A NOTE ABOUT DERIVED CATEGORIES

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ABSTRACT. These are my notes from the introductory talks at the Derived categories and moduli spaces conference at Cornell University, given by Rachel Webb and Tudor Pădurariu.

1. Reconstruction of some schemes from their derived categories (Rachel Webb)

The references are

- Caldararu, *Derived categories of sheaves: a skimming*.
- Bondal-Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalence*.
- Bondal-Kapranov, *Representable functors, Serre functors, and mutations*.

Let X be a scheme and let $D(X)$ be the bounded derived category of coherent sheaves.

Question 1.1. *Let X and Y be schemes such that $D(X) \simeq D(Y)$. When do we have $X \simeq Y$?*

Theorem 1.2 (Bondal-Orlov)**.** *If X is smooth projective and ω_X or ω_X^{-1} is ample, then $D(X)$ classifies X up to isomorphism.*

Roughly speaking, Serre duality is an invariant of $D(X)$, so if $D(X) \simeq D(Y)$, then

$$\bigoplus H^0(X, \omega_X^{\otimes i}) \simeq \bigoplus H^0(Y, \omega_Y^{\otimes i}),$$

so if ω_X and ω_Y are ample, we can take Proj to obtain $X \simeq Y$.

1.1. Serre duality.

Recall that if X is a smooth projective variety, then $\omega_X = \det(T_X^*)$. If $\mathcal{E}$ is locally free, then

$$H^i(X, \mathcal{E}) = H^{n-i}(X, \mathcal{E}^\vee \otimes \omega_X)^\vee.$$

Moreover, if $\mathcal{E}, \mathcal{F} \in D(X)$, we have

$$\mathrm{Hom}_{D(X)}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_{D(X)}(\mathcal{F}, \mathcal{E} \otimes \omega_X[n])^\vee.$$

When $\mathcal{E}, \mathcal{F}$ are locally free, we see that

$$\mathrm{Hom}_{D(X)}(\mathcal{E}, \mathcal{F}) = H^0(X, \mathcal{E}^\vee \otimes \mathcal{F}) = H^n(X, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \omega)^\vee = \mathrm{Hom}(\mathcal{E}, \mathcal{F} \otimes \omega_X[n])^\vee.$$

Date: April 29, 2022.

Definition 1.3 (Bondal-Kapranov)**.** Let C be a k-linear category. A *Serre functor* is an equivalence $S\colon C \to C$ together with an isomorphism

$$\phi_{AB}\colon \operatorname{Hom}(A,B) \xrightarrow{\sim} \operatorname{Hom}(B,SA)^\vee.$$

Example 1.4. Let $C = D(X)$. Then we define $SA = A \otimes \omega_X[n]$.

Theorem 1.5 (Bondal-Kapranov)**.** *Let C be as above. A Serre functor on C is unique, exact, and if $F\colon C \simeq D$ is an equivalence and S_C is a Serre functor on C, then D has a Serre functor S_D and $F \circ S_C = S_D \circ F$.*

Proof. Clearly Serre functors are unique (Homs into SA are uniquely described). If $F\colon C \simeq C$ is an autoequivalence, we note that

$$\begin{aligned}\operatorname{Hom}(A, S(FB)) &= \operatorname{Hom}(FB,A)^\vee \\ &= \operatorname{Hom}(B,F^{-1}A)^\vee \\ &= \operatorname{Hom}(F^{-1}A, SB) \\ &= \operatorname{Hom}(A, FSB).\end{aligned}$$ □

It is **not true** in general that $F\colon D(X) \simeq D(Y)$ implies that $F(\omega_X) = \omega_Y$. More closely, we actually have $F(\mathcal{O}_X \otimes \omega_X[n]) = F(\mathcal{O}_X) \otimes \omega_Y[n]$.

1.2. **Reconstruction.** This subsection is dedicated to the proof of the following theorem.

Theorem 1.6 (Bondal-Orlov)**.** *Let X be a smooth projective variety such that ω_X is ample (or anti-ample). Let Y be any smooth quasiprojective variety. Then if $D(X) \simeq D(Y)$, $X \simeq Y$.*

There is a closely related result that is of interest.

Theorem 1.7 (Bondal-Orlov)**.** *Let X be as above. Then up to natural transformation, every autoequivalence of $D(X)$ is a composition of shifts, twists, and automorphisms of X.*

Example 1.8 (Mukai)**.** Let E be an elliptic curve. Then there exists $\Phi\colon D(E) \xrightarrow{\sim} D(E)$ such that $\Phi \circ \Phi = \iota[-1]$.

We will need two key lemmas, describing which objects in the derived category are points and which objects are line bundles.

Definition 1.9. A object $P \in D(X)$ is a *point object of codimension* s if

(i) $S_D(P) \simeq P[s]$, were S_D is the Serre functor on $D(X)$;
(ii) $\operatorname{Hom}(P, P[i]) = 0$ for $i < 0$;
(iii) $\operatorname{Hom}(P,P) = k(P)$ is some field.

Definition 1.10. Let X be a smooth projective variety of dimension n and suppose one of $\omega_X^{\pm 1}$ is ample. Then point objects in $D(X)$ are $\mathcal{O}_x[i]$.

Definition 1.11. An object $L \in D(X)$ is *invertible* if for all point objects $P \in D(X)$, there exists $s \in \mathbb{Z}$ such that

(1) $\mathrm{Hom}(L, P[s]) = k(P)$;

(2) $\mathrm{Hom}(L, P[i]) = 0$ for all $i \neq s$.

Lemma 1.12. *Let X be a smooth variety such that all point objects are $\mathcal{O}_x[i]$. Then invertible objects are shifts of line bundles $\mathcal{L}[i]$.*

Sketch of Theorem 1.7. Let $F\colon D(X) \xrightarrow{\sim} D(X)$ be an autoequivalence. Then $F(\mathcal{O})$ is invertible, so $F(\mathcal{O}) = \mathcal{L}[i]$ for some line bundle $\mathcal{L}$. Now if we replace F by $F(-) \otimes \mathcal{L}^{-1}[i]$, we can assume $F(\mathcal{O}) = \mathcal{O}$. This implies that $F(\omega_X) = \omega_X$. This gives an automorphism

$$\bigoplus H^0(X, \omega_X^{\otimes i}) \xrightarrow{\sim} H^0(X, \omega_X^{\otimes i}).$$

Taking the proj, we obtain $f \in \mathrm{Aut}(X)$. Finally, we can assume that F, fixes $\mathcal{O}$, ω_X and induces the identity on X. With some work, we can finally show that F is the identity functor. □

Sketchier summary of Theorem 1.6. The proof proceeds in several steps.

(1) The first step is to show that point objects in $D(Y)$ are precisely structure sheaves $\mathcal{O}_y[i]$. This is because on X, if P, Q are point objects, either $P = Q[i]$ or $\mathrm{Hom}(P, Q[i]) = 0$ for all i. But then if $P \in D(Y)$ is not $\mathcal{O}_Y[i]$, then for all $y \in Y, i \in \mathbb{Z}$, $\mathrm{Hom}(P, \mathcal{O}_Y[i]) = 0$, so $P = 0$.

(2) The second step is to note that invertible objects on Y are shifts of line bundles $\mathcal{L}[i]$.

(3) We now want to define a morphism $f\colon |X| \to |Y|$ of topological spaces. Modifying F a little bit, we can enforce $F(\mathcal{O}_x[0]) \to \mathcal{O}_Y[0]$ by choosing a line bundle $\mathcal{L}_X$ and $F\mathcal{L}_X =: \mathcal{L}_Y$.

(4) Next, f is a homeomorphism. If $\mathcal{M}, \mathcal{N}$ are line bundles and we have $\mathcal{N} \to \mathcal{M}$, then

$$\{x \mid \mathrm{Hom}(\mathcal{M}, \mathcal{O}_x) \to \mathrm{Hom}(\mathcal{N}, \mathcal{O}_x) \text{ is zero}\}$$

is a closed set, and the complements of such sets form a basis for the topology.

(5) Now we know that Y is a smooth projective variety of the same dimension as X, so using a topological argument, we see that ω_Y is ample. This gives us an isomorphism of pluricanonical rings, so $X \simeq Y$. □

2. The Fourier-Mukai transform (Tudor Pădurariu)

Following from the Bondal-Orlov reconstruction theorem, we now consider examples of nonisomorphic varieties X, Y such that $D^b(X) \simeq D^b(Y)$. These are related by Fourier-Mukai transforms, and our strategy will be to start with X with some properties and construct Y as a moduli space of objects on X.

Let X, Y be smooth projective varieties and consider $\mathcal{E} \in D^b(X \times Y)$. We of course have the following diagram:

$$\begin{array}{ccc} & X \times Y & \\ {}^{\pi_X}\swarrow & & \searrow^{\pi_Y} \\ X & & Y. \end{array}$$

Then we will define the *Fourier-Mukai transform with kernel* $\mathcal{E}$

$$\Phi = \Phi^{\mathcal{E}}_{X \to Y} \coloneqq \pi_{Y*}(\mathcal{E} \otimes \pi_X^*(-)) \colon D^b(X) \to D^b(Y).$$

Examples 2.1.

(1) The identity functor $\mathrm{id} \colon D^b(X) \xrightarrow{\sim} D^b(X)$ is given by the Fourier-Mukai transform with kernel $\mathcal{E} = \mathcal{O}_\Delta$.

(2) The shift $[1] \colon D^b(X) \to D^b(X)$ is given by the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta[1]$.

(3) Let $f \colon X \to Y$ be a morphism and let $Z \subset X \times Y$ be the graph of f. Then $f_* \colon D^b(X) \to D^b(Y)$ is given by the Fourier-Mukai transform with kernel $\mathcal{O}_Z$ and f^* is given by the Fourier-Mukai transform in the opposite direction with the same kernel.

Fourier-Mukai transforms are interesting because of the following theorem.

Theorem 2.2 (Orlov)**.** *Let* X, Y *be smooth and projective over* $\mathbb{C}$ *and* $\Phi \colon D^b(X) \xrightarrow{\sim} D^b(Y)$ *be a derived equivalence. Then there exists* $\mathcal{E} \in D^b(X \times Y)$ *such that* $\Phi = \Phi^{\mathcal{E}}_{X \to Y}$.

Here are some properties of Fourier-Mukai transforms:

(1) We can compose Fourier-Mukai transforms. If $\mathcal{E} \in D^b(X \times Y), \mathcal{F} \in D^b(Y \times Z)$, we would like $\mathcal{G} \in D^b(X \times Z)$ such that

$$\Phi^{\mathcal{F}}_{Y \to Z} \circ \Phi^{\mathcal{E}}_{X \to Y} = \Phi^{\mathcal{G}}_{X \to Z}.$$

Consider the diagram

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & {}^{\pi_{XY}}\swarrow & \downarrow{\scriptstyle \pi_{XZ}} & \searrow^{\pi_{YZ}} & \\ X \times Y & & X \times Z & & Y \times Z. \end{array}$$

Then we define

$$\mathcal{G} \coloneqq \pi_{XZ*}(\pi^*_{YZ}(\mathcal{F}) \otimes \pi^*_{XY}(\mathcal{E})).$$

(2) If $\mathcal{E} \in D^b(X \times Y)$, then $\Phi^{\mathcal{E}}_{X \to Y}$ has left adjoint $\Phi^{\mathcal{E}^\vee \otimes \omega_Y[\dim Y]}_{Y \to X}$ and right adjoint $\Phi^{\mathcal{E}^\vee \otimes \omega_X[\dim X]}_{Y \to X}$.

Now we will construct examples of derived equivalent varieties that are nonisomorphic.The first example is due to Mukai. Let A be an abelian variety. The dual abelian variety $A^\vee$ parameterizes degree 0 line bundles on A, so we have a universal line bundle $\mathcal{P}$ on $A \times A^\vee$ such that $\mathcal{P}|_{A\times p} = \mathcal{L}_p$ is the line bundle corresponding to $p \in A^\vee$.

Theorem 2.3 (Mukai)**.** *The Fourier-Mukai transform* $\Phi^{\mathcal{P}}_{A\to A^\vee}\colon D^b(A) \to D^b(A^\vee)$ *is an equivalence.*

The second example is also due to Mukai. Let S be a smooth projective surface with $\omega_X \simeq \mathcal{O}_X$. Let M be a moduli space of stable sheaves on S (with no strictly semistable sheaves). Then of course there exists a universal sheaf $\mathcal{U}$ on $S \times M$, where a point $p \in M$ corresponds to a stable sheaf $\mathcal{U}_p$ on S. Then we have $\mathcal{U}|_{S\times p} = \mathcal{U}_p$.

Theorem 2.4 (Mukai)**.** *If* $\dim M = 2$*, which says that if* M *parameterizes sheaves with Mukai vector* $(r, \beta, d) \in H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$*, then* $\beta^2 - 2rd = 0$*, then* $\Phi^{\mathcal{U}}_{M\to S}\colon D^b(M) \to D^b(S)$ *is a derived equivalence.*

A useful lemma is the following:

Lemma 2.5 (Mukai, Bondal-Orlov, Bridgeland)**.** *Let* X, Y *be smooth and projective,* $\Phi\colon D^b(X) \to D^b(Y)$*,* $x \in X$*, and* $\mathcal{P}_x \coloneqq \Phi(\mathcal{O}_x)$*.*

(1) Φ *is fully faithful if and only if*

$$\mathrm{Ext}^1_Y(\mathcal{P}_x, \mathcal{P}_y) = \begin{cases} 0 & x \neq y \text{ or } i \notin [0, \dim X] \\ \mathbb{C} & x = y \text{ and } i = 0. \end{cases}$$

(2) Φ *is an equivalence if, in addition,* $\mathcal{P}_x \otimes \omega_Y \cong \mathcal{P}_x$*.*

Proof of Theorem 2.4. First, note that $\Phi^{\mathcal{U}}(\mathcal{O}_p) = \mathcal{U}_p$. The second condition of the lemma is clear because S is Calabi-Yau, so we only need to check the first condition. Clearly $\mathrm{Hom}(\mathcal{U}_p, \mathcal{U}_p) = \mathbb{C}$ because $\mathcal{U}_p$ is stable, so for points p, q we need to consider $\mathrm{Ext}^i(\mathcal{U}_p, \mathcal{U}_q)$. This vanishes for $i \leqslant 1$ or $i \geqslant 3$ because $\mathcal{U}_p, \mathcal{U}_q$ is an honest sheaf. We can see that

$$\mathrm{Ext}^0(\mathcal{U}_p, \mathcal{U}_q) = \mathrm{Hom}(\mathcal{U}_p, \mathcal{U}_q) = 0$$
$$\mathrm{Ext}^2(\mathcal{U}_p, \mathcal{U}_q) = \mathrm{Hom}(\mathcal{U}_q, \mathcal{U}_p)^\vee = 0$$
$$\mathrm{Ext}^1(\mathcal{U}_p, \mathcal{U}_q) = 0,$$

where the first equality is because $\mathcal{U}_p, \mathcal{U}_q$ are different stable sheaves, the second follows from Serre duality, and the third follows from Riemann-Roch. □