi_i, \phi_j) = (\phi_i, \phi_j) + \langle\!\langle \phi_i, \frac{\phi_j}{z-\psi} \rangle\!\rangle_{0,2}^{X,T}.$$

1.3.3 Shift operators

Let $k \colon \mathbb{C}^\times \to T$ be a cocharacter of T. Then define a $\widehat{T}$-action ρ_k on X by

$$\rho_k(t,x)x = tu^k \cdot x$$

for $t \in T, u \in \mathbb{C}^\times, x \in X$. Under the group automorphism

$$\phi_k \colon \widehat{T} \to \widehat{T} \qquad \phi_k(t,u) = (tu^{-k}, u),$$

the identity map $(X,\rho_0)\to(X,\rho_k)$ is $\widehat{T}$-equivariant, so we obtain isomorphisms

$$\Phi_k\colon H^*_{\widehat{T},\rho_0}(X)\to H^*_{\widehat{T},\rho_k}(X).$$

Now define the bundle

$$E_k=(X\times(\mathbb{C}^2\setminus 0))/\mathbb{C}^\times,$$

where $\mathbb{C}^\times$ acts by

$$s\cdot(x,v_1,v_2)=(s^kx,s^{-1}v_1,s^{-1}v_2).$$

This is an X-bundle over $\mathbb{P}^1$ with an action on $\widehat{T}$ by

$$(t,u)\cdot[x,(v_1,v_2)]=[t\cdot x,(v_1,uv_2)].$$

Setting $0=[1,0]$ and $\infty=[0,1]$, we see that $\widehat{T}$ acts on X_0 by ρ_0 and X_∞ by ρ_k.

Definition 1.3.1. A cocharacter $k\colon\mathbb{C}^\times\to T$ is *seminegative* if all weights of $H^0(X,\mathcal{O})$ are nonpositive with respect to k and is *negative* if all nonzero weights of $H^0(X,\mathcal{O})$ are negative.

Lemma 1.3.2. *If k is seminegative, then E_k is semiprojective.*

Now let $\pi\colon E_k\to\mathbb{P}^1$ be the projection. We now consider *section classes*, which are those effective classes in $H_2(E_k,\mathbb{Z})$ satisfying $\pi_*d=[\mathbb{P}^1]$. For the $\mathbb{C}^\times$-action on X given by k, there is a unique fixed component F_{min} whose normal weights are all positive (one way to see this is to consider the moment map of the corresponding circle action). Therefore, there is a minimal section class σ_{min} corresponding to F_{min}.

Lemma 1.3.3. *Given $\tau\in H^*_T(X)$, there exists $\widehat{\tau}\in H^*_{\widehat{T}}(E_k)$ such that $\widehat{\tau}|_{X_0}=\tau$ and $\widehat{\tau}|_{X_\infty}=\Phi_k(\tau)$.*

Lemma 1.3.4. *If k is seminegative, then*

$$\mathrm{Eff}(E_k)^{\mathrm{sec}}=\sigma_{\mathrm{min}}+\mathrm{Eff}(X).$$

Definition 1.3.5. Let $k\colon\mathbb{C}^\times\to T$ be seminegative. Given $\tau\in H^*_T(X)$, we define the *shift operator*

$$\widetilde{\mathbb{S}}_k\colon H^*_{\widehat{T},\rho_0}(X)[\![Q]\!]\to H^*_{\widehat{T},\rho_k}(X)[\![Q]\!]$$

by the formula

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha,\beta)=\sum_{\widehat{d}\in\mathrm{Eff}(E_k)^{\mathrm{sec}}}\frac{Q^{\widehat{d}-\sigma_{\mathrm{min}}}}{n!}\langle(\iota_0)_*\alpha,(\iota_\infty)_*\beta,\widehat{\tau},\ldots,\widehat{\tau}\rangle^{E_k,\widehat{T}}_{0,n+2,\widehat{d}},$$

where $\alpha\in H^*_{\hat{t},\rho_0}(X)$ and $\beta\in H^*_{\widehat{T},\rho_k}(X)$. We also define

$$\mathbb{S}_k(\tau)=\Phi_k^{-1}\circ\widehat{\mathbb{S}}_k(\tau).$$

Theorem 1.3.6. *We have the formula*

$$M(\tau)\circ\mathbb{S}_k(\tau)=\mathcal{S}_k\circ M(\tau),$$

where $\mathcal{S}_k$ is defined via the commutative diagram

$$\begin{tikzcd}
H^*_{\widehat{T}}(X)_{\mathrm{loc}} \arrow[r, "\mathcal{S}_k"] \arrow[d] & H^*_{\widehat{T}}(X)_{\mathrm{loc}} \arrow[d, "\iota^*"] \\
H^*_{\widehat{T}}(X^T)_{\mathrm{loc}} \arrow[r, "\bigoplus_i \Delta_i(k)e^{-2k\delta_\lambda}"] & H^*_{\widehat{T}}(X^T)_{\mathrm{loc}}.
\end{tikzcd}$$

Here, we define

$$\Delta_i(k) = Q^{\sigma_i - \sigma_{\min}} \prod_{\alpha} \prod_{j=1}^{\operatorname{rk} N_{i,\alpha}} \frac{\prod_{c=-\infty}^{0}(\rho_{i,\alpha,j} + \alpha + cz)}{\prod_{c=-\infty}^{-\alpha\cdot k}(\rho_{i,\alpha,j} + \alpha + cz)} \in H^*_{\widehat{T}}(F_i)_{\mathrm{loc}}[\![Q]\!],$$

where

$$N_i = N_{F_i/X} = \bigoplus_{\alpha} N_{i,\alpha}$$

is the normal bundle of F_i in X and $\rho_{i,\alpha,j}$ are its Chern roots.

The idea of the proof is to decompose

$$E^{\widehat{T}}_{k,0,n+2,\widehat{d}} = \bigsqcup_{i} \bigsqcup_{I_1 \cup I_2 = [n+2]} \bigsqcup_{d_0 + d_\infty + \widehat{\sigma} = \widehat{d}} (X_0)^T_{0,I_1 \sqcup p, d_0} \times_{F_i} (X_\infty)^T_{0, I_2 \sqcup q, d_\infty}.$$

Using the exact sequence

$$0 \to \mathrm{Aut}(C,x) \to \mathrm{Def}(f) \to T^1 \to \mathrm{Def}(C,x) \to \mathrm{Obs}(f) \to T^2 \to 0,$$

we obtain the explicit formulae

$$\begin{aligned} \mathrm{Aut}(C,x)^m &= \mathrm{Aut}(C_0,x_0)^m + \mathrm{Aut}(C_\infty,x_\infty)^m \\ \mathrm{Def}(C,x)^m &= \mathrm{Def}(C_0,x_0)^m \oplus \mathrm{Def}(C_0,x_0)^m \oplus T_pC_0 \otimes T_p\mathbb{P}^1 \oplus T_qC_\infty \otimes T_q\mathbb{P}^1. \end{aligned}$$

This gives the virtual normal bundle, and using virtual localization, we obtain

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha, \beta) = (\widetilde{\mathbb{S}}_k M(\tau,z)\alpha, M'(\tau',-z)\beta),$$

where

$$M'(\tau',z) = \Phi_k \circ M(\tau,z) \circ \Phi_k^{-1}.$$

Using the unitarity property of M, we obtain the desired result.

1.4 Orbifold stuff (Patrick, Apr 04)

1.4.1 Orbifold Gromov-Witten theory Let X be a smooth and separated Deligne-Mumford stack of finite type over $\mathbb{C}$.

Definition 1.4.1. The *inertia stack* of X is the fiber product in the diagram

$$\begin{tikzcd} IX \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

More concretely, we may think about IX as parameterizing pairs (x,g), where $x \in X$ and $g \in \mathrm{Aut}(x)$. There is another description of IX if X lives over $\mathbb{C}$. In general, IX is disconnected. We will write

$$IX = \bigsqcup_{i \in I} X_i.$$

It also has an important morphism inv: $IX \to IX$ given by $(x,g) \mapsto (x,g^{-1})$.

Definition 1.4.2. A morphism $X \to Y$ of algebraic stacks is *representable* if for all schemes S and morphisms $S \to Y$, the fiber product $X \times_S Y$ is an algebraic space.

Theorem 1.4.3. *Let*

$$I_\mu X := \bigsqcup_{r\geq 0} \mathrm{Hom}_{\mathrm{rep}}(B\mu_r, X)$$

denote the stack of representable morphisms from classifying stacks of roots of unity to X (the cyclotomic inertia stack). Then $I_\mu X \simeq IX$.

We need to make one more definition, which will appear as a degree shift on cohomology. Let $(x,g) \in X_i$. Because $\langle g \rangle \subset \mathrm{Aut}(x)$ is cyclic, there is a decomposition

$$T_x X = \bigoplus_{0\leq \ell < r_i} V_\ell,$$

where V_ℓ is the eigenspace with eigenvalue $e^{2\pi\sqrt{-1}\frac{\ell}{r_i}}$ and r_i is the order of g. Then the function

$$\mathrm{age} := \frac{1}{r_i} \sum_{0\leq \ell < r_i} \ell \cdot \dim V_\ell$$

is constant on X_i, so we denote its value by $\mathrm{age}(X_i)$.

Recall that by the Keel-Mori theorem, X (which has finite inertia) has a coarse moduli space $|X|$, which is an algebraic space satisfying two properties:

- The morphism $\pi\colon X \to |X|$ is bijective on k-points whenever k is an algebraically closed field;
- $|X|$ is initial for morphisms from X to any algebraic space.

From now on, we will assume that $|X|$ is quasiprojective, and in particular that it is a scheme.

Moduli of stable maps

Definition 1.4.4. The moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ parameterizes objects

$$\begin{tikzcd}
(C, \{\Sigma_i\}) \arrow[r, "f"] \arrow[d] & X \\
T, &
\end{tikzcd}$$

where

1. C is a prestable balanced twisted curve of genus g. This means that C has stacky structure only at nodes and marked points, and the nodes are formally locally $[(\mathrm{Spec}\,\mathbb{C}[x,y]/xy)/\mu_r]$, where μ_r acts by $\zeta(x,y) = (\zeta x, \zeta^{-1} y)$;
2. $\Sigma_i \subset C$ is an étale cyclotomic gerbe over T with a trivialization for all i;
3. $f\colon C \to X$ is representable and the induced morphism between coarse moduli spaces is a stable map of degree β with n marked points.

We see that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has evaluation maps $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to IX$. It is also disconnected, with the connected components being indexed by components of IX. Let

$$\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\ldots,i_n) := \bigcap_{j=1}^{n} \mathrm{ev}_j^{-1}(X_{i_j}).$$

Then

$$\overline{\mathcal{M}}_{g,n}(X,\beta) = \bigsqcup_{i_1,\dots,i_n} \overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots,i_n).$$

Each component has a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots i_n)]^{\mathrm{vir}} \in H_*(\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots,i_n),\mathbb{Q})$$

of virtual dimension

$$\int_\beta c_1(X) + (1-g)(\dim X - 3) + n - \sum_{j=1}^n \mathrm{age}(X_{i_j}).$$

given by the relative perfect obstruction theory $(R\pi_* f^* TX)^\vee$, where $\pi\colon C \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ is the universal curve, over the moduli stack $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ of prestable twisted curves. Because we chose to work with trivialized gerbe markings, we need to multiply the virtual fundamental class as follows. Note that the j-th marked point is

$$\Sigma_j \cong \overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots,i_n) \times B\mu_{r_{i_j}}.$$

Here, if $x = [B\mu_r \to X] \in X_{i_j} \subset IX$, then $r_{i_j} = r$. Then set

$$[\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots i_n)]^w := \left(\prod_{j=1}^n r_{i_j}\right)[\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots i_n)]^{\mathrm{vir}}.$$

Now consider the morphism $p\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(|X|,\beta)$ given by taking the coarse moduli space. Let $C_{|X|} \to \overline{\mathcal{M}}_{g,n}(|X|,\beta)$ be the universal curve and $\sigma_{i,|X|}$ be the marked points. Then the descendant classes[2] are defined to be

$$\psi_j := p^* c_1(\sigma_j^* \omega_{C_{|X|}/\overline{\mathcal{M}}_{g,n}(|X|,\beta)}).$$

Quantum cohomology We are now able to define Gromov-Witten invariants. Let $\alpha_j \in H^{p_j}(X_{i_j},\mathbb{C})$. Then define

$$\langle \alpha_1\psi^{k_1},\dots,\alpha_n\psi^{k_n}\rangle^X_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots,i_n)]^w} \prod_{j=1}^n \mathrm{ev}_j^* \alpha_j \psi_j^{k_j}.$$

We are still able to form generating series $\mathcal{F}_g, J_X, \dots$ as before, and the invariants satisfy the string, dilaton, and divisor equations (although we have to be careful that the marked point we delete is a scheme point), so the orbifold Gromov-Witten theory has a Lagrangian cone $\mathcal{L}_X \subset \mathcal{H}$.

The *orbifold Poincaré pairing* is defined by the formula

$$(\alpha,\beta) := \int_{IX} \alpha \cup \mathrm{inv}^* \beta,$$

where $\cup$ denotes the usual cup product. This is well-defined because of the formula

$$\mathrm{age}(X_i) + \mathrm{age}(X_{\mathrm{inv}(i)}) = \dim X - \dim X_i$$

when X is proper. When X is not proper, we will assume we are working equivariantly. Now we may define the *quantum product* by the formula

$$(a \star_\tau b, c) := \sum_{n,\beta} \frac{Q^\beta}{n!} \langle a,b,c,\tau,\dots,\tau\rangle^X_{0,n+3,\beta}$$

---

[2] Most people call these $\overline{\psi}$, but I am extremely lazy.

for $a, b, c, \tau \in H^*(IX, \mathbb{C})$. Restricting to the degree 0 part and setting $\tau = 0$, we obtain the *orbifold cup product*, which is given by

$$(a \star b, c) = \langle a, b, c \rangle^X_{0,3,0}.$$

Denote $H^*_{\mathrm{CR}}(X) := (H^*(IX, \mathbb{C}), \star)$. The orbifold cup product is graded for the grading $\deg(a) = p + 2\operatorname{age}(X_i)$ for $a \in H^p(X_i)$. Using the quantum product, we may define the quantum connection and its fundamental solution.

1.4.2 Toric Deligne-Mumford stacks

We will assume the reader is familiar with the fan presentation of a toric variety. If you are not, there are many references.

Definition 1.4.5. An *extended stacky fan* is a quadruple $\boldsymbol{\Sigma} = (N, \Sigma, \beta, S)$ of

1. A finitely generated abelian group N of rank n;
2. A rational simplicial fan Σ in $N_{\mathbb{R}} = N \otimes \mathbb{R}$;
3. A homomorphism $\beta \colon \mathbb{Z}^m \to N$. We will write $b_i = \beta(e_i) \in N$ for the image of the standard basis vector $e_i \in \mathbb{Z}^m$ and $\overline{b}_i$ for its image in $N_{\mathbb{R}}$;
4. A subset $S \subset \{1, \ldots, m\}$

satisfying the following conditions:

1. The set $\Sigma(1)$ of 1-dimensional cones is exactly the set $\left\{\mathbb{R}_{\geq 0} \cdot \overline{b}_i \mid i \notin S\right\}$;
2. For all $i \in S$, $\overline{b}_i \in |\Sigma|$.

We will now assume that $|\Sigma|$ is convex and full-dimensional and, that there is a strictly convex piecewise linear function $f \colon |\Sigma| \to \mathbb{R}$ which is linear on each cone, and that β is surjective. From this data, we will now obtain a GIT presentation. Define $\mathbb{L}$ by the exact sequence

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\beta} N \to 0.$$

Then define $K := \mathbb{L} \otimes \mathbb{C}^\times$. Then define $D_i \in \mathbb{L}^\vee$ to be the image of the i-th standard basis vector in $(\mathbb{Z}^m)^\vee$ under the last arrow in the exact sequence

$$0 \to N^\vee \to (\mathbb{Z}^m)^\vee \to \mathbb{L}^\vee$$

Finally, set

$$\mathcal{A}_\omega = \left\{I \subset \{1, \ldots, m\} \mid S \subset I, \sigma_{\overline{I}} \text{ is a cone of } \Sigma\right\}.$$

Choose a stability condition

$$\omega \in C_\omega := \bigcup_{I \in \mathcal{A}_\omega} \left\{\sum_{i \in I} a_i D_i \mid a_i \in \mathbb{R}_{>0}\right\}.$$

Then we define

$$X_{\boldsymbol{\Sigma}} := [(\mathbb{C}^m)^s / K].$$

The ample cone is $C'_\omega \subset \mathbb{L}^\vee_{\mathbb{R}} / \sum_{i \in S} \mathbb{R} D_i \cong H^2(X_{\boldsymbol{\Sigma}}, \mathbb{R})$, which is defined in the same way as C_ω after deleting S from the extended stacky fan, and the cone of effective curve classes is its dual.

Orbifold cohomology First, we will describe the equivariant cohomology of $X_{\mathbf{\Sigma}}$. Let $\mathcal{Q} = (\mathbb{C}^\times)^m/K$. Then if u_i is Poincaré dual to $(x_i = 0 \subset (\mathbb{C}^m)^s)/K$, we have

$$H^*_{\mathcal{Q}}(X_{\mathbf{\Sigma}}, \mathbb{C}) = H^*_{\mathcal{Q}}(\mathrm{pt}, \mathbb{C})[u_1, \ldots, u_m]/(\mathfrak{I} + \mathfrak{J}),$$

where

$$\mathfrak{I} := \langle \chi - \sum_{i=1}^{m} \langle \chi, b_i \rangle u_i \mid \chi \in N^\vee_{\mathbb{C}} \rangle$$
$$\mathfrak{J} := \langle \prod_{i \notin I} u_i \mid I \notin \mathcal{A}_\omega \rangle.$$

There is a combinatorial description of the components of the inertia stack $IX_{\mathbf{\Sigma}}$. Because $X_{\mathbf{\Sigma}}$ is a global quotient, the components of the inertia stack correspond to elements $g \in K$ such that $((\mathbb{C}^m)^s)^g$ is nonempty. Equivalently, if we define

$$\mathbb{K} := \left\{ f \in \mathbb{L} \otimes \mathbb{Q} \mid \left\{ i \in \{1, \ldots, m\} \mid D_i \cdot f \in \mathbb{Z} \right\} \in \mathcal{A}_\omega \right\},$$

then the components of $IX_{\mathbf{\Sigma}}$ are in bijection with $\mathbb{K}/\mathbb{L}$. To give a description in terms of the fan, for any $\sigma \in \Sigma(n)$, define

$$\mathrm{Box}(\sigma) := \left\{ v \in N \mid \overline{v} = \sum_{\rho_i \subseteq \sigma} a_i \overline{b}_i \mid 0 \le a_i < 1 \right\}$$

and then

$$\mathrm{Box}(\mathbf{\Sigma}) := \bigcup_{\sigma \in \Sigma(n)} \mathrm{Box}(\sigma).$$

Then there is a natural bijection $\mathbb{K}/\mathbb{L} \cong \mathrm{Box}(\mathbf{\Sigma})$. For any $f \in \mathbb{K}/\mathbb{L}$, X_f is a toric DM stack with $K, \mathbb{L}, \omega$ the same as for X_ω and characters D_i for i such that $D_i \cdot f \in \mathbb{Z}$. At the level of fans, this corresponds to killing the minimal cone of Σ containing the corresponding $\overline{v}$.

We will now give the orbifold cohomology of $X_{\mathbf{\Sigma}}$. Define the *deformed group ring* $\mathbb{C}[N]^{\mathbf{\Sigma}}$ as the vector space $\mathbb{C}[N]$ with product given by

$$y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1 + c_2} & \text{there exists } \sigma \in \Sigma \text{ such that } \overline{c}_1, \overline{c}_2 \in \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Then there is an isomorphism of rings [BCS05]

$$H^*_{\mathrm{CR}}(X_{\mathbf{\Sigma}}) \cong \frac{\mathbb{C}[N]^{\mathbf{\Sigma}}}{\langle \sum_{i \notin S} \chi(b_i) y^{b_i} \mid \chi \in N^\vee \rangle}.$$

Remark 1.4.6. This result also works in families over a base B [Jia08], where $\mathbb{C}^m$ is replaced by a direct sum of m line bundles on B. Then we need to add a $c_1(L_\chi)$ to the relations and obtain

$$H^*_{\mathrm{CR}}(X^B_{\mathbf{\Sigma}}) := \frac{H^*(B)[N]^{\mathbf{\Sigma}}}{\langle c_1(L_\chi) + \sum_{i \notin S} \chi(b_i) y^{b_i} \mid \chi \in N^\vee \rangle}.$$

1.5 Gamma-integral structure (Patrick, Apr 04)

Let $IX = \bigsqcup_{v \in \mathrm{B}} X_v$ and $q_v \colon X_v \to X$ be the restriction of $IX \to X$. Let E be a T-equivariant vector bundle on X. Recall that v corresponds to some $g_v \in K$, so we obtain an eigenbundle decomposition

$$q_v^* E = \bigoplus_{0 \le f < 1} E_{v,f},$$

where $E_{v,f}$ is the subbundle where g_v acts by $e^{2\pi i f}$. We now define the orbifold Chern character to be

$$\widetilde{\mathrm{ch}}(E) = \bigoplus_{v\in \mathsf{B}} \sum_{0\le f<1} e^{2\pi i f}\,\mathrm{ch}(E_{v,f}).$$

Now let $\delta_{v,f,j}$ be the Chern roots of $E_{v,f}$. We define the orbifold Todd class to be

$$\widetilde{\mathrm{Td}}(E) := \bigoplus_{v\in \mathsf{B}} \left(\prod_{0<f<1} \prod_j \frac{1}{1-e^{-2\pi i f-\delta_{v,f,j}}} \right) \prod_j \frac{\delta_{v,0,j}}{1-e^{-\delta_{v,0,j}}}.$$

The $\widehat{\Gamma}$-class should be a square root of this and is defined by

$$\widehat{\Gamma}(E) = \bigoplus_{v\in \mathsf{B}} \prod_{0\le f<1} \prod_j \Gamma(1-f+\delta_{v,f,j}),$$

where we expand Γ around $1-f$. The reflection formula for the Γ-function implies that the X_v-component of $\widehat{\Gamma}(E^\vee)\cup\widehat{\Gamma}(E)$ is given by

$$[\widehat{\Gamma}(E^\vee)\cup\widehat{\Gamma}(E)]_v = (2\pi i)^{\mathrm{rk}(q_v^* E)^{\mathrm{mov}}} \left[e^{-\pi i(\mathrm{age}(q^*E)+c_1(q^*E))} (2\pi i)^{\frac{\deg_0}{2}} \widetilde{\mathrm{Td}}(E) \right]_{\mathrm{inv}(v)}.$$

Here, $\deg_0$ is the grading operator given by the degree without age shifting.

Definition 1.5.1. Define the *K-group framing* $\mathfrak{s}\colon K_T(X) \to H^*_{\mathrm{CR},T}(X) \otimes_{R_T} R_T[\log z](\!(z^{-\frac{1}{k}})\!)[\![Q,\tau]\!]$ by the formula

$$\mathfrak{s}(E)(\tau,z) := \frac{1}{(2\pi)^{\frac{\dim X}{2}}} L(\tau,z) z^{-\mu} z^{\rho} \widehat{\Gamma}_X \cup (2\pi i)^{\frac{\deg_0}{2}}\, \mathrm{inv}^*\, \widetilde{\mathrm{ch}}(E),$$

where $L(\tau,z)$ is the fundamental solution to the quantum connection, μ is the usual grading operator given by $\frac{1}{2}(\deg - \dim X)$ on homogeneous elements, and $\rho = c_1(TX) \in H^2(X)$.

Proposition 1.5.2. *Define the equivariant Euler pairing by*

$$\chi(E,F) := \sum_j (-1)^k \mathrm{ch}^T(\mathrm{Ext}^k(E,F))$$

and the modified version $\chi_z(E,F)$ by replacing the equivariant parameters λ_j by $\frac{2\pi i \lambda_j}{z}$. Then

$$(\mathfrak{s}(E)(\tau, e^{-i\pi} z), \mathfrak{s}(F)(\tau,z)) = \chi_z(E,F).$$

Remark 1.5.3. Everything we have discussed so far makes sense for toric DM stacks after specializing $Q = 1$.

2

Crepant transformation conjecture

2.1 Toric wall-crossings in GW theory (Davis, Apr 11)

Our goal is to study GIT wall-crossing in toric Gromov-Witten theory. For toric varieties $X_\pm$ related across a wall, we want to define $\mathcal{M} \supset U_\pm$ such that $X_\pm \in U_\pm$, as well as quantum connections on $U_\pm$ that restrict to $X_\pm$ appropriately.

2.1.1 GIT wall-crossing

Recall that the data of an (extended) stacky fan is equivalent to GIT data consisting of

1. A vector space V of dimension m;
2. A torus K;
3. Characters $D_i \in \operatorname{char}(K)$ for $i = 1, \dots, m$;
4. A stability condition $\omega \in \operatorname{char}_{\mathbb{R}}(K)$;
5. A set of anticones $A_\omega = \left\{ I \subset \{1, \dots, m\} \mid \omega \in \sum_{i \in I} a_i D_i, a_i \in \mathbb{R}_{>0} \right\}$.

Remark 2.1.1. The DM stack X_ω is defined to be $[U_\omega / K]$, where

$$U_\omega := \bigcup_{I \in A_\omega} (\mathbb{C}^\times)^I \times \mathbb{C}^{I^c}$$

is the semistable locus. Therefore, if $\omega_1, \dots, \omega_2$ are stability conditions, there is a birational map $X_{\omega_1} \dashrightarrow X_{\omega_2}$ induced by identifying the dense open tori.

Suppose that $\omega_1 \in C_1, \omega_2 \in C_2$ are separated by a wall W in the space of stability conditions. Choose $\omega_0 \in W \cap C_1 = W \cap C_2$ and let

$$X_0 = [U_{\omega_0} / K],$$

which may not be Deligne-Mumford. Choose a resolution $\widetilde{X}$ of $X_{\omega_1}, X_{\omega_2}$ completing the diagram

$$\begin{tikzcd} & X_0 & \\ X_{\omega_1} \arrow[ur, hook, "g_1"] & & X_{\omega_2} \arrow[ul, hook', "g_2"] \\ & \widetilde{X}. \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

We will assume that $\sum_{i=1}^{M} D_i \in W$. Under this assumption, the wall-crossing is crepant:

$$f_1^* K_{X_{\omega_1}} = f_2^* K_{X_{\omega_2}}.$$

Example 2.1.2. If e is a vector perpendicular to W, the $\mathbb{G}_m$ it generates may not be fixed. For example, let $K = \mathbb{G}_m^2$ and $V = \mathbb{C}^3$. Let $D_1 = (1,0)$, $D_2 = (1,2)$, and $D_3 = (0,2)$. Therefore,

$$\sum a_i D_i = (a_1 + a_2, 2a_2 + 2a_3).$$

Let $\omega := (\omega_x, \omega_y)$. If $\omega_x, \omega_y > 0$, then $A_\omega \supset \{\{1,2,3\},\{1,3\}\}$. Then D_2 appears if either

$$(\omega_x, \omega_y) = a_1(1,0) + a_2(1,2) \text{ if } \omega_x > \frac{\omega_y}{2} > 0$$

$$(\omega_x, \omega_y) = a_2(1,2) + a_3(0,2) \text{ if } \frac{\omega_y}{2} > \omega_x > 0.$$

Therefore, there is a wall givven by $2\omega_x = \omega_y$. In the chamber $\omega_y > 2\omega_x > 0$, we have

$$A_\omega = \{\{1,2,3\}\{1,3\},\{2,3\}\}$$

and therefore $U_\omega = (\mathbb{C}^2 \setminus 0) \times \mathbb{C}^\times$. Taking the quotient, we see that

$$U_\omega / K = \mathbb{P}^1 / B_{\mu_2}.$$

In the other chamber, we have

$$A_\omega = \{\{1,2,3\}\{1,3\},\{1,2\}\},$$

which implies that $U_\omega = \mathbb{C}^\times \times (\mathbb{C}^2 \setminus 0)$, so

$$X_\omega = \mathbb{P}(2,2).$$

For a stability condition ω_0 on the wall, we will obtain

$$U_0 = (\mathbb{C}^2 \setminus 0) \times \mathbb{C}^\times \cup \mathbb{C}^\times \times (\mathbb{C}^2 \setminus 0) \cup \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}.$$

The middle $\mathbb{C}^\times$ is fixed by the cocharacter $(2,-1)$, so X_0 is non-DM.

2.1.2 The secondary variety

Definition 2.1.3. The wall and chamber structure on $\mathrm{char}_{\mathbb{R}}(K)$ defines a fan on $\mathrm{char}_{\mathbb{R}}(K)$ called the *secondary fan*. The associated toric variety is called the *secondary toric variety*. For us, we will consider the subfan consisting of cones $C_{\omega_1}, C_{\omega_2}$, and their faces. Call the corresponding toric variety $\mathcal{M}$. This is a moduli space of Landau-Ginzburg models mirror to the X_i.

Unfortunately, $\mathcal{M}$ is generally singular, so we will consider a smooth finite cover $\mathcal{M}_{\mathrm{reg}}$. Explicitly, we see that

$$M = \operatorname{Spec} \mathbb{C}[C_1^\vee \cap \mathrm{cochar}(K)] \cup \operatorname{Spec} \mathbb{C}[C_2^\vee \cap \mathrm{cochar}(K)].$$

Call the charts U_+, U_-. Define

$$\mathbb{K}_i := \left\{ f \in \mathrm{cochar}_{\mathbb{Q}}(K) \mid D_i f \in \mathbb{Z} \text{ for all } i \in I \Rightarrow I \in A_\omega \right\}$$

and let $\widetilde{\mathbb{L}}_i$ be the free $\mathbb{Z}$-submodule of $\mathrm{cochar}_{\mathbb{Q}}(K)$ generated by $\mathbb{K}_i$.

Example 2.1.4. Continung with the previous example, recall that

$$A_{\omega_1} = \{\{1,2,3\}\{1,3\},\{2,3\}\}$$

If we want $(f_x, f_y) \cdot (0,2) \in \mathbb{Z}$, then we see that $f_y \in \frac{1}{2}\mathbb{Z}$, so $\widetilde{\mathbb{L}}_1 = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$.

Remark 2.1.5. Recall that

$$H^2(X_\omega,\mathbb{R}) \simeq \mathrm{char}_{\mathbb{R}}(K) / \sum_{i\in S} \mathbb{R}D_i,$$

where

$$S = \{i \mid i^c \notin A_\omega\}$$

is the set of indices contained in every anticone. Therefore, we can split

$$\mathrm{char}_{\mathbb{R}}(K) = \bigcap_{j\in S} \ker(\xi_j) \oplus \bigoplus_{j\in S} \mathbb{R}D_j.$$

Claim 2.1.6. *There exist* $\xi_j^\pm$ *such that*

1. $\xi_j^+|_W = \xi_j^-|_W$ *for all* $j \in S_+ \cap S_-$;
2. *For all* $j \in S_+ \Delta S_-$, $\xi_j^\pm|_W = 0$;
3. *All* $\xi_j^\pm \in \mathbb{K}_\pm$.

Claim 2.1.7. *We have a decomposition*

$$\widetilde{\mathbb{L}}^\vee_\pm = (H^2(X_\pm,\mathbb{R}) \cap \widetilde{\mathbb{L}}^\vee_\pm) \oplus \bigoplus_{j\in S_\pm} \mathbb{Z}D_j.$$

Furthermore, $W \cap \widetilde{\mathbb{L}}^\vee_+ = W \cap \widetilde{L}^\vee_-$.

Proof. If $j \in S_+$, then $j \in I$ for all $I \in A_{\omega_+}$, so $D_j f \in \mathbb{Z}$. Thus $D_j \in \widetilde{L}_i^+$. In the other direction, if $v \in \widetilde{L}^\vee_+$, $v\xi_i \in \mathbb{Z}$ because $\xi_j \in \mathbb{K}_+$. Therefore,

$$w = v - \sum_{i\in S_+} \langle v, \xi_i \rangle D_i \in \bigcap_{j\in S} \ker(\xi_j) \cap \widetilde{L}^\vee_+. \qquad \square$$

Corollary 2.1.8. *There exists an integral basis for* $\widetilde{L}_i^\vee$ *of the form*

$$\{p_1^\pm, \ldots, p_\ell^\pm\} \cup \{D_j\}_{j\in S}$$

such that $p_k^\pm \in \overline{C}'_\pm$ *live in the closure of the ample cones of* X_i *and* $p_k^+ = p_k^-$ *as classes of* C'_W.

Now we have maps

$$\mathbb{C}[C^\vee_\pm \cap \mathrm{cochar}(K)] \to \mathbb{C}[y_1^\pm, \ldots, y_{\ell_\pm}^\pm, \{x_j\}_{j\in S}]$$

given by the formulae

$$y^d \mapsto \prod_j^{\ell_\pm} y_j^{\pm p_j^\pm \cdot d} \prod_{j\in S} x_j^{D_j\cdot d}$$

for cocharacters $d \in \mathrm{cochar}(K)$.

Remark 2.1.9. We may reorder the basis p_i as $\{q_1, \ldots, q_r\}$ such that q_r is the unique vector not on the wall W. Dually, we may write $\{y_i, x_j\}$ as $\{z_1, \ldots, z_r\}$ such that z_r is dual to q_r. In these coordinates, we can write

$$U_+ = \operatorname{Spec} \mathbb{C}[z_1^+, \ldots, z_{r-1}^+, z_r^{+\frac{1}{B}}]$$
$$U_- = \operatorname{Spec} \mathbb{C}[z_1^-, \ldots, z_{r-1}^-, z_r^{-\frac{1}{A}}].$$

Let $\widehat{\mathcal{M}}_{\mathrm{reg}}$ be the analytification of $\mathcal{M}_{\mathrm{reg}} = U_+ \cup U_-$.

2.1.3 Extending the I-functions

Let

$$d = \overline{d} + \sum_{j \in S_\pm} (D_j, d) \xi_j$$

be the decomposition of some $d \in \mathrm{cochar}(K)$. Then the I-functions of $X_\pm$ are given by

$$I_\pm^{\text{temp}} = z e^{\frac{\sigma}{z}} \sum_{d \in \mathbb{K}_\pm} e^{\sigma \overline{d}} \prod_{j \in S_\pm} x_j^{D_j \cdot d} \left(\prod_{j=1}^{m} \frac{\prod_{a \leq 0, \langle a \rangle = \langle D_j \cdot d \rangle} (u_j + az)}{\prod_{a \leq D_j \cdot d, \langle a \rangle = \langle D_j \cdot d \rangle} (u_j + az)} \right).$$

Precisely, we have

$$I_\pm \in H^*_{\mathrm{CR},T}(X_{\omega_\pm}) \otimes_{R_T} R_T (\!(z^{-1})\!) [Q, \sigma, x].$$

In the above formula, we will make the substitutions $Q = 1$ and

$$\sigma_\pm = \sum_{j=1}^{\ell_\pm} \theta^\pm (p_j^\pm) \log(y_j) - \sum_{j \in S_\pm} \lambda_j \log x_j + c_0 \lambda.$$

Here, $\lambda_j = \theta^+(D_j)$, where θ^+ is the restriction to H^2, and $c_0 \lambda = \sum \lambda_i$.

A more explicit formula is given by

$$I_+ = z e^{\frac{\sigma_+}{z}} \sum_{d \in \mathbb{K}_+} y^d \prod_{j \in S_\pm} x_j^{D_j \cdot d} \left(\prod_{j=1}^{m} \frac{\prod_{a \leq 0, \langle a \rangle = \langle D_j \cdot d \rangle} (u_j + az)}{\prod_{a \leq D_j \cdot d, \langle a \rangle = \langle D_j \cdot d \rangle} (u_j + az)} \right) \mathbb{1}_{[-d]},$$

where $[-d] \in \mathbb{K}/\mathbb{L}$ is the twisted sector.

Claim 2.1.10. *The I-function I_+ defines a convergent power series.*

Idea of proof. Choose $f \in \mathbb{K}/\mathbb{L}$ and d such that $[-d] = f$. Then $X_+^f \subset IX_+$ is a connected component and a closed substack of X_+. The restriction

$$H_T^*(X_+^f) \to H_T^*(X_+^{f,T})$$

is injective. Then

$$I^+|_{X_f^T} \propto \Phi(\beta, X),$$

where $\beta_j = \frac{u_j}{z}$, $x = y_r^{p_r^+ \cdot e}$, and

$$\Phi(\beta, x) = \sum_{k \in \mathbb{Z}} x^k \frac{\prod_{a \leq 0, \langle a \rangle = \langle D_j \cdot d \rangle} (\beta_j + a)}{\prod_{a \leq D_j \cdot d + k D_j \cdot e, \langle a \rangle = \langle D_j \cdot d \rangle} (\beta_j + a)}.$$

For $k \ll 0$, because $R_j = \langle D_j \cdot d \rangle = 0$ for $j \in \delta \subset I_f$ and $D_j \cdot e > 0$, the x^k-term vanishes. Convergence follows by the ratio test. □

Because Φ solves an explicit differential equation[1] with singularities only at $0, \infty, \prod (D_j \cdot e)^{D_j \cdot e}$, it can be analytically continued to

$$\widetilde{\mathcal{M}}_+ = \left(\widehat{U}_+ \setminus \left\{ v^e = \text{conifold} \right\} / \text{Deck} \right)^{\text{univ cover}}.$$

For just X_+, there are differential operators P_i such that

$$(z^{-1} P_1 I_+, \ldots, z^{-1} P_2 I_+, \ldots) = L^{-1} \Gamma,$$

where $L^{-1} = 1 + O(z^{-1})$ and $\Gamma = \text{const} + O(z)$. Therefore, we may determine L^{-1} by Birkhoff factorization. This gives us quantum connections on $\widetilde{\mathcal{M}}_\pm$.

---

[1] We can write the coefficient of x^k as a ratio of products of gamma functions.

2.2 Crepant transformation conjecture for toric complete intersections (Davis, Apr 18)

Our goal is now to relate $I_\pm$ to each other. More precisely, they will differ by a gauge transformation, which is a Fourier-Mukai transform. The main tool will be to compute in localized equivariant cohomology.

2.2.1 Gauge transformation

First, we will rewrite the I-functions as ratios of Γ-functions:

$$I_+(y,z) = ze^{\frac{\sigma_+}{z}} \sum_{d \in \mathbb{K}_+} \frac{y^d}{z^{\sum D_i \cdot d}} \prod_{j=1}^{m} \frac{\Gamma\left(1 + \frac{u_j}{z} - \langle D_j \cdot d \rangle\right)}{\Gamma\left(1 + \frac{u_j}{z} + D_j \cdot d\right)} \frac{\mathbb{1}_{[-d]}}{z^{\iota_{[-d]}}},$$

where $\iota_{[-d]}$ is the age. We will not work directly with these. Instead, we will consider

$$H_+(y) = e^{(2\pi i)^{-1}\sigma_+} \Sigma_{d \in \mathbb{K}_+} y^d \prod_{j=1}^{m} \frac{1}{\Gamma(1 + u_j (2\pi i)^{-1} + D_j \cdot d)} \mathbb{1}_{[-d]}.$$

Using the $\widehat{\Gamma}$ class, it is related to the I-function by the formula

$$z^{-1} I_+(y,z) = z^{\frac{-c_0 \lambda}{2\pi i} - \frac{\dim X_+}{2}} z^{-\mu^+} z^{\rho^+} (\widehat{\Gamma}_{X_+} \cup (2\pi i)^{\frac{\deg_0}{2}} \operatorname{inv}^* H_+(z^{-\frac{\deg(y)}{2}} y)).$$

Recall that T-fixed points on I_{X_+} correspond to pairs (δ, f), where δ is a minimal anticone and $f \in \mathbb{K}_+/\mathbb{L}$. Then we compute

$$\iota_{(\delta,f)}^* H_+ = \sum_{\substack{d \in \mathbb{K}_+ \\ [d]=f}} \frac{y^d}{\prod_{j \in \delta} \Gamma(1 + (2\pi i)^{-1} u_j(\delta) + D_j \cdot d)} \cdot \frac{e^{(2\pi i)^{-1}\sigma_+ \delta}}{\prod_{j \notin \delta} \Gamma(1 + (2\pi i)^{-1}) u_j(\delta) + D_j \cdot d}.$$

Note that for $j \in \delta$, $u_j(\delta) = 0$. In addition, $D_j \cdot d \in \mathbb{Z}$, so if it is negative, then the corresponding Γ-function vanishes. Therefore, we define

$$\delta^\vee := \left\{d \in \mathbb{L}_\mathbb{Q} \mid D_j \cdot d \in \mathbb{Z}_{\geq 0} \text{ for all } j \in \delta\right\}$$

and the I-function becomes

$$\iota_{(\delta,f)}^* H_+ = \sum_{\substack{d \in \delta^\vee \\ [d]=f}} \frac{y^d}{\prod_{j \in \delta} \Gamma(1 + (2\pi i)^{-1} u_j(\delta) + D_j \cdot d)} \cdot \frac{e^{(2\pi i)^{-1}\sigma_+ \delta}}{\prod_{j \notin \delta} \Gamma(1 + (2\pi i)^{-1}) u_j(\delta) + D_j \cdot d}.$$

If $\delta \in \mathcal{A}_{\omega_+} \cap \mathcal{A}_{\omega_-}$, then $\iota^*(\delta, f) H_+ = \iota_{(\delta,f)}^* H_-$. Otherwise, δ has the form $\left\{j_1, \ldots, j_{r-1}, j_+\right\}$, where all $D_{j_i} \in W$ for $i < r$ and $D_{j_+} \cdot e > 0$ for the normal vector to W pointing towards ω_+.

Definition 2.2.1. A pair $(\delta_\pm \in \mathcal{A}_{\omega_\pm})$ are *next to each other* if both are of the form

$$\delta_\pm = \left\{j_1, \ldots, j_{r-1}, j_\pm\right\},$$

where $\pm D_{j_\pm} \cdot e > 0$. Likewise, $(\delta_\pm, f_\pm)$ are *next to each other* if $(\delta_\pm)$ are and $f_- = f_+ + \alpha e \in \mathbb{L}_\mathbb{Q}/\mathbb{L}$.

Lemma 2.2.2. *If δ_+ is next to δ_-, for all j and for any $j_- \in \delta_- \cap \delta_+^c$,*

$$u_j(\delta_+) - u_j(\delta_-) = \frac{D_j \cdot e}{D_{j_-} \cdot e} u_{j_-}(\delta_+).$$

Proof. Expand

$$D_j = \sum_{i=1}^{r-1} c_j D_{j_i} + c_- D_{j_-}.$$

Then $D_j \cdot e = c_- D_{j_-} \cdot e$. Because

$$\begin{aligned}\theta(D_j) &= u_j - \lambda_j \\ &= \sum c_i (u_{j_i} - \lambda_{j_i}) + c_i (u_{j_-} - \lambda_{j_-})\end{aligned}$$

and $u_{j_-}(\delta_-) = 0$, we see that

$$\begin{aligned}u_j(\delta_+) - \lambda_j &= -\sum c_i \lambda_{j_i} + c_- (u_{j_-}(\delta_+) - \lambda_{j_-}) \\ u_j(\delta_-) - \lambda_j &= -\sum c_i \lambda_{j_i} + c_- (u_{j_-}(\delta_-) - \lambda_{j_-}).\end{aligned}$$

This implies that

$$u_j(\delta_+) - u_j(\delta_-) = c_- u_{j_-}(\delta_+).$$

□

Claim 2.2.3. *We have the formula*

$$\iota^*_{(\delta_+, f_+)} H_+ = \sum_{\substack{(\delta_-, f_-) \\ \textit{next to} \\ (\delta_+, f_+)}} c^{\delta_-, f_-}_{\delta_+, f_+} \iota^*_{(\delta_-, f_-)} H_-,$$

where c is some explicit matrix[2] *made of products of things like*

$$\prod_{\substack{j | D_j \cdot e < 0 \\ j \neq j_-}} \frac{\sin \pi ((2\pi i)^{-1} u_j(\delta_+) + D_j \cdot f_+)}{\sin \pi ((2\pi i)^{-1} u_j(\delta_-) + D_{j_-} \cdot f_-)}.$$

Proof. Idea of proof If $d \in \delta_+$, then $D_j \cdot d \geq 0$ for all $j \in \delta_+$, so we can write $d = D_+ + ke$, where $d_+ \in \delta_+^{vee}$ and $D_+ - e \notin \delta_+^\vee$. We now obtain

$$\iota^*_{(\delta_+, f_+)} H_+ = \sum_{d_+} y^{d_+} \sum_{k=0}^{\infty} \frac{(y^e)^k e^{(2\pi i)^{-1} \sigma_+ \delta_+}}{\prod_{j=1}^m \Gamma(1 + (2\pi i)^{-1} u_j(\delta_+) + D_j \cdot d_+ + k D_j \cdot e)}.$$

Using the reflection formula

$$\begin{aligned}\Gamma(1 + (2\pi i)^{-1} u_j(\delta_+) + D_j \cdot d_+ + k D_j \cdot e) =& (-1)^{k D_j \cdot e} \frac{\sin \pi (-(2\pi i)^{-1} u_j \delta_+ - D_j \cdot d_+)}{\pi} \\ & \times \Gamma(-(2\pi i)^{-1} u_j \delta_+ - D_j \cdot d_+ - k D_j \cdot e),\end{aligned}$$

we may write

$$\iota^*_{(\delta_+, f_+)} H_+ = \mathrm{Res}_{s=k} (\cdots) \Gamma(s) \Gamma(1-s) \prod_{D_j \cdot e < 0} \Gamma(-(2\pi i)^{-1} u_j(\delta_+) - D_j \cdot d_+ - s D_j \cdot e).$$

We obtain poles at the following locations:

(1) $\Gamma(1-s)$ gives poles at $s = 1, 2, 3, \ldots$;

(2) $\Gamma(s)$ gives poles at $s = 0, -1, -2, \ldots$;

---

[2] The reason the formulae are so complicated is because we are using the wrong basis.

(3) The last term gives poles at $s = (-D_{j_-} \cdot e)^{-1}((2\pi i)^{-1} u_j(\delta_+) + D_{j_-} \cdot d_+ - n)$.

We can then rewrite this as a contour integral where the contour is such that all poles of type (1) are on the right and all poles of types (2) and (3) except the one at 0 are on the left. For small y, we have

$$\iota^*_{\delta_+, f_+} H_+ = \oint_{\text{right}},$$

and for large y, we have

$$\iota^*_{\delta_+, f_+} H_+ = \oint_{\text{left}} = \sum c^{\delta_-, f_-}_{\delta_+, f_+} \iota^*_{\delta_-, f_-} H_-.$$

Choose some j_- for which $D_{j_-} e < 0$, we obtain

$$\mathrm{Res}_{j_-} = \prod_{D_j \cdot e < 0, j \neq j_-} \frac{\Gamma}{\sin}.$$

Setting $\delta_- = \{j_1, \dots, j_{r-1}, j_-\}$ and

$$d_- = d_+ + \frac{D_{j_-} \cdot d_+ - 1}{-D_{j_i} \cdot e} e,$$

we use the lemma to rewrite some $\Gamma(\dots u_j(\delta_+))$ in terms of $u_j(\delta_-)$ and conclude. □

We have therefore obtained the formula

$$\iota^*_{(\delta_+, f_+)} H_+ = \sum_{\substack{(\delta_-, f_-) \\ \text{next to} \\ (\delta_+, f_+)}} \sum_{\substack{d_- \in \delta_-^\vee \\ \text{minimal} \\ [d_-] = f_-}} y^{d_-} \sum_{k=0}^{\infty} \frac{e^{(2\pi i)^{-1} \sigma_i \delta_- (y^e)^{-k}}}{\prod_{j=1}^m \Gamma(1 + (2\pi i)^{-1} u_j(\delta_-) + D_j \cdot d_- - k D_j \cdot e)} c^{\delta_-, f_-}_{\delta_+, f_+} \iota^*_{\delta_-, f_-} H_-.$$

Now define

$$\mathbb{U}_H(\alpha) := \sum_{\delta \in \mathcal{A}_+ \cap \mathcal{A}_-, f} (\iota^*_{\delta, f} \alpha) \frac{\mathbb{1}_{\delta, f}}{e^T(N_{\delta, f})} + \sum_{\delta_+, f_+} \sum_{\substack{\delta_-, f_- \\ \text{next to} \\ \delta_+, f_+}} c^{\delta_-, f_-}_{\delta_+, f_+} (\iota^*_{\delta_-, f_-} \alpha) \frac{\mathbb{1}_{\delta_-, f_-}}{e^T(N_{\delta_-, f_-})}.$$

We have a commutative diagram

$$\begin{tikzcd}
H^*_T(IX_i)^{\text{comp}}_{\text{loc}} \arrow[r, "\mathbb{U}_H"] \arrow[d, "z^{-\mu_-} z^{-\rho_-} (\Gamma_{X_-} \widehat{\cup (2\pi i)}^{\frac{\deg_0}{2}} \mathrm{inv}^* \alpha)"] & H^*_T(IX_+)^{\text{comp}}_{\text{loc}} \arrow[d] \\
H^*_{\mathrm{CR}, T}(X_-)_{\text{loc}}[\log z](\!(z^{-\frac{1}{k}})\!) \arrow[r, "\mathbb{U}"] & H^*_{\mathrm{CR}, T}(X_+)_{\text{loc}}[\log z](\!(z^{-\frac{1}{k}})\!).
\end{tikzcd}$$

2.2.2 The Fourier-Mukai transform

We will now prove that $\mathbb{U}$ is given by the Fourier-Mukai transform. Here, let $\widetilde{X}$ be a common toric blowup of $X_\pm$ as in

$$\begin{tikzcd}
& \widetilde{X} \arrow[dl, "f_+"'] \arrow[dr, "f_-"] & \\
X_+ & & X_-
\end{tikzcd}$$

and $\mathbb{FM} := (f_+)^*(f_-)^* := K^0_T(X_-) \to K^0_T(X_+)$. We see that $K^0_T(X_\pm)$ is generated by $[L_{\pm(\rho)}]$, where $\rho \in \mathrm{char}(K)$ and $K^0_T(\widetilde{X})$ is generated by $[L_\pm(\rho, n)]$. We will now move to the fixed point basis and see that

$$e_{\delta, \rho} = L_\pm(\rho) \prod_{i \notin \delta} (1 - L(D_i)^{-1} \otimes e^{-\lambda_i}).$$

Remark 2.2.4. Let x_δ be a fixed point of $X_\pm$ and ρ be in its isotropy. Then $e_{\delta,\rho} = (\iota_\delta)_* \rho$.

We now compute $\mathbb{FM}(e_{\delta,\rho})$.

1. If $\delta \in \mathcal{A}_- \cap \mathcal{A}_+$, then $\mathbb{FM}(e_{\delta,\rho}) = e_{\delta,\rho}$;
2. Otherwise, we have a complicated formula.

The computation involves doing localization and relating anticones of $X_\pm$ to those of $\widetilde{X}$. We then obtain a commutative diagram

$$\begin{tikzcd} K_T^0(X_-) \arrow[r, "\mathbb{FM}"] \arrow[d, "\widetilde{\mathrm{ch}}"'] & K_T^0(X_+) \arrow[d, "\widetilde{\mathrm{ch}}"] \\ H_T^*(IX_i) \arrow[r, "\mathbb{U}_H"] & H_T^*(IX_+) \end{tikzcd} \tag{2.1}$$

by direct computation.

2.2.3 Crepant transformation conjecture

We are now ready to state the crepant transformation conjecture.

Theorem 2.2.5. *Let $\mathcal{H}(X_\pm) := H^*_{\mathrm{CR},T}(X_\pm)(\!(z^{-1})\!)$. Then there exists a degree-preserving symplectic transformation*

$$\mathbb{U} := \mathcal{H}(X_-) \to \mathcal{H}(X_+)$$

such that

1. *$I_+ = \mathbb{U} I_-$ after analytic continuation;*
2. *If $g_\pm : X_\pm \overline{X}_=$ is the map to the toric blowdown, then*

   $$\mathbb{U} \circ (g_-^* v \cup) = (g_+^* v \cup) \circ \mathbb{U}$$

   for all $v \in H_T^2(\overline{X}_0)$;
3. *$\mathbb{U}$ fits into the commutative diagram (2.1).*

Theorem 2.2.6 (Crepant transformation conjecture for toric DM stacks)**.** *Let $(F^\pm, E^\pm, \nabla^\pm)$ be the quantum connections on $X_\pm$. Then there exists a gauge transformation*

$$\Theta \in \mathrm{Hom}(H^*_{\mathrm{CR},T}(X_-), H^*_{\mathrm{CR},T}(X_+)) \otimes_{R_T} (\mathcal{O}_{U^0} \otimes R_T)[z][\![y_1, \dots, y_{r-1}]\!]$$

such that

1. *$\nabla^+ \Theta = \Theta \nabla^-$;*
2. *Θ is homogeneous of degree 0;*
3. *Θ preserves the orbifold Poincaré pairings.*

The proof starts by applying differential operators to $\mathbb{U} I_- = I_+$ to obtain

$$z^{-1} \vec{P}_\pm I_\pm = e^{\frac{\sigma_\pm}{z}} L_\pm^{-1} \gamma_\pm.$$

We then set $\Theta = \gamma_+ \gamma_-^{-1}$ and obtain

$$(e^{\frac{\sigma_+}{z}} L_+(y,z)^{-1}) \Theta(y,z) = \mathbb{U}(e^{\frac{\sigma_-}{z}} L_-(y,z)^{-1}).$$

2.3 Wall-crossing for Grassmannian flops (Kostya, Apr 25)

We will now consider a wall-crossing in a nonabelian situation. This relies on reducing to the abelian case and citing CIJ.

2.3.1 Geometric setup Let B be a smooth projective variety and let E, F be vector bundles on B which split as direct sums of line bundles

$$E = \bigoplus_{i=1}^{n} L_i, \qquad F = \bigoplus_{i=1}^{n} M_i.$$

Now let

$$V = \underline{\mathrm{Hom}}(F, \mathbb{C}^r) \oplus \underline{\mathrm{Hom}}(\mathbb{C}^r, E)$$

and $G = GL_r$. G acts on the total space of V by the formula

$$(X, Y) \circ g = (g^{-1}X, Yg).$$

Let $X_\pm = V /\!\!/_{\pm \det} G$ be the corresponding GIT quotients of V. We obtain

$$\begin{tikzcd}
X_+ = \mathrm{tot}(S_+ \otimes p^* F^\vee) \arrow[r] & \mathrm{Gr}(r, E) \arrow[d, "p"] \\
 & B \\
X_- = \mathrm{tot}(S_- \otimes p^* E) \arrow[r] & \mathrm{Gr}(r, F^\vee). \arrow[u, "p"']
\end{tikzcd}$$

The birational transformation here is known as a Grassmannian flop.

Example 2.3.1. Consider the quiver given by

$$\boxed{k} \longrightarrow \textcircled{r} \longrightarrow \textcircled{k} \longrightarrow \boxed{n}.$$

The corresponding moduli of semistable representations is

$$(\mathbb{C}^k \to \mathbb{C}^r) \times (\mathbb{C}^r \times \mathbb{C}^k) \times (\mathbb{C}^k \to \mathbb{C}^n) /\!\!/ \mathrm{GL}_r \times \mathrm{GL}_k\,.$$

Then

$$X_+ = \mathrm{tor}(\widehat{S}_r^{\oplus k}) \to \mathrm{Fl}(r, k, n), \qquad X_- = \mathrm{tot}(\overline{S}_r^\vee \otimes S_k) \to \mathrm{Gr}(r, k) \times \mathrm{Gr}(k, n).$$

There is an action of $\mathbb{T} = (\mathbb{C}^\times)^{2n}$ on the total space of $E \oplus F$, which induces an action of $\mathbb{T}$ on V. Denote

$$x_i = c_1^{\mathbb{T}}(L_i^\vee) \qquad z_i = c_1^{\mathbb{T}}(M_i^\vee).$$

Let $R = \mathbb{C}^r$, viewed as a right G-representation. Then $R \times V \to V$ is a G-equivariant vector bundle on V which descends to $X_\pm$ and are the pullbacks of $S_+^\vee$ and S_-, repsectively. Now denote by

$$y_1, \dots, y_r$$

the Chern roots of $R^\vee$. Then

$$H_{\mathbb{T}}^*(X_\pm) = H_{\mathbb{T}}^*(B)[X_i, z_i][Y_k]^{S_n} / I_\pm,$$

where I_+ is the ideal given by

$$I_+ = \left\{ \left[\frac{\prod_{i=1}^n (1 - x_i)}{\prod_{j=1}^r (1 + y_j)} \right]_\ell \;\middle|\; \ell > n - r \right\}$$

and I_- is defined similarly. Let $R_\mathbb{T} = H^*_\mathbb{T}(\mathrm{pt})$ and $S_\mathbb{T} = R_{\mathbb{T},\mathrm{loc}}$. Finally, let $\widehat{S}_\mathbb{T}$ be a completion of $S_\mathbb{T}$ containing exponentials and Γ-functions. The $\mathbb{T}$-fixed loci in $X_\pm$ correspond to $r \times r$ minors of the corresponding Grassmannians. We encode them by

$$\delta \colon \{1, \ldots, r\} \to \{1, \ldots, n\}$$

which are strictly order-preserving, as in $\delta_i := \delta(i) < \delta(j) =: \delta_j$ if $i < j$. In other words,

$$X_\pm^\mathbb{T} = \bigsqcup_{\delta^\pm} B_{\delta^\pm} \cong B.$$

Lemma 2.3.2. *We have*

$$R|_{B_{\delta^-}} = \bigoplus_{i=1}^r M^\vee_{\delta^-_i}$$
$$R|_{B_{\delta^+}} = \bigoplus_{i=1}^r L^\vee_{\delta^+_i}.$$

This implies that

$$y_i|_{B_{\delta^-}} = -z_{\delta^-_i}$$
$$y_i|_{B_{\delta^+}} = -x_{\delta^+_i}.$$

2.3.2 Abelianization

Let $(C^\times)^r \subset G = \mathrm{GL}_r$ be the maximal torus which acts diagonally on $\mathbb{C}^r$. Then the abelianized quotients are

$$\begin{aligned} X_{T,-} &= V /\!\!/_{\theta_-} T \\ &= \mathrm{tot}(\mathcal{O}_{\mathbb{P}(F^\vee)}(-1) \otimes p^*E) \times_B \cdots \times_B \mathrm{tot}(\mathcal{O}_{\mathbb{P}(F^\vee)}(-1) \otimes p^*E) \\ X_{T,+} &= V /\!\!/_{\theta_+} T \\ &= \mathrm{tot}(\mathcal{O}_{\mathbb{P}(E)}(-1) \otimes p^*F^\vee) \times_B \cdots \times_B \mathrm{tot}(\mathcal{O}_{\mathbb{P}(E)}(-1) \otimes p^*F^\vee). \end{aligned}$$

Note that T has many more stability conditions than T. Denote

$$\theta_j := \prod_{k=1}^j t_k \prod_{k=j+1}^r t_k^{-1}$$

and the GIT quotient

$$X_{T,j} := V /\!\!/_{\theta_j} T.$$

Therefore, $X_{T,0} = X_{T,-}$ and $X_{T,r} = X_{T,+}$.

The cohomology of $X_{T,j}$ is given by

$$H^0_\mathbb{T}(X_{T,j}) = H^*_\mathbb{T}(B)[x_i, z_i][y_k]/I^{\mathrm{ab}}_j,$$

where the ideal I^{ab}_j is given by

$$I^{\mathrm{ab}}_j = \left\{ \prod_{i=1}^n (-x_i - y_k) \;\middle|\; 1 \le k \le j \right\} \cup \left\{ \prod_{i=1}^n (z_i + y_k) \;\middle|\; j + 1 \le k \le r \right\}.$$

The $\mathbb{T}$-fixed loci are indexed by points in $\mathbb{P}^{n-1}$. These are denoted by arbitrary functions

$$f\colon \{1,\dots,r\} \to \{1,\dots,n\}.$$

Note that f is not required to be injective. Thus we have a decomposition

$$X_{T,j}^{\mathbb{T}} = \bigsqcup_f B_f \cong B.$$

The first abelianization in cohomology is

$$H_{\mathbb{T}}^*(X_\pm) \cong H_{\mathbb{T}}^*(V^{\pm\mathrm{ss}}/T)^W.$$

Next, note the diagram

$$\begin{tikzcd}
X_\pm & \\
V^{\pm\mathrm{ss}}(G)/T \arrow[u, two heads] \arrow[r, hook] & X_{T,\pm} \arrow[ul, dashed]
\end{tikzcd}$$

gives us a rational map between the abelian and nonabelian quotients. This gives us a surjection

$$H_{\mathbb{T}}^*(X_{T,\pm}) \twoheadrightarrow V^{\pm\mathrm{ss}}(G)/T.$$

The kernel of this morphism are those f which are not injective.

2.3.3 Fourier-Mukai transform

Consider the mutual blowup

$$\begin{tikzcd}
& \widetilde{X} \arrow[dl, "f_+"'] \arrow[dr, "f_-"] & \\
X_+ & & X_- \arrow[ll, dashed]
\end{tikzcd}$$

and let

$$f_+ \coloneqq f_{+,*} f_-^* \colon K_{\mathbb{T}}(X_-) \to K_{\mathbb{T}}(X_+).$$

Here,

$$\widetilde{X} = (F \to \mathbb{C}^r) \times (\mathbb{C}^r \to \mathbb{C}^r) \times (\mathbb{C}^r \to E) /\!\!/_{(\det^{-1},\det)} \mathrm{GL}_r^2.$$

Then the morphisms are

$$f_+((X,U,Y),(g_1,g_2)) = ((X,YU),g_1), \qquad f_-((X,U,Y),(g_1,g_2)) = ((UX,Y),g_2).$$

The fixed loci in $\widetilde{X}$ are indexed by δ^+,δ^-.

Lemma 2.3.3. *The projections $f_\pm$ map $B_{(\delta^+,\delta^-)}$ isomorphically to B_{δ^+} or B_{δ^-}, respectively. Moreover,*

$$R_1|_{B_{(\delta^+,\delta^-)}} = \bigoplus_{i=1}^r M_{\delta_i^-}^\vee$$
$$R_2|_{B_{(\delta^+,\delta^-)}} = \bigoplus_{i=1}^r L_{\delta_i^+}^\vee.$$

Lemma 2.3.4. *The localized $\mathbb{T}$-equivariant K-theory of X_- is spanned by classes of the form*

$$\pi^*(A) \otimes e_{\delta_-} = \pi^* A \otimes \prod_{j \notin \delta^-} \Lambda^*(M_j^\vee \otimes R^\vee)^\vee$$

for $A \in K(B)$ and similarly for X_+.

Lemma 2.3.5. *We have*

$$\mathrm{FM}(e_{\delta^-}) = \prod_{j \notin \delta^-} \Lambda^*(M_j^\vee \otimes R^\vee)^\vee \in K_{\mathbb{T}}(X_+).$$

Proposition 2.3.6. *The following diagram commutes:*

$$\begin{tikzcd}
K_{\mathbb{T}}(X_-) \arrow[r, "\mathrm{FM}"] \arrow[d, "\mathrm{ch}_-^{\mathbb{T}}"] & K_{\mathbb{T}}(X_+) \arrow[d, "\mathrm{ch}_+^{\mathbb{T}}"] \\
H^*_{\mathbb{T}}(X_-) \otimes \widehat{S}_{\mathbb{T}} \arrow[r, "\mathbb{U}_H"] & H^*(X_+) \otimes \widehat{S}_{\mathbb{T}}.
\end{tikzcd}$$

Here, $\mathbb{U}_H$ *is an explicit* $\widehat{S}_{\mathbb{T}}$*-linear operator given by the formula*

$$\mathbb{U}_H\left(\frac{\alpha}{e_{\mathbb{T}}(N_{\delta^-})}\right) = \sum_{\delta^+} C_{\delta^-,\delta^+} \frac{\phi_{\delta^+,\delta^-}(\alpha)}{e_{\mathbb{T}}(N_{\delta^+})},$$

where

$$C_{\delta^-,\delta^+} = \prod_{i=1}^{r} e^{(n-r)(x_{\delta_i^+} - z_{\delta_i^-})/2} \prod_{j_- \in \delta^-} \frac{\sin\left(\frac{x_{\delta_i^+} - z_{j_-}}{2i}\right)}{\sin\left(\frac{z_{\delta_i^-} - z_{j_-}}{2i}\right)}.$$

The strategy to prove this result is first to replace cohomology by the symplectic space and replace the Chern character by the K-group framing. Then we use abelianization for I-functions [Web23] and repeatedly apply CIJ to each wall-crossing from the X_j. We then compose the symplectic transformations from CIJ and check that it equals the one given above. Finally, a statement about deforming the contour into the Weyl-invariant part gives the desired result.

3

Quantum cohomology of projective bundles

3.1 Mirror theorem (Che, Feb 22)

3.1.1 Setup Let X be a smooth projective variety, $\{\phi_i\}_{i=0}^s$ be a basis of $H^*(X)$, $\{\phi^i\}_{i=0}^s$ be the dual basis, and

$$\tau = \sum_{i=0}^s \tau^i \phi_i \in H^*(X).$$

We will let

$$J_X(\tau) = 1 + \frac{\tau}{z} + z^{-1} \sum_{d,n} \sum_{j=0}^s \langle \tau, \ldots, \tau \frac{\phi_j}{z-\psi} \rangle_{0,n+1,d}^X \frac{Q^d}{n!},$$

which is the J-function in Definition 1.1.3 multiplied by z^{-1}.[1]. Also, recall the inverse of the fundamental solution of the quantum D-module

$$M_X(\tau) \in \mathrm{End}(H^*(X))[z^{-1}][\![Q,\tau]\!],$$

which is defined by

$$(\overline{M}_X(\tau)\phi_i, \phi_j) = (\phi_i, \phi_j)_X + \sum_{d,n} \langle \phi_i, \tau, \ldots, \tau, \frac{\phi_j}{z-\psi} \rangle_{0,n+2,d}^X \frac{Q^d}{n!}.$$

Remark 3.1.1. By the string equation, we have

$$J_X(\tau) = M_X(\tau) \cdot 1.$$

3.1.2 The vector bundle case Now let $V \to B$ be a vector bundle with $\mathrm{rk}\, V \geq 2$. This has an action of $\mathbb{C}^\times$ scaling the fibers. Then we have

$$H^*_{\mathbb{C}^\times}(V) = H^*(B) \otimes \mathbb{C}[\lambda].$$

Now we may take $\tau^0, \ldots, \tau^s$ to be $\mathbb{C}[\lambda]$-valued coordinates.

Remark 3.1.2. Equivariant localization is required to define the Gromov-Witten invariants of V, which lie in $\mathbb{C}[\lambda, \lambda^{-1}]$.

---

[1] This is in fact the older definition of the J-function, but the one in Definition 1.1.3 lies on the Lagrangian cone

In order to avoid this issue, we will assume that $V^\vee$ is globally generated. This implies that V is semiprojective, meaning that the evaluation maps $\mathrm{ev}\colon V_{0,n,d} \to V$ are proper. As before, we may define the fundamental solution

$$M_V(\tau) \in \mathrm{End}(H^*(B))[\lambda, z^{-1}][\![Q,\tau]\!]$$

and the J-function

$$J_V^\lambda(\tau) = M_V(\tau)\cdot 1.$$

Because the evaluation maps are proper, they can be defined without localization.

3.1.3 Statement and discussion of the mirror theorem

Theorem 3.1.3. *Define the $H^*(\mathbb{P}(V))$-valued function*

$$I_{\mathbb{P}(V)}(\tau,t) = \sum_{k=0}^{\infty} \frac{e^{pt/z} q^k e^{kt}}{\prod_{c=1}^{k} \prod_\delta (p+\delta+cz)} J_V^{p+kz}(\tau),$$

where δ are the Chern roots of V, q is the Novikov variable, and $p = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$. Then $zI_{\mathbb{P}(V)}(\tau,t)$ lies on the Lagrangian cone of $\mathbb{P}(V)$.

Let $\mathcal{L}_X^{\mathrm{orig}}$ be the Lagrangian cone for X, which has the explicit form

$$-z + t(z) + \sum_{d,n}\sum_{k\geq 0}\sum_{i=0}^{s} \frac{\phi^i}{(-z)^{k+1}} \langle t(\psi),\ldots,t(\psi),\phi_i\psi^k\rangle_{0,n+1,d}^X \frac{Q^d}{n!}. \tag{3.1}$$

Definition 3.1.4. For a set of variables $x = (x_1, x_2, \ldots)$, we say that $f \in \mathcal{H}_X[\![x]\!]$ is a $\mathbb{C}[\![Q,x]\!]$*-valued point* on $\mathcal{L}_X^{\mathrm{orig}}$ if f is of the form 3.1 for some $t(z) \in \mathcal{H}_+[\![x]\!]$ with $t(z)|_{Q=x=0} = 0$.

Example 3.1.5. The point $zJ_X(\tau)|_{z\mapsto -z}$ is a $\mathbb{C}[\![Q,\tau]\!]$-valued point on $\mathcal{L}_X^{\mathrm{orig}}$.

Given this, define $\mathcal{L}_X := \mathcal{L}_X^{\mathrm{orig}}|_{z\mapsto -z}$. By Theorem 1.1.10, we obtain

$$L_X = \bigcup_\tau zM_X(\tau)\mathcal{H}_+,$$

which means that any $\mathbb{C}[\![Q,x]\!]$-valued point on $\mathcal{L}_X$ can be written as $zM_X(\tau)f$ for some $\tau \in H^*(X)[\![Q,x]\!]$ and $f \in \mathcal{H}_+[\![x]\!]$ such that $\tau|_{Q=x=0} = 0$ and $f|_{Q=x=0} = 1$. This property will be used to construct the Fourier transform later.

3.1.4 Proof of Theorem 3.1.3

We will now sketch a proof of Theorem 3.1.3. First, we will need Quantum-Riemann-Roch for a vector bundle $W \to X$ in two cases:

(a) When the vector bundle W is convex, which means that $H^1(C, f^*W) = 0$ for all stable maps $f\colon C \to X$ of genus 0, and $\mathbf{c} = e(\lambda)$ is the equivariant Euler class, which corresponds to setting

$$s_k = \begin{cases} \log\lambda & k=0 \\ (-1)^{k-1}(k-1)!\lambda^{-k} & k>0. \end{cases}$$

(b) When W is globally generated and $\mathbf{c} = e_\lambda^{-1}$.

In the first case, we obtain the Gromov-Witten invariants of the zeroes of a regular section $Z \subset X$ of W via

$$\lim_{\lambda\to 0}\langle \alpha_1\psi^{k_1},\dots,\alpha_n\psi^{k_n}\rangle_{0,n,d}^{X,(W,e_\lambda)} = \sum_{i_*d'=d}\langle i^*\alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n}\rangle_{0,n,d'}^{Z}.$$

In the second case, we obtain the Gromov-Witten invariants of W via

$$\langle \alpha_1\psi^{k_1},\dots,\alpha_n\psi^{k_n}\rangle_{0,n,d}^{X,(W,e_\lambda^{-1})} = \langle i^*\alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n}\rangle_{0,n,d}^{W}.$$

We are now ready to begin the proof. Because $V^\vee$ is globally generated, there is a surjection

$$\mathcal{O}^{\oplus N}\to V^\vee.$$

This gives an exact sequence

$$0\to V\to\mathcal{O}^{\oplus N}\to Q\to 0$$

embedding $\mathbb{P}(V)\hookrightarrow B\times\mathbb{P}^{N-1}$. By a result of Brown-Elezi, we have

$$J_{B\times\mathbb{P}^{N-1}}(\tau,t)=\sum_{k=0}^{\infty}\frac{e^{pt/z}q^k e^{kt}}{\prod_{c=1}^{k}(p+cz)^N}J_B(\tau).$$

Now define

$$Q(1):=\pi_1^*Q\otimes\pi_2^*\mathcal{O}(1)$$

on $B\times\mathbb{P}^1$. This has a section s given by

$$\pi_2^*\mathcal{O}(-1)\to\mathcal{O}^{\oplus N}_{B\times\mathbb{P}^{N-1}}\to\pi_1^*Q$$

which satisfies $s^{-1}(0)=\mathbb{P}(V)$. Because $Q(1)$ is convex, we use Quantum-Riemann-Roch in case (a) to relate the Gromov-Witten theory of $\mathbb{P}(V)$ to the $(Q(1),e_\lambda)$-twisted Gromov-Witten theory. We now require two more techinical ingredients.

Moving points on the Lagrangian cone via differential operators

Lemma 3.1.6. *Let* $x=(x_1,x_2,\dots)$ *and* $y=(y_1,y_2,\dots)$ *be formal variables. Let*

$$F\in\mathbb{C}[z][\![x]\!]\langle z\partial_{x_1},z\partial_{x_2},\dots\rangle[\![Q,y]\!]$$

be a differential operator. Then $\exp(F/z)$ *preserves* $\mathbb{C}[\![Q,x,y]\!]$*-valued points on* $\mathcal{L}_X$*.*

Definition 3.1.7. A $\mathbb{C}[\![Q,\tau,y]\!]$-valued point f on $\mathcal{L}_X$ is called a *miniversal slice* if

$$f|_{Q=y=0}=z+\tau+\mathcal{O}(z^{-1}).$$

For example, the J-function is a miniversal slice.

Lemma 3.1.8. *Any miniversal slice on* $\mathcal{L}_X$ *can be obtained from* $zJ_X(\tau)$ *be applying* $\exp(F/z)$ *for some differential operator* F *as in the previous lemma satisfying* $F|_{Q=y=0}=0$.

The rest of the proof (ignoring convergence issues) First, we introduce

$$\Delta_W^\lambda := e^{\mathrm{rk}(W)(\lambda\log\lambda-\lambda)/z}\Delta_{(W,e_\lambda^{-1})}.$$

Because $\log\Delta_W^\lambda$ and $\log\Gamma(x)$ have similar asymptotic expansions, we have

$$\Delta_W^{\lambda+kz}/\Delta_W^\lambda = \prod_{c=1}^{k}\prod_{\delta}(\lambda+\delta+cz).$$

Using the exact sequence

$$0\to V\to\mathcal{O}^{\oplus N}\to Q\to 0,$$

we see that

$$\Delta_V^\lambda\Delta_Q^\lambda=\Delta_{\mathcal{O}^{\oplus N}}^\lambda,$$

which preserves the Lagrangian cone $\mathcal{L}_B$. We see that

$$\Delta_Q^\lambda\colon\mathcal{L}_{B,(V,e_\lambda^{-1})}\to\mathcal{L}_B.$$

Applying Quantum-Riemann-Roch in case (b), we see that

$$zJ_V^\lambda(z)\in L_{B,(V,e_\lambda^{-1})},$$

and thus

$$\Delta_Q^\lambda zJ_F^\lambda(z)\in\mathcal{L}_B.$$

By Lemma 3.1.8, there exists F such that

$$\Delta_Q^\lambda zJ_V^\lambda(z)=e^{F(\lambda)/z}zJ_B(\tau).$$

By Lemma 3.1.6, we obtain

$$e^{F(\lambda+z\partial_t)/z}J_{B\times\mathbb{P}^{N-1}}(\tau,t)\in\mathcal{L}_{B\times\mathbb{P}^{N-1}}.$$

Now we compute

$$\begin{aligned}
I^\lambda(\tau,t) &:= (\Delta_{Q(1)}^\lambda)^{-1}e^{F(\lambda+z\partial_t)/z}J_{B\times\mathbb{P}^{N-1}}(tau)\\
&=\sum_{k=0}\frac{e^{pt/z}q^ke^{kt}}{\prod_{c=1}^k(p+cz)^N}(\Delta_{Q(1)}^\lambda)^{-1}e^{F(\lambda+p+kz)}J_B(\tau)\\
&=\sum_{k=0}\frac{e^{pt/z}q^ke^{kt}}{\prod_{c=1}^k(p+cz)^N}(\Delta_{Q(1)}^\lambda)^{-1}(\Delta_Q^{\lambda+p})^{-1}\Delta_Q^{\lambda+p+kz}J_V^{\lambda+p+kz}\\
&=\sum_{k\geq 0}e^{pt/z}q^ke^{kt}\frac{\prod_{c=1}^k\prod_\varepsilon(\lambda+p+\varepsilon+cz)}{\prod_{c=1}^k(p+cz)^N}J_V^{\lambda+p+kz},
\end{aligned}$$

where ε runs over the Chern roots of Q. Taking the non-equivariant limit $\lambda\to 0$, we obtain the $I(\tau,t)$ in the statement of Theorem 3.1.3.

3.2 Fourier transform (Kostya, Feb 29)

Technically, there are two different Fourier transforms:

1. The discrete Fourier transform $\mathrm{QDM}_{S^1}(V)\xrightarrow{\mathrm{FT}}\mathrm{QDM}(\mathbb{P}(V))$;
2. The continuous Fourier transform $\mathrm{QDM}(\mathbb{P}(V))_{\mathrm{loc}}\to\bigoplus_{i=0}^{r-1}\mathrm{QDM}(B)$.

3.2.1 Quantum D-modules and symplectic spaces

Let $V \to B$ be a rank r vector bundle. The quantum D-module of V will be

$$H^*_{S^1}(V) \otimes \mathbb{C}[z, \lambda][\![Q, \boldsymbol{\tau}]\!],$$

where λ is the equivariant variable, Q is the Novikov variable, and $\boldsymbol{\tau} = \{\tau^{i,k}\}$ for i counting a basis of $H^*(V)$ and k records the degree of λ. It is equipped with the Dubrovin connection

$$\nabla \colon \mathrm{QDM}_{S^1}(V) \to z^{-1}\mathrm{QDM}_{S^1}(V)$$

given by

$$\begin{aligned}
\nabla_{\tau^{i,k}} &= \frac{\partial}{\partial \tau^{i,k}} + z^{-1}\lambda^k(\phi_i \star -) \\
\nabla_{\xi Q\partial_Q} &= \xi Q\partial_Q + z^{-1}(\xi \star -) \\
\nabla_{z\partial_z} &= z\partial_z - z^{-1}(E_{S^1} \star -) + \mu_{S^1}.
\end{aligned}$$

Note the last line is only $\mathbb{C}$-linear. Recall the fundamental solution $M_V^{-1}(\boldsymbol{\tau}, z)$ is a fundamental solution in the cohomology directions (but not the conformal direction) in the sense that

$$\begin{aligned}
\partial_{\tau^{i,k}} M_V &= M_V \nabla_{\tau^{i,k}} \\
(\varepsilon Q\partial_Q + z^{-1\xi})M_V &= M_V \nabla_{\varepsilon Q\partial_Q}
\end{aligned}$$

and intertwines the shift operator by

$$\mathcal{S} M_V = M_V \mathbb{S}(\boldsymbol{\tau}),$$

where

$$\mathcal{S} = e_\lambda(V) e^{z\partial_\lambda}.$$

Now, the symplectic space for V is

$$\mathcal{H}_V^{S^1} := H^*_{S^1}(V)(\!(z^{-1})\!)[\![Q, \tau]\!]$$

with its Lagrangian cone $\mathcal{L}_V$. It has the important property that $f(\boldsymbol{\tau})$ is in $\mathcal{L}_V$ means that there exists $\widehat{\tau}(\boldsymbol{\tau})$ and $\widetilde{f} \in \mathrm{QDM}_{S^1}(V)$ such that

$$f = zM_V(\widehat{\tau}(\boldsymbol{\tau}), z)\widetilde{f},$$

which can be seen as a Birkhoff factorization.

We now turn to $\mathbb{P}(V)$. There is a decomposition

$$H^*(\mathbb{P}(V)) = H^*(V)[p] / \prod_\delta (\delta + p),$$

where δ runs over the Chern roots of V. This receives the Kirwan map

$$\kappa \colon H^*_{S^1}(V) \twoheadrightarrow H^*(\mathbb{P}(V)), \qquad \kappa(\lambda) = p.$$

Thus the quantum D-module for $\mathbb{P}(V)$ is

$$\mathrm{QDM}(\mathbb{P}(V)) = H^*(\mathbb{P}(V)) \otimes \mathbb{C}[z, q][\![Q, \widehat{\boldsymbol{\tau}}]\!],$$

where $\widehat{\boldsymbol{\tau}} = \{\widehat{\tau}_i\}$ is a basis for $H^*(\mathbb{P}(V))$ and q is the Novikov variable of the fiber curve class. It is equipped with the connections $\nabla_{\widehat{\tau}_i}, \nabla_{\xi Q\partial_Q}, \nabla_{q\partial_q}, \nabla_{z\partial_z}$.

Remark 3.2.1. Note there are no shift operators, but there is an additional q-direction in the quantum D-module for $\mathbb{P}(V)$

We also have the symplectic space $\mathcal{H}_{\mathbb{P}(V)}$, the Lagrangian cone $\mathcal{L}_{\mathbb{P}(V)}$, and the fundamental solution $M_{\mathbb{P}(V)}(\widehat{\boldsymbol{\tau}}, z)$. Finally, we will recall the mirror theorem in the form that

$$I_{\mathbb{P}(V)} = \sum_{k\geq 0} \kappa(\mathcal{S}^{-k} J^{\lambda+kz}) q^k$$

lies on $\mathcal{L}_{\mathbb{P}(V)}$.

3.2.2 Discrete Fourier transform

Definition 3.2.2. The *discrete Fourier transform* $\mathcal{H}_V \to \mathcal{H}_{\mathbb{P}(V)}$ is the transform

$$J^\lambda \mapsto \widehat{J} = \sum_{k\geq 0} \kappa(\mathcal{S}^{-kJ^{\lambda+kz}}) q^k.$$

In this framing, the mirror theorem states that the discrete Fourier transform of the J-function of V lies on the Lagrangian cone of $\mathbb{P}(V)$.

Theorem 3.2.3. *There exists a "mirror map"*

$$\widehat{\boldsymbol{\tau}} = \widehat{\boldsymbol{\tau}}(\boldsymbol{\tau}) \in H^*(\mathbb{P}(V))[q][\![Q, \tau]\!]$$

and an isomorphism

$$\mathrm{FT}\colon\ \mathrm{QDM}_{S^1}(V) \to \widehat{\boldsymbol{\tau}}^* \,\mathrm{QDM}(\mathbb{P}(V))$$

of $\mathbb{C}[z][\![Q, \boldsymbol{\tau}]\!]$*-modules intertwining the connections in the natural ways.*

Remark 3.2.4. One has to be careful with the Novikov variables and think about approximately eight other points of the theorem, but we will ignore these for now.

Because the Fourier transform intertwines the connections, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{QDM}_{S^1}(V) \arrow[r, "\mathrm{FT}"] \arrow[d, "M_V(\boldsymbol{\tau})"] & \boldsymbol{\tau}^* \,\mathrm{QDM}(\mathbb{P}(V)) \arrow[d, "M_{\mathbb{P}(V)}(\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau}))"] \\
\mathcal{H}_V \arrow[r, "J \to \widehat{J}"] & \mathcal{H}_{\mathbb{P}(V)}.
\end{tikzcd}$$

Idea of proof. The idea of the proof is to start from the mirror theorem (the bottom row) and apply Birkhoff factorization. The mirror theorem states that

$$(M_V(\boldsymbol{\tau})1)^\wedge = M(\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau}))\Upsilon \in \mathcal{L}_{\mathbb{P}(V)}$$

for some mirror map $\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau})$ and $\Upsilon \in \mathrm{QDM}(\mathbb{P}(V))$. Using the intertwining properties of M, we see that

$$(M_V(\boldsymbol{\tau})(\phi_i \lambda^k))^\wedge = M(\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau})) z \boldsymbol{\tau}^* \nabla_{\frac{\partial}{\partial t^{i,k}}} \Upsilon.$$

Defining

$$\mathrm{FT}(\phi_i \lambda^k) := z \boldsymbol{\tau}^* \nabla_{\partial_{\tau^{i,k}}} \Upsilon$$

and $\widehat{\boldsymbol{\tau}}$ to be the mirror map appearing in the Birkhoff factorization, we are done. □

Remark 3.2.5. The mirror map satisfies

$$\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau})|_{q=Q=0} = \kappa(\boldsymbol{\tau})$$

and the Fourier transform satisfies

$$\mathrm{FT}(\phi_i \lambda^k)|_{Q=\tau=0} = \phi_i p^k.$$

Remark 3.2.6. The Fourier transform intertwines the natural pairings on the quantum D-modules.

3.2.3 Continuous Fourier transform

Definition 3.2.7. Define

$$\mathrm{QDM}(\mathbb{P}(V))_{\mathrm{loc}} := \mathrm{QDM}(\mathbb{P}(V)) \otimes \mathbb{C}[z](\!(q^{-\frac{1}{r'}}[\![Q,\widehat{\boldsymbol{\tau}}]\!],$$

where $r' = r$ or $2r$ depending on parity.

Theorem 3.2.8. *For $j = 0, \ldots, r-1$, there exist maps $H^*(\mathbb{P}(V)) \to H^*(B)$ given by*

$$\widehat{\boldsymbol{\tau}} \mapsto \zeta_j(\widehat{\boldsymbol{\tau}}) \in -c_1(V)\log\left(e^{\frac{2\pi\sqrt{-1}j}{r}} q^{\frac{1}{r}}\right) + H^*(B)(\!(q^{-\frac{1}{r}}[\![Q,\widehat{\boldsymbol{\tau}}]\!]$$

and an isomorphism

$$\Phi\colon\ \mathrm{QDM}(\mathbb{P}(V))_{\mathrm{loc}} \cong \bigoplus_{j=0}^{r-1} \zeta_j^* \,\mathrm{QDM}(B)_{\mathrm{loc}}$$

intertwining the pairings and quantum connections in a natural way, namely that

$$\Phi\Delta = \bigoplus \zeta_j^* \Delta\Phi.$$

Writing $\Phi = (\Phi_0, \ldots, \Phi_j)$, *we have*

$$\Phi_j(\phi_i p^k)|_{Q=\widehat{\boldsymbol{\tau}}=0} = \frac{1}{\sqrt{r}} \lambda_j^{k-\frac{r-1}{2}} (\phi_i + O(q^{-\frac{1}{r}})).$$

Idea of proof. We use another realization of the Fourier transform on $\mathrm{QDM}_{S^1}(V)$ and

$$\mathrm{FT}\colon\ \mathrm{QDM}_{S^1} \cong \widehat{\boldsymbol{\tau}}^* \,\mathrm{QDM}(\mathbb{P}(V)).$$

If we consider Δ_V^λ arising from Quantum Riemann-Roch, it is given as

$$\Delta_V^\lambda \asymp \prod_\rho \sqrt{\frac{z}{2\pi}} z^{\frac{\lambda+\rho}{z}} \Gamma\left(\frac{\rho+\lambda}{z}+1\right).$$

Shifting by $-z$, we see that

$$\Delta_V^{\lambda - z} = \Delta_V^\lambda \prod_\rho \frac{1}{\rho+\lambda} = \Delta_V^\lambda \frac{1}{e_{S^1}(V)}.$$

We now consider the transformation

$$s \mapsto \int q^{\frac{\lambda}{z}} (\Delta_V^\lambda)^{-1} M_V(\boldsymbol{\tau}) \cdot s \, \mathrm{d}\lambda$$

for $s \in \mathrm{QDM}_{S^1}(V)$. Because this integral intertwines $\mathbb{S}$ with q and λ with $z\nabla_{q\partial_q}$, it formally gives a solution to $\mathrm{QDM}(\mathbb{P}(V))$.

To make sense of this terrible expression, we use the stationary phase expansion of the integral. Setting

$$I(s) = \int e^{-\frac{\varphi(\lambda)}{z}} \lambda^{-\frac{c_1(V)}{z}} \lambda^{-\frac{r}{2}} (\widetilde{\Delta}_V^\lambda)^{-1} J^\lambda \, \mathrm{d}\lambda,$$

where $\frac{\varphi(\lambda)}{z}$ is the Stirling asymptotics of Δ_V^λ, given by

$$\varphi(\lambda) = r(\lambda\log\lambda - \lambda) - \lambda\log q.$$

The critical points of φ are given by

$$\frac{\partial}{\partial\lambda}\varphi(\lambda) = r(\log\lambda) - \log q = 0,$$

which tells us that $\lambda^r = q$. Thus, we obtain r solutions

$$\lambda_j = e^{\frac{2\pi\sqrt{-1}j}{r}} q^{\frac{1}{r}}.$$

We now consider the formal expansions around λ_j. These produce a "continuous Fourier transform"

$$J \mapsto \mathcal{F}_j(J)$$

such that

$$I(M^{-1}J) \asymp \sqrt{2\pi z} e^{r\frac{\lambda_j}{z}} \mathcal{F}_j(J).$$

These intertwine the quantum connection and multiplication by λ, as in

$$\mathcal{F}_j(\lambda J) = (\lambda_j + zq\partial_q)\mathcal{F}_j(J)$$
$$\mathcal{F}_j(\mathcal{S}J) = q\mathcal{F}_j(J),$$

so $z\mathcal{F}_j(J_V(\boldsymbol{\tau}))$ is on the Lagrangian cone of B. We then use the following result:

Proposition 3.2.9. *We have*

$$\mathcal{F}_j(J_V(\boldsymbol{\tau})) = M_B(\sigma_j(\boldsymbol{\tau}))v_j$$

for some σ_j, v_j.

Unfortunately, $\mathcal{F}_j$ does not intertwine $\nabla_{z\partial_z}$ correctly. To fix this, define

$$\zeta_j(\widehat{\boldsymbol{\tau}}) = \sigma_j(\boldsymbol{\tau}(\widehat{\boldsymbol{\tau}})) + r\lambda_j$$

and Φ_j by a shift of v_j. □

3.2.4 Discrete equals continuous

Warning 3.2.10. **Everything in this subsection may be false.**

Consider the Fourier transform

$$\int \prod_\rho \Gamma\left(-\frac{\rho+\lambda}{z}\right) J_V^\lambda q^{\frac{\lambda}{z}}\, \mathrm{d}\lambda.$$

This can be computed either using residues or using stationary phase asymptotics. Using residues, we obtain

$$\sum_{k\geq 0} \mathrm{Res}_{p=0}\Gamma\left(-k - \frac{\rho+\lambda}{z}\right) J_V^{\lambda+kz} q^{\frac{p}{z}} q^k,$$

which is precisely

$$\frac{1}{\prod_{c=0}^k e_{p+\lambda z}(V)}\Gamma\left(-\frac{\rho-\lambda}{z}\right).$$

Using stationary phase asymptotics, we obtain the $I(s)$ defined previously.

4

Quantum cohomology of blowups

4.1 Setup (Kostya, Mar 07)

Let X be a smooth projective variety, $Z \subset X$ be a smooth closed subvariety of codimension r, and

$$\varphi\colon \widetilde{X} = \mathrm{Bl}_Z X \to X$$

be the blowup of X with center Z. Denote the exceptional divisor by

$$j \coloneqq D \cong \mathbb{P}(N_{Z/X}) \hookrightarrow \widetilde{X}.$$

As a vector space, there is an isomorphicm

$$H^*(X) = H^*(X) \oplus \bigoplus_{i=0}^{r-2} H^*(Z).$$

Our goal is to upgrade this to the level of quantum cohomology. However, this is very tricky because of convergence issues, so it will be a corollary of a decomposition theorem for quantum D-modules.

Recall that the quantum D-module of X is given by

$$\mathrm{QDM}(X) = H^*(X)[z][\![Q,\tau]\!]$$

with the Dubrovin connection

$$\nabla_{\partial_\tau}, \nabla_{\xi Q\partial_Q}, \nabla_{z\partial_z}$$

and pairing

$$(f(z), g(z)) = \int_X f(-z)g(z).$$

4.1.1 Statement of the theorem

The first problem we need to fix is that the cohomological and Novikov variables of $X, Z, \widetilde{X}$ are different.

Definition 4.1.1. The *extended Novikov ring* is defined as

$$\mathbb{C}[\![\mathbb{Q}]\!] \coloneqq \mathbb{C}[\![Q, xy^{-1}, Q^{\varphi_*\widetilde{d}} y^{-[D]\cdot\widetilde{d}}]\!],$$

where $d \in \mathrm{NE}_{\mathbb{N}}(X)$ and $\widetilde{d} \in \mathrm{NE}_{\mathbb{N}}(\widetilde{X})$.

Introducing another formal variable q, we can embed the Novikov rings of $Z, X, \widetilde{X}$ into $\mathbb{C}(\!(q^{-\frac{1}{s}})\!)[\![\mathfrak{Q}]\!]$, where s is either $r-1$ or $2(r-1)$ depending on the parity of r (we want s to be even):

- $\mathbb{C}[\![Q]\!]$ embeds as $Q^d \mapsto Q^d$;
- $\mathbb{C}[\![\widetilde{Q}]\!]$ embeds as $\widetilde{Q}^{\widetilde{d}} \mapsto Q^{\varphi_* \widetilde{d}} q^{-[D]\cdot \widetilde{d}}$;
- $\mathbb{C}[\![Q_Z]\!]$ embeds as $Q_Z^d \mapsto Q^{\iota_* d} q^{-\frac{\rho_Z \cdot d}{r-1}}$, where $\rho_Z = c_1(N_{Z/X})$.

Later, we will see that $q = yS^{-1}$, where S^{-1} will be an equivariant variable for a $\mathbb{C}^*$ action. Denote

$$\mathrm{QDM}(X)^{\mathrm{La}} := \mathrm{QDM}(X) \otimes_{\mathbb{C}[\![Q]\!]} \mathbb{C}[\![\mathfrak{Q}]\!].$$

We can now state the main result.

Theorem 4.1.2. *There exists a formal invertible change of variables*

$$H^*(\widetilde{X}) \to H^*(X) \oplus H^*(Z)^{\oplus r-1}$$

denoted by

$$\widetilde{\tau} \to \left(\tau(\widetilde{\tau}), \{\zeta_j(\widetilde{\tau})\}_{j=0}^{r-2}\right)$$

and an isomorphism

$$\Psi\colon\ \mathrm{QDM}^{\mathrm{La}}(\widetilde{X}) \xrightarrow{\cong} \tau^* \mathrm{QDM}(X)^{\mathrm{La}} \oplus \bigoplus_{j=0}^{r-2} \zeta_j^* \mathrm{QDM}(Z)^{\mathrm{La}}$$

such that Ψ *intertwines the quantum connections and the pairings.*

4.1.2 The master space

Definition 4.1.3. Define the *master space*

$$W := \mathrm{Bl}_{Z\times\{0\}} X \times \mathbb{P}^1 \xrightarrow{\widehat{\varphi}} X \times \mathbb{P}^1$$

to be the degeneration to the normal cone of $Z \subset X$. This is endowed with a $T = \mathbb{C}^\times$-action extending the action

$$\lambda \cdot (x, u) = (x, \lambda u)$$

for $(x, u) \in X \times \mathbb{P}^1$. See Figure 4.1 for a picture of W.

The fixed locus of the $\mathbb{C}^*$-action is given by

$$W^{\mathbb{C}^*} = X \sqcup \widetilde{X} \sqcup Z.$$

We can also see that

$$\begin{aligned} N_T^1(W) &= \widehat{\varphi}^* N_T^1(X \times \mathbb{P}^1) \oplus \mathbb{Z}[\widetilde{D}] \\ &\cong N^1(X) \oplus \mathbb{Z}^3 \ni (\omega, t, \varepsilon, a). \end{aligned}$$

Note here that

$$N_T^1(X \times \mathbb{P}^1) = \mathrm{pr}_1^* N^1(X) \oplus \mathbb{Z}[X] \oplus \mathbb{Z}\lambda,$$

so a general class in $N_1^T(W)$ can be written as

$$\widehat{\varphi}^* \mathrm{pr}_1^* \omega + t[X] - \varepsilon[\widehat{D}] + a \cdot \lambda.$$

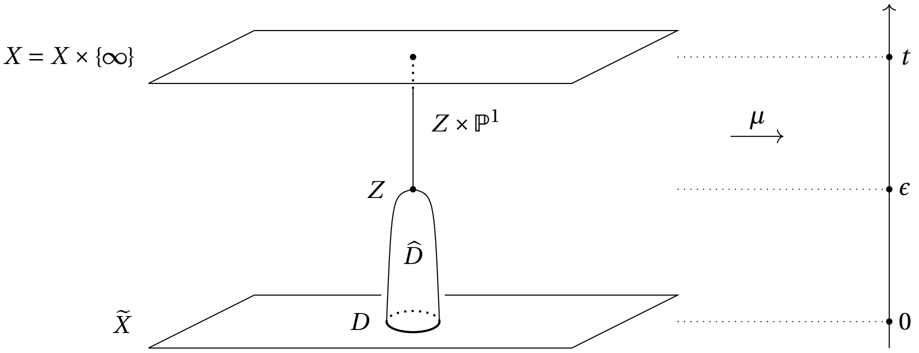

Figure 4.1: $W = \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{P}^1)$ and a moment map $\mu\colon W \to \mathbb{R}$

The dual notion is the group of 1-cycles, which is given by

$$N_1^T(W) \cong N_1(X) \oplus \mathbb{Z}^3 \ni (d, k, \ell, m) =: \beta,$$

whose Novikov variable is

$$Q^d x^k y^\ell S^m.$$

Note that

$$[X] \cdot \beta = k, \qquad -[\widehat{D}] \cdot \beta = \ell.$$

The effective curve classes are those which are equivariant, so they either lie in the fixed loci or are 1-dimensional orbits. There are classes C_1 lying entirely inside X, classes C_2, which are $x \times \mathbb{P}^1$ for $x \notin Z$, classes $C_3 \subset Z \times \mathbb{P}^1$, and classes $C_4 \subset \widehat{D}$. The corresponding Novikov variables are Q^d, x, xy^{-1}, and y, respectively.

Lemma 4.1.4. *The monoid of effective curve classes is generated by* C_1, C_2, C_3, C_4, S.

Example 4.1.5. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $Z = (0,0)$. Then a toric diagram for W with C_1, C_2, C_3, C_4 is given in Figure 4.2.

The T-ample cone $C_T(W)$ of W is the dual to the cone of effective curve classes. Then $\widehat{\omega} \in C_T(W)$ if the set of $\widehat{\omega}$-stable points under the $\mathbb{C}^*$-action is nonempty. Then there is a decomposition

$$\overline{C_T(W)} = \overline{C}_X \cup \overline{C}_{\widetilde{X}}$$

into pieces where the GIT quotient

$$W /\!\!/_{\widehat{\omega}}\, T \cong X \text{ or } \widetilde{X},$$

respectively. The stable points are

- For $\widehat{\omega} \in C_X$, the stable points are $X \times \mathbb{C}^*$;
- For $\widehat{\omega} \in C_{\widetilde{X}}$, the stable points are $\mathcal{O}_{\widetilde{X}}(-\widehat{D}) \setminus \widehat{X}$.

The dual cones $C_X^\vee$ and $C_{\widetilde{X}}^\vee$ correspond to embeddings of the effective cones of X and $\widetilde{X}$, respectively, into the effective cone of W as in Figure 4.3.

Recall the *Kirwan map*

$$\kappa_Y \colon H_T^*(W) \twoheadrightarrow H_T^*(W^s) = H^*(Y)$$
$$\kappa_Y^* \colon \mathrm{NE}_{\mathbb{N}}(Y) \to N_1^T(W),$$

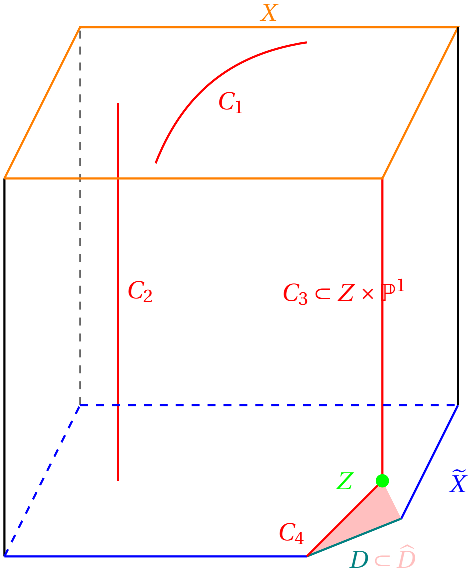

Figure 4.2: Toric diagram of W for $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $Z = (0,0)$ with curve classes C_1, C_2, C_3, C_4.

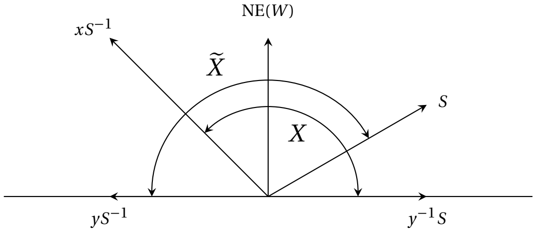

Figure 4.3: A schematic picture of the cones $C_X^\vee$ and $C_{\widetilde{X}}^\vee$ in $N_1^T(W)$.

where Y is either X or $\widetilde{X}$.

Now consider the equivariant quantum D-module

$$QDM_T(W) = H_T^*(W)[z][\![\mathcal{Q},\theta]\!]$$

endowed with the quantum connection and the action of the shift operators

$$\widehat{\mathbb{S}}_\beta(\theta), \qquad \beta \in N_1^T(W) \cong N_1(X) \oplus \mathbb{Z}^3,$$

which are defined by

$$\widehat{\mathbb{S}}^\beta(\theta) = Q^d x^k y^\ell \mathbb{S}(\theta)^m,$$

where β is identified with (d,k,ℓ,m) under the factorization $N_1(X) \oplus \mathbb{Z}^3$.

We can obtain the quantum D-modules for X and $\widetilde{X}$ from the equivariant quantum D-module of W by taking affine charts and completing. Here, we think of $\mathrm{QDM}_T(W)$ as a global Kähler moduli space.

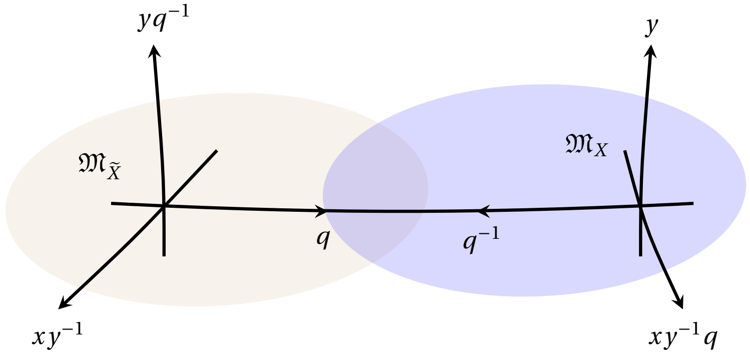

Figure 4.4: Global Kähler moduli space associated with $\mathrm{QDM}_T(W)$. See also Figure 4.3.

4.1.3 Fourier transforms

To construct the map in Theorem 4.1.2, Iritani uses both the discrete and continuous Fourier transforms. For $f \in \mathcal{H}_W$, the discrete Fourier transform is given by

$$F_Y(f) := \sum_{k \in \mathbb{Z}} S^k \kappa_Y(\mathcal{S}^{-k} f) \in \mathcal{H}_Y^{\mathrm{ext}}.$$

The continuous Fourier transform for $Z \in \pi_0(W^T)$ is given by sending $f \in \mathcal{H}_W$ to the element of $\mathcal{H}_Z$ given by the formal asymptotic expansion of

$$\int e^{\lambda \log \frac{S}{z}} \Delta_Z^{-1} f|_Z \, \mathrm{d}\lambda.$$

Remark 4.1.6. We can think of X and $\widetilde{X}$ either as fixed components $X, \widetilde{X} \in \pi_0(W^T)$ (corresponding to the continuous Fourier transform) or as GIT quotients $W /\!\!/_{\widehat{\omega}} T$, corresponding to the discrete Fourier transform. These are in fact equal up to a factor.

Note that Z is not a GIT quotient of W, but is a fixed component, so there is a continuous Fourier transform to W. In total, we have three morphisms

$$\begin{tikzcd}
 & \mathrm{QDM}(\widetilde{X}) \\
\mathrm{QDM}_T(W) \arrow[ur, "\cong"] \arrow[r] \arrow[dr, "r-1"] & \mathrm{QDM}(X) \\
 & \mathrm{QDM}(Z).
\end{tikzcd}$$

4.2 Fourier analysis for blowups (Sam, Mar 21)

In this section, denote the Chern roots of $\mathcal{N}_{Z/X}$ by $\varepsilon_1, \ldots, \varepsilon_r$.

4.2.1 Extended Kirwan maps

We will begin by describing the Kirwan maps in coordinates. On $H_T^2(W)$, we have

$$\begin{aligned}
\kappa_X : \widehat{\varphi}^* \operatorname{pr}_1^* \alpha &\mapsto \alpha \\
[X] &\mapsto 0 \\
-[\widehat{D}] &\mapsto 0 \\
\lambda &\mapsto 0
\end{aligned}$$

for $Y = X$

$$\begin{aligned}\kappa_{\widetilde{X}} \colon \widehat{\varphi}^* \operatorname{pr}_1^* \alpha &\mapsto \varphi^* \alpha \\ [X] &\mapsto 0 \\ -[\widehat{D}] &\mapsto -[D] \\ \lambda &\mapsto [D]\end{aligned}$$

for $Y = \widetilde{X}$. The dual Kirwan maps on $N_1(Y)$ are given by

$$\begin{aligned}\kappa_X^* \colon &N_1(X) \to N_1^T(W) \cong N_1(X) \oplus \mathbb{Z}\lambda^\vee \oplus \mathbb{Z}[X]^\vee \oplus \mathbb{Z}(-[\widehat{D}])^\vee \\ &d \mapsto (d, 0, 0, 0) \\ \kappa_{\widetilde{X}}^* \colon &N_1(\widetilde{X}) \to N_1^T(W) \cong N_1(X) \oplus \mathbb{Z}\lambda^\vee \oplus \mathbb{Z}[X]^\vee \oplus \mathbb{Z}(-[\widehat{D}])^\vee \\ &\widetilde{d} \mapsto (\varphi_* \widetilde{d}, 0, -[D] \cdot \widetilde{d}, [D] \cdot \widetilde{d})\end{aligned}$$

when $Y = X$ and $Y = \widetilde{X}$, respectively. It may appear that we don't see the equivariant parameters in the dual Kirwan map, but we will fix this.

Definition 4.2.1. The *extended Givental space* is

$$\mathcal{H}_Y^{\text{ext}} := H^*(Y)[z^{\pm}][\![C_{Y,N}^\vee]\!],$$

which is a base change of $\mathcal{H}_Y$.

The shift operators on $\mathcal{H}_W^{\text{rat}} = H_{\text{loc}}^*(W)[Z^{\pm}][\![Q]\!]$ are now given by

$$\mathcal{S}\iota_*(f_X, f_Z, f_{\widetilde{X}}) = (\mathcal{S}f|_X, \mathcal{S}f|_Z, \mathcal{S}f|_{\widetilde{X}}),$$

where

$$\begin{aligned}\mathcal{S}^k f_X &= x^k \frac{\prod_{c=-\infty}^0 (-\lambda + cz)}{\prod_{c=-\infty}^k (-\lambda + cz)} e^{-kz\partial_\lambda} f_X \\ \mathcal{S}^k f_Z &= y^k \frac{\prod_{c=-\infty}^0 \prod_{i=1}^r (\varepsilon_i - \lambda + cz) \prod_{c=-\infty}^0 (\lambda + cz)}{\prod_{c=-\infty}^k \prod_{i=1}^r (\varepsilon_i - \lambda + cz) \prod_{c=-\infty}^k (\lambda + cz)} e^{-kz\partial_\lambda} f_X \\ \mathcal{S}^k f_{\widetilde{X}} &= \frac{\prod_{c=-\infty}^0 ([D] + \lambda + cz)}{\prod_{c=-\infty}^k ([D] + \lambda + cz)} e^{-kz\partial_\lambda} f_{\widetilde{X}}.\end{aligned}$$

4.2.2 Discrete Fourier transform

We are now able to make the following definition.

Definition 4.2.2. The *discrete Fourier transform* is

$$F_Y \colon \mathcal{H}_W^{\text{rat}}[Q^{-1}] \dashrightarrow \mathcal{H}_Y^{\text{ext}}[Q^{-1}],$$

defined by

$$F_Y(f) = \sum_{k \in \mathbb{Z}} S^k \kappa_Y(\mathcal{S}^{-k} f).$$

Because we invert the equivariant parameters, this may not necessarily be well-defined. However, we do have the following result.

Proposition 4.2.3. *The discrete Fourier transform F_Y is well-defined on tangent spaces to $\mathcal{L}_W$.*

To prove this result, we need various ingredients:

1. Some regularity at $\lambda = 0$, which follows from properness;
2. Not needing arbitrarily high powers of Q^{-1} (landing in the target).

As in the case of projective bundles, the shift operators satisfy various nice properties. For example, because

$$[\lambda, \mathcal{S}^k] = z^k \mathcal{S},$$

we obtain the formulae

$$\begin{aligned} F_Y(\mathcal{S}^\ell f) &= S^\ell F_Y(f) \\ F_Y(\xi f) &= (z\xi Q\partial_Q + \kappa_Y(\xi))F_Y(f) \end{aligned}$$

for any $\xi \in H_2^T(W)$.

4.2.3 Continuous Fourier transform

Let $F \in \{X, \widetilde{X}, Z\}$ be a fixed component in W^T. Denote the Chern roots of $\mathcal{N}_{F/W}$ by $\rho_1, \ldots, \rho_n$. Now define

$$G_F := \prod_\rho \frac{1}{\sqrt{-2\pi z}}(-z)^{-\frac{\rho}{z}}\Gamma\left(\frac{-\rho}{z}\right).$$

More specifically, for each individual fixed component, we have

$$\begin{aligned} G_X &= \frac{1}{-2\pi z}(-z)^{\frac{\lambda}{z}}\Gamma\left(\frac{\lambda}{z}\right) \\ G_Z &= \frac{1}{\sqrt{-2\pi z}^{r+1}}(-z)^{-\frac{\lambda}{z}}\Gamma\left(\frac{-\lambda}{z}\right)\prod_{i=1}^r (-z)^{\frac{\lambda-\varepsilon_i}{z}}\Gamma\left(\frac{\lambda-\varepsilon_i}{z}\right) \\ G_{\widetilde{X}} &= \frac{1}{\sqrt{-2\pi z}}(z)^{\frac{[D]-\lambda}{z}}\Gamma\left(\frac{[D]-\lambda}{z}\right). \end{aligned}$$

This G_F actually intertwines all $\widehat{\mathcal{S}}^\beta$, which are defined for all

$$\beta \in N_1^T(W) \to H_2(BT) \ni \overline{\beta}.$$

Given F, we have

$$\sigma_k(F) \in N_1(E_k) \to N_1^T(W),$$

which are obtained by section classes on E_k as in Figure 4.5. Now G_F satisfies the equation

$$G_F(\widehat{\mathcal{S}}^\beta f)_F = (Q^{\beta+\sigma_F(-\overline{\beta})}e^{-z\overline{\beta}\partial_\lambda})G_F f,$$

which follows from the same argument as in the projective bundle case.

We are now able to describe the continuous Fourier transform formally as an integral which has the same properties as the discrete one. Define

$$\mathfrak{FT}^\infty \colon f \mapsto \int e^{\lambda \log \frac{S_f}{z}} G_F f_F \, \mathrm{d}\lambda,$$

where $S_F = \sigma_1(F)$. Formally, we see that

$$\begin{aligned} \mathfrak{FT}^\infty(\widehat{\mathcal{S}}^\beta f) &= \int e^{\lambda \log \frac{S_F}{z}} Q^{\beta+\sigma_1(-\overline{\beta})} e^{z\overline{\beta}\log\frac{S_F}{z}} G_F f_F \, \mathrm{d}\lambda \\ &= \widehat{\mathcal{S}^\beta}\mathfrak{FT}^\infty(f), \end{aligned}$$

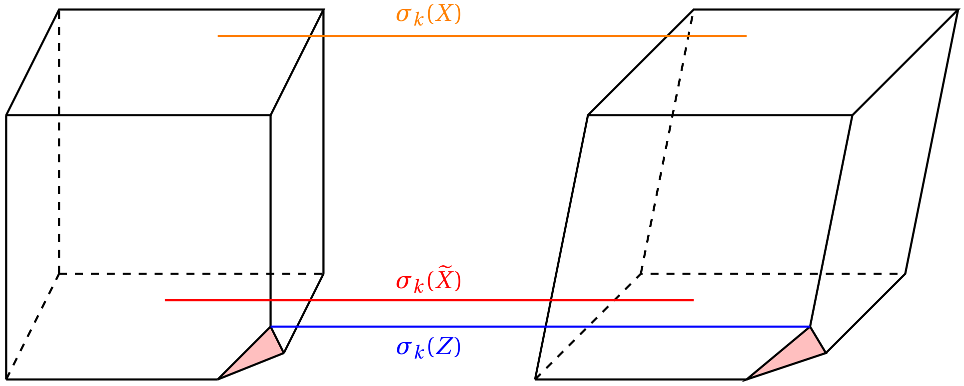

Figure 4.5: Toric diagram of E_k when X, Z are as in Figure 4.2.

which suggests that this is the right object to study the asymptotics of. Recall the Stirling approximation

$$\begin{aligned}\log G_F &\asymp \sum_{\rho_i} -\frac{\rho_i \log \rho_i - \rho_i}{z} - \frac{1}{2}\log \rho_i - \sum_{n=2}^{\infty} \frac{B_n}{n(n-1)}\left(\frac{z}{\rho_i}\right)^{n-1}, \\ &= \sum_{\alpha \in \mathrm{wts}(\mathcal{N}_{F/W})} \left(-r_\alpha \frac{\alpha \log \alpha - \alpha}{z} - \log \Delta_\alpha\right),\end{aligned}$$

where we have the decomposition

$$\mathcal{N}_{F/W} = \bigoplus_{\alpha} \mathcal{N}^{\alpha},$$

$\rho_\alpha = c_1(\mathcal{N}^\alpha)$ is the first Chern class, $r_\alpha = \mathrm{rk}\mathcal{N}^\alpha$, and Δ_α is the quantum Riemann-Roch operator. We now obtain

$$\mathfrak{F}\mathfrak{T}^{\infty} \asymp \int e^{\frac{\lambda \log S_F - \sum_\alpha r_\alpha(\alpha \log \alpha - \alpha)}{z}} \prod_{\alpha} \Delta_\alpha^{-1} f_F \, \mathrm{d}\lambda.$$

We have a critical point

$$\lambda_0 = S_F^{\frac{1}{c}} \prod_{\alpha} w_\alpha^{-r_\alpha \frac{w_\alpha}{c_F}},$$

where

$$c_F = \begin{cases} -1 & F = X \\ 1 & F = \widetilde{X} \\ -(r-1) & F = Z. \end{cases}$$

Also, for $j = 0, \ldots, |c_F| - 1$, we have critical points

$$\lambda_j = e^{\frac{2\pi\sqrt{-1}j}{c_F}} \lambda_0.$$

Definition 4.2.4. The *continuous Fourier transform* is given by the asymptotic expansion

$$\int e^{\lambda \log \frac{S_F}{z}} G_F f_F \, \mathrm{d}\lambda \asymp \sqrt{2\pi z} e^{r\frac{\lambda_j}{z}} \mathfrak{F}_{F,j}(f),$$

where we take the asymptotics as $z \to 0$ and substitute

$$\lambda = \lambda_j \exp\left(\frac{u}{\sqrt{c\lambda_j}}\right)$$

and expand in u-powers as $u \to 0$.

Via taking a formal asymptotic analysis of what we did before, we have

Proposition 4.2.5.

1. *The continuous Fourier transform intertwines the shift operators as*

$$\mathfrak{F}_{F,j}(\widehat{\mathbb{S}}^\beta f) = \widehat{S}^\beta \mathfrak{F}_{F,j}(f);$$

2. *The continuous Fourier transform intertwines the equivariant parameter via*

$$\mathfrak{F}_{F,j}(\lambda f) = (zS + \lambda_j)\mathfrak{F}_{F,j}(f).$$

3. *There is a similar property for the Euler vector field.*

Now let

$$\widehat{J}^{\mathrm{tw}}_{F,j} = \mathfrak{F}_{F,j}(\iota_* J^{\mathrm{tw}}_F(t)).$$

Proposition 4.2.6. *There exists*

$$\tau(t) \in H^*(F)[\![S_F^{-\frac{1}{c}}, Q_F S^{-\frac{\rho_F}{c}}, tS_F^{\frac{*}{c}}]\!]$$

and

$$v(t) \in H^*(F)[z][\![S_F^{-\frac{1}{c}}, Q_F S^{-\frac{\rho_F}{c}}, tS_F^{\frac{*}{c}}]\!]$$

such that

$$\widehat{J}^{\mathrm{tw}}_{F,j} = S_F^{-\frac{\rho_F}{cz}} M_F(\tau(t) Q_F S_F^{-\frac{\rho_F}{cz}})v.$$

In other words, we can recover QDM_F from $\mathfrak{F}_{F,j}$ after a change of basis and an étale cover of the Kähler moduli space.

Proposition 4.2.7. *The discrete and continuous Fourier transforms agree for $Y \in \{X, \widetilde{X}\}$. In other words, we have*

$$e^{S_Y^{\frac{1}{c_Y}}} \mathfrak{F}_{Y,0}(f) = c_Y^{-1} S_Y^{-\frac{\rho_Y}{c_Y z}} F_Y(f).$$

4.3 Decomposition (Sam, Mar 28)

4.3.1 A general picture We begin by stating some general conjectures that the blowup result fits into.

Conjecture 4.3.1 (Dubrovin)**.** *Let X be smooth and projective. If there exists a semiorthogonal decomposition*

$$D^b(X) = \langle D^b(X_1), \ldots, D^b(X_n) \rangle,$$

then there is a decomposition

$$\mathrm{QDM}(X) = \bigoplus_i \mathrm{QDM}(X_i).$$

More specifically, if X admits a full exceptional collection, $\mathrm{QDM}(X)$ *can be extended to* $\mathcal{O}^{\mathrm{an}}[\![z]\!]$*. In addition the Gram matrix of the pairing* $\chi(-,-)$ *on* $K(X)$ *is recovered as the Stokes matrix*

$$S = \Phi_{\arg z \in (-\pi, 0+\varepsilon)} \Phi^{-1}_{\arg z \in (0,\pi)}$$

of flat sections of the irregular connection

$$z\partial_z + \frac{1}{z} E_X \star_\tau + \mu_X.$$

On the level of derived categories, we have a decomposition

$$D^b(\widetilde{X}) = \langle D^b(X), D^b(Z)_0, \ldots, D^b(Z)_{r-2} \rangle,$$

so we expect a decomposition

$$\mathrm{QDM}(\widetilde{X}) = \bigoplus_{\mu \in \mathrm{Spec}(E_{\widetilde{X}} \star_\tau)} \mathrm{QDM}(\widetilde{X})_\mu.$$

In the limit $Q\tau \to 0$, we obtain $\mathrm{QDM}(X)$ at $\mu = 0$ and $r-1$ copies of $\mathrm{QDM}(Z)$ at roots of unity. The shift away from $\mu = 0$ will correspond to the shift of saddle points in the Fourier transform.

In the equivariant setting, we have semiorthogonal decompositions

$$D^b_T(W) = \langle D^b(W /\!\!/_\theta T), \ldots \rangle.$$

We then expect

Conjecture 4.3.2. *Setting* $W /\!\!/_\theta T =: Y$*, then*

$$I := \sum_\beta \kappa_Y(\widehat{S}^{-\beta} J_W(\tau)) \widehat{S}^\beta$$

lies on the Lagrangian cone of Y*.*

4.3.2 Decomposition of the quantum D-module

First, we would like to give a more precise formulae for the quantities appearing in Proposition 4.2.6. First, τ is given by

$$\tau|_{Q=0} = h_{F,j} + \cdots,$$

where

$$h_{F,j} = 2\pi i j \frac{c_1(N_{F/W})}{C_F} + \sum_\alpha \left(\mathrm{rk}_\alpha \, w_\alpha \frac{c_1(N_{F/W})}{c_F} - c_1(N^\alpha_{F/W}) \right).$$

Then we have

$$v|_{Q=0} = q_{F,j}(1 + \cdots),$$

where

$$q_{F,j} = \sqrt{c_F^{-1} \lambda_j} \prod_\alpha (w_{\alpha \lambda_j})^{\frac{\mathrm{rk}_\alpha}{2}}$$

for $\lambda_j = e^{\frac{1\pi i j}{c_F}} \lambda_0$.

For $X, \widetilde{X}$, the continuous Fourier transform $\mathcal{F}_{F,j}$ intertwines S^β and λ with $\xi S \frac{\partial}{\partial S}$ and $E^T_W \star_\tau + \mu_W$ with $E_{\widetilde{X}} \star_\tau + \mu_X + \frac{1}{2}$. On the other hand, for $F = Z$, the critical points are different, so there is a shift in λ. For example, we have

$$\mathcal{F}_{F,j}(\lambda f) = \left(zS \frac{\partial}{\partial S} + \lambda_j \right) \mathcal{F}_{F,f}(f)$$

because the integral satisfies the formula

$$zS\frac{\partial}{\partial S}\int e^{\log\frac{S_F}{z}(\lambda-\lambda_j)} fG\,\mathrm{d}\lambda = \int(\lambda-\lambda_j)e^{\log\frac{S_F}{z}(\lambda-\lambda_j)} fG\,\mathrm{d}\lambda.$$

A similar argument yields

$$\mathcal{F}_{F,j}\left(\left(z\partial_z+\mu_Z+\frac{1}{2}\right)f\right)=(z\partial_z+z^{-1}(c_1(F)+c_F\lambda_j+\mu_F))\mathcal{F}_{F,j}(f).$$

We next need a space on which to compare $\mathrm{QDM}(X)$, $\mathrm{QDM}(\widetilde{X})$, and $\mathrm{QDM}_T(W)$ via the dual Kirwan maps. Define

$$\mathrm{QDM}(Y)^{\mathrm{ext}} := \mathrm{QDM}(Y)\otimes\mathbb{C}[\![C^\vee_{Y,\mathbb{N}}]\!]$$

and extend ∇ trivially on $C^\vee_{Y,\mathbb{N}}$. As in Figure 4.4, we have $\mathfrak{M}_{\widetilde{X}}$ and $\mathfrak{M}_X$, but we also need

$$\mathfrak{M}_0 = \operatorname{Spf}\mathbb{C}[z][\![\mathrm{NE}^T_{\mathbb{N}}(W)]\!][\![\theta]\!].$$

The three charts can be compared on the chart

$$\mathfrak{U} = \operatorname{Spf}\mathbb{C}[z][\![\mathrm{NE}^T_{\mathbb{N}}(W)]\!][\![\theta]\!](\!(q^{-\frac{1}{s}})\!).$$

The comparison of the effective cones of curves is given in Figure 4.3.

The comparison also requires a completion and localization of $\mathrm{QDM}_T(W)$. In order to compare with $\mathrm{QDM}(\widetilde{X})$ we need the action of extended shift operators $\mathbb{S}^\beta$. Define

$$\mathrm{QDM}_T(W)_{\widetilde{X}} := \mathbb{C}[C^\vee_{\widetilde{X},\mathbb{N}}]\cdot\mathrm{QDM}_T(W)\subset\mathrm{QDM}_T(W)(Q^{-1}).$$

The completion is given by

$$\widehat{\mathrm{QDM}}_T(W)_{\widetilde{X}} = \mathrm{QDM}_T(W)[\![S, yS^{-1}]\!].$$

Now let

$$\begin{aligned}
\tau(\theta) &= xS+\cdots\\
v(\theta) &= 1+\cdots\\
\widetilde{\tau}(\theta) &= S+\cdots\\
\widetilde{v}(\theta) &= 1+\cdots
\end{aligned}$$

such that

$$\begin{aligned}
F_X(J_w(\theta)) &= M_X(\tau(\theta))\,v(\theta)\\
F_{\widetilde{X}}(J_w(\theta)) &= M_{\widetilde{X}}(\widetilde{\tau}(\theta))\,\widetilde{v}(\theta).
\end{aligned}$$

By taking derivatives $\partial_{\theta^{ik}}$, the Fourier transform is lifted to a map of quantum D-modules.

Theorem 4.3.3. *There is an isomorphism*

$$\widehat{\mathrm{FT}}_{\widetilde{X}}\colon \widehat{\mathrm{QDM}}_T(W)_{\widetilde{X}}\to\widetilde{\tau}^*\,\mathrm{QDM}(\widetilde{X})^{\mathrm{ext}}$$

and a projection

$$\widehat{\mathrm{FT}}_X\colon \widehat{\mathrm{QDM}}_T(W)_{\widetilde{X}}\to\tau^*\,\mathrm{QDM}(X)^{\mathrm{ext}}.$$

They intertwine the quantum connection and the pairing up to a $\frac{1}{2}$ shift in μ.

These extend to the completions, but we will not prove this here.

For $F = Z$ and any j, there are coordinates

$$\sigma_j(\theta) = h_{Z,j} + \cdots$$
$$u_j j(\theta)^{=} q_{Z,j} + \cdots$$

such that

$$q^{\frac{c_1(N_{Z/W})}{(r-1)z}} \mathcal{F}_{Z,j}(J_w(\theta)) = M(\sigma_j(\theta))u_j(\theta).$$

Theorem 4.3.4. *There are projections*

$$\widehat{\mathrm{FT}}_{Z,j} \colon \widehat{\mathrm{QDM}}_T(W)_{\widetilde{X}} \to \sigma_j^* \,\mathrm{QDM}(Z)^{\mathrm{ext,loc}}$$

which intertwine λ with $z\nabla_{S\partial_S} + \lambda_j$ and $z\nabla_{\xi Q\partial_Z}$ with $z\nabla_{\widehat{S}\partial_{\widehat{S}}} + \iota_{\mathrm{pt}}^* \xi|_{\lambda=\lambda_i}$. Here, $\mathrm{QDM}(Z)^{\mathrm{ext,loc}}$ *is defined by the extension*

$$\mathbb{C}[z][\![Q_Z, \theta]\!] \to \mathbb{C}[z][\![q^{-\frac{1}{s}}]\!][\![Q, \theta]\!] \qquad Q_Z^d \mapsto Q^{(\iota_Z)_* d} q^{-\frac{c_1(N_{Z/W})\cdot d}{r-1}}.$$

We will now shift our variables in order to make our comparison. Let

$$\varsigma_j(\theta) = \sigma_j(\theta) - (r-1)\lambda_j.$$

Then

$$\widehat{\mathrm{FT}}^{\varsigma}_{Z,j} \colon \widehat{\mathrm{QDM}}_T(W) \to \varsigma_j^* \,\mathrm{QDM}(Z)^{\mathrm{ext,loc}}$$

intertwines the quantum connections up to a $\frac{1}{2}$ shift in μ_Z. Combining the Fourier transforms for the different fixed loci, we obtain

Theorem 4.3.5 (Main theorem)**.** *The diagram*

$$\begin{tikzcd}
 & & \tau^* \,\mathrm{QDM}(X)^{\mathrm{ext}} \\
\widetilde{\tau}^* \,\mathrm{QDM}(\widetilde{X})^{\mathrm{ext}} \arrow[r, "\widehat{\mathrm{FT}}_X^{-1}"] & \widehat{\mathrm{QDM}}_T(W)_{\widetilde{X}} \arrow[ur, "\widehat{\mathrm{FT}}_X"] \arrow[dr, "\oplus_j \widehat{\mathrm{FT}}_{Z,j}"] & \\
 & & \bigoplus_j \varsigma_j^* \,\mathrm{QDM}(Z)^{\mathrm{ext,loc}}
\end{tikzcd}$$

induces an isomorphism

$$\widetilde{\tau}^* \,\mathrm{QDM}(\widetilde{X})^{\mathrm{ext}} \simeq \tau^* \,\mathrm{QDM}(X)^{\mathrm{ext}} \oplus \bigoplus_j \varsigma_j^* \,\mathrm{QDM}(Z)^{\mathrm{ext,loc}}$$

over $\mathfrak{U}$ that intertwines the quantum connections and Poincaré pairings.

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