THE COMPLEX COBORDISM RING

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Being the universal complex oriented cohomology theory, complex cobordism is a very powerful generalized cohomology theory. In this paper, we define complex cobordism and compute the complex cobordism ring of a point. We assume that the reader has knowledge of basic algebraic topology, including Steenrod squares; see my notes[1][2] for some of the relevant background.

1. (B, f)-STRUCTURES

Before we define what complex bordism even means, we first need to define what it means for a manifold to have a certain structure. Let k be a natural number and denote by CW_* the category of based CW-complexes.

Definition 1.1. A S^k-(B, f)-*structure* is a triple $\mathfrak{B} = (B, f, \lambda)$ consisting of a functor $B\colon k\mathbb{N} \to CW_*$ with Serre fibrations $f_{kn}\colon B_{kn} \to BO(kn)$ and based maps $\lambda\colon B_{kn} \to B_{k(n+1)}$ such that the diagram

$$\begin{array}{ccc} B_{kn} & \xrightarrow{\lambda} & B_{k(n+1)} \\ \downarrow f & & \downarrow f \\ BO(kn) & \hookrightarrow & BO(k(n+1)) \end{array}$$

commutes. A *multiplicative* (B, f)-*structure* in addition has maps $\mu\colon B_{kn} \times B_{kn'} \to B_{k(n+n')}$ satisfying:

- (Compatibility). The diagram

  $$\begin{array}{ccc} B_{kn} \times B_{kn'} & \xrightarrow{\mu} & B_{k(n+n')} \\ \downarrow f\times f & & \downarrow f \\ BO(kn) \times BO(kn') & \longrightarrow & BO(k(n+n')) \end{array}$$

  commutes.
- (Associativity). The diagram

  $$\begin{array}{ccc} B_{kn} \times B_{kn'} \times B_{kn''} & \xrightarrow{\mu\times 1} & B_{k(n+n')} \times B_{kn''} \\ \downarrow 1\times\mu & & \downarrow \mu \\ B_{kn} \times B_{k(n'+n'')} & \xrightarrow{\mu} & B_{k(n+n'+n'')} \end{array}$$

  commutes.

[1] https://math.columbia.edu/~plei/docs/AT1.pdf
[2] https://math.columbia.edu/~plei/docs/AT2.pdf

- (Unit). The diagram

$$\begin{array}{ccccc} B_{kn} & \longrightarrow & B_{kn} \times B_{kn'} & \longleftarrow & B_{n'} \\ & \searrow^{\lambda} & \downarrow \mu & \swarrow_{\lambda} & \\ & & B_{k(n+n')} & & \end{array}$$

commutes.

- (More compatibility). The diagram

$$\begin{array}{ccc} B_{kn} \times B_{kn'} & \xrightarrow{(1,\lambda)} & B_{kn} \times B_{k(n'+n'')} \\ \downarrow \mu & & \downarrow \mu \\ B_{k(n+n')} & \xrightarrow{\lambda} & B_{k(n+n'+n'')} \\ \mu \uparrow & & \mu \uparrow \\ B_{kn} \times B_{kn'} & \xrightarrow{(\lambda,1)} & B_{k(n+n'')} \times B_{kn'} \end{array}$$

commutes.

Some examples are $BO = n \mapsto BO(n)$ (orthogonal group), $BSO = n \mapsto BSO(n)$ (special orthogonal group), $EO = n \mapsto EO(n)$, $BU = 2n \mapsto BU(n)$ (unitary group), and $BSp = 4n \mapsto BSp(n)$ (symplectic group). It is easy to see that any family of Lie subgroups of $O(n)$ defines a (B,f)-structure. In this article, we will focus on the structure BU.

Definition 1.2. Let $\mathfrak{B} = (B, f, \lambda)$ be a (B,f)-structure. A *manifold with $\mathfrak{B}$-structure* is a triple (M^n, e, g) of a closed smooth manifold M^n with an embedding $e\colon M^n \to \mathbb{R}^k$ and $g\colon M \to B_{k-n}$ lifting (up to homotopy) the classifying map $M \to BO(k-n)$ of the normal bundle νM. If $\mathfrak{B}$ is multiplicative, then $M_1 \times M_2$ has a product $\mathfrak{B}$-structure in the obvious way.

Example 1.3. An EO-structure is the same as a framing of the normal bundle, and a BSO-structure is the same thing as an orientation.

Theorem 1.4. *All complex manifolds have a BU-structure.*

Definition 1.5. We will declare two $\mathfrak{B}$-structures $(M, e_1, g_1), (M, e_2, g_2)$ on a manifold M^n to be *equivalent* if $e_2\colon M \xrightarrow{e_1} \mathbb{R}^{k_1} \hookrightarrow \mathbb{R}^{k_2}$ and $g_2\colon M \xrightarrow{g_1} B_{k-n} \to B_{k-n}$ up to homotopy.

2. Bordism

In this section, we will define bordism for any (B,f)-structure $\mathfrak{B}$, construct a spectrum $M\mathfrak{B}$ representing bordism with $\mathfrak{B}$-structure, and finally state the Pontryagin-Thom theorem, which will allow us to reduce the computation of cobordism rings to stable homotopy theory.

Consider the maps

$$e_I\colon I \to \mathbb{R}^2 \qquad t \mapsto (\cos \pi t, \sin \pi t)$$
$$g_I\colon I \to \mathbb{R}^2 \qquad t \mapsto (\cos \pi t, \sin \pi t).$$

Note g_I is a framing of νI. Set $-(M^n, e, g) = (M^n \times I, e \times e_I, g \times g_I)\big|_{M^n \times I}$. Now we may define bordism.

Definition 2.1. Two manifolds with $\mathfrak{B}$-structure $(M^n, e, g), (N^n, e', g')$ are *bordant* if there exists a manifold (W^{n+1}, E, G) such that $\partial W^{n+1} = M \sqcup -N$.

This is an equivalence relation, so we may consider the set $\Omega_*^{\mathfrak{B}}$ of bordism classes of manifolds with $\mathfrak{B}$-structure.

Proposition 2.2.

(a) The set $\Omega_*^{\mathfrak{B}}$ is a graded abelian group under the operation of disjoint union. Here, we set $[M^n, e, g] + [N^n, f, h] = [M^n \sqcup N^n, e \sqcup f, g \sqcup h]$.

(b) If $\mathfrak{B}$ is multiplicative, then $\Omega_*^{\mathfrak{B}}$ is a graded ring with product the Cartesian product. Here, $[M, e, g] \cdot [N, f, h] = [M \times N, e \times f, \mu \circ (g \times h)]$.

Now recall that if $\pi\colon E \to B$ is a vector bundle with disk bundle $D(\pi)$ and sphere bundle $S(\pi)$, then the *Thom space* of π is $M(\pi)$. If ν is the normal bundle of some embedding $M \to \mathbb{R}^k$, we have a homeomorphism $N_\varepsilon(M)/\partial N_\varepsilon(M) \cong M(\nu)$, where $N_\varepsilon(M)$ is the ε-neighborhood of $M \subset \mathbb{R}^k$. Also, if $\pi_{k-n}^{\mathfrak{B}} = f^*(\gamma_{k-n} \to BO(k-n))$, then we have a map $g\colon \nu \to \pi_{k-n}^{\mathfrak{B}}$ because $g\colon M \to B_{k-n}$ lifts the classifying map of ν.

Now we will construct a spectrum $M\mathfrak{B}$ for any (B, f)-structure $\mathfrak{B}$. Note that for $k < k'$, we have

$$\lambda^*(\pi_{k'}^{\mathfrak{B}}) = \lambda^* f^*(\gamma_{k'}) = f^* j^* \gamma_{k'},$$

where $j\colon BO(k) \to BO(k')$ is the natural inclusion given by the map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ induced by $\mathbb{R}^k \subset \mathbb{R}^{k'}$. Now we see that $j^*\gamma_{k'} = \gamma_k \oplus \mathbb{R}^{k'-k}$, and therefore we obtain a map

$$\mathbb{R}^{k'-k} \oplus \pi_k^{\mathfrak{B}} \to \pi_{k'}^{\mathfrak{B}}.$$

Taking the Thom spaces and recalling that $M(E \oplus \mathbb{R}) = \Sigma M(E)$, we obtain maps

$$\Sigma^{k'-k} M(\pi_k^{\mathfrak{B}}) \to M(\pi_{k'}^{\mathfrak{B}}).$$

This defines the *Thom spectrum* $M\mathfrak{B}$.

Example 2.3. If $\mathfrak{B} = BU$, then note that if $j\colon BU(k) \to BU(k+1)$ is the natural inclusion, then $j^*\gamma_{k+1} = \mathbb{C} \oplus \gamma_k$, and thus we obtain maps $\Sigma^2 M(\gamma \to BU(n)) \to M(\gamma \to BU(n+1))$. This defines the Thom spectrum MU.

For a manifold (M^n, e, g) with $\mathfrak{B}$-structure, we want to construct an element of $\pi_* M\mathcal{B}$. Consider the morphism

$$\xi(k, k-n)\colon S^k \to \overline{N_e(M)}/\partial N_e(M) \cong M(\nu) \to M(\pi_{k-n}^{\mathfrak{B}}),$$

where $S^k \to \overline{N_e(M)}/\partial N_e(M)$ collapses everything outside of $N_e(M)$ to the basepoint. One can check that up to equivalence of $\mathfrak{B}$-structures, this gives a well-defined element of the stable homotopy group $\pi_n(M\mathfrak{B})$. One can also check that this assignment is well-defined on bordism classes, which suggests the following result:

Lemma 2.4. *The map $\xi_n \colon \Omega_n^{\mathfrak{B}} \to \pi_n(M\mathfrak{B})$ is a group homomorphism. If $\mathfrak{B}$ is multiplicative, then ξ_* is a morphism of graded rings.*

It will turn out that ξ_* is an isomorphism, but first we will define a candidate inverse $\zeta_* \colon \pi_*(M\mathfrak{B}) \to \Omega_*^{\mathfrak{B}}$. Let $x \in \pi_n(M\mathfrak{B})$ be represented by $F \colon S^{k+n} \to M\mathfrak{B}_k$. Then define $F' \coloneqq Mf \circ F \colon S^{k+n} \to M(\gamma \to BO(k))$. By compactness, we obtain a map $F'' \colon S^{k+1} \to M(\gamma \to Gr(k, N))$ for some N. Consider the (open) disk bundle $D_r(\gamma \to Gr(k, N))$, where r is the radius, and define

$$V_r^F = (F'')^{-1}(D_r(\gamma \to Gr(k, N))).$$

Now modify F'' by a homotopy such that D_r, V_r^F are smooth manifolds for $\frac{3}{4} < r < 1$ and if $V = \partial V^F_{13/16}$, then

$$V^F_{15/16} - V^F_{13/16} = V \times [13/16, 15/16).$$

Finally, we require that F'' corresponds to the map

$$V \times [13/16, 15/16) \ni (x, t) \mapsto \left(t + \frac{3}{16}\right) f(x).$$

Now we can represent x by $g \colon S^{k+n} \to M(\gamma \to Gr(k, N))$ that is smooth on $V^g_{15/16}$. Finally, we can modify g so it is transverse to $Gr(k, N)$, and now we define

$$M^n = g^{-1}(Gr(k, N)) \qquad e \colon M^n \hookrightarrow V^g_{15/16} \hookrightarrow \mathbb{R}^{n+k}.$$

Finally, $T(g)$ maps the normal bundle νM isomorphically to the normal bundle $\gamma \to Gr(k, N)$ of $Gr(k, N) \subset D(\gamma \to Gr(k, N))$, so $\nu M = \gamma\big|_M$. Now $T(g) \colon M^n \to BO(k)$ is homotopic to the classifying map of νM by [Koc96, Proposition 1.3.2], and so by homotopy lifting we obtain a map $G \colon M \to \mathcal{B}_k$. Now we may set

$$\zeta_n(x) = [M^n, e, G].$$

Theorem 2.5 (Pontryagin-Thom)**.** *The map ζ_* is a well-defined homomorphism of graded abelian groups, and if $\mathfrak{B}$ is multiplicative, then ζ_* is a well-defined morphism of graded rings. Furthermore, ξ_*, ζ_* are inverse isomorphisms.*

Now we have translated the problem of computing the bordism ring $\Omega_*^{\mathfrak{B}}$ into the problem of computing the stable homotopy groups $\pi_*(M\mathfrak{B})$. Later, we will see that there is a spectral sequence computing the homotopy groups of sufficiently nice spectra.

3. Homology

In this section, we first define a Steenrod algebra for any prime p. The cohomology of any space or spectrum will be a module over this algebra. Next, we will build up the homology of some simpler spaces as a comodule over the dual coalgebra, and finally we will give an expression for the homology of MU.

3.1. **Steenrod algebra.** For any prime p, there is an algebra $\mathcal{A}_p$ of stable cohomology operations on $\mathbb{Z}/p$-cohomology. We will call this algebra the *mod* p *Steenrod algebra*. We assume that the reader is already familiar with the algebra $\mathcal{A}_2$. For example, see [Hat02, §4.L] for a discussion.

Let p be an odd prime. Let β be the Bockstein homomorphism associated to the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

Now there exist stable cohomology operations

$$P^n \colon H^k(X, \mathbb{Z}/p) \to H^{k+2n(p-1)}(X, \mathbb{Z}/p)$$

for any space X satisfying the following properties for $x \in H^*(X, \mathbb{Z}/p)$:

(1) If $2n > |x|$, then $P^n(x) = 0$.

(2) If $|x| = 2n$, then $P^n(x) = x^p$.

(3) (Cartan formula). For $x, y \in H^*(X, \mathbb{Z}/p)$, we have

$$P^n(xy) = \sum_{k=0}^{n} P^k(x) P^{n-k}(y).$$

(4) (Adem relations). We have the two relations

$$P^a P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} P^{a+b-j} P^j$$

and

$$P^a \beta P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j + \sum_j (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j.$$

Now we may define the algebra $\mathcal{A}_p = \mathbb{Z}/p \langle \beta, P^n \mid n \geqslant 1 \rangle / \text{Adem relations}$. This is a Hopf algebra, where the coproduct is

$$\Delta(Sq^n) = \sum Sq^k \otimes Sq^{n-k}$$

for $p = 2$, and

$$\Delta(P^n) = \sum P^k \otimes P^{n-k} \qquad \Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta$$

for odd primes p. Note that if X is a space, $H^*(X, \mathbb{Z}/p)$ is a module over $\mathcal{A}_p$, and $H_*(X, \mathbb{Z}/p)$ is a comodule over the dual Steenrod algebra $\mathcal{A}_p^*$. Later, we will need a more refined description of $\mathcal{A}_p^*$, due to Milnor.

Theorem 3.1 (Milnor)**.** *Write*

$$\xi_n = (Sq^{2^{n-1}} \cdots Sq^2 Sq^1)^*$$

for $p = 2$ *and*

$$\xi_n = (P^{p^{n-1}} \cdots P^p P^1)^*$$

for p *an odd prime. When* p *is an odd prime, additionally write*

$$\tau_n = (P^{p^{n-1}} \cdots P^p P^1 \beta)^*.$$

Now we have

$$\mathcal{A}_2^* = \mathbb{Z}/2[\xi_1, \ldots, \xi_n, \ldots]$$

and

$$\mathcal{A}_p^* = \mathbb{Z}/p[\xi_1, \ldots, \xi_n, \ldots] \otimes \bigwedge\nolimits_{\mathbb{Z}/p}[\tau_0, \ldots, \tau_m, \ldots]$$

with coproduct

$$\Delta(\xi_n) = \sum \xi_{n-k}^{p^k} \otimes \xi_k$$

for all p *and*

$$\Delta(\tau_n) = \tau_n \otimes 1 + \sum \xi_{n-k}^{p^k} \otimes \tau_k$$

for p *odd.*

3.2. **Homology of** MU**.** For a spectrum E, define $H_k(E; G) = \varinjlim H_{k+n}(E_n; G)$ over the system of morphisms

$$H_{k+n}(E_n; G) \xrightarrow{\sim} H_{k+n+r}(\Sigma^r E_n; G) \to H_{k+n+r}(E_{n+r}; G).$$

We will now give expressions for the homology of simpler spaces before giving an expression for the homology of MU. Recall that $BU(n) = Gr(n, \infty)$ and $H^*(BU(n)) = \mathbb{Z}[c_1, \ldots, c_n]$, where $|c_i| = 2i$. Therefore, if BU is the classifying space of the infinite unitary group U, then $H^*(BU) = \mathbb{Z}[c_1, \ldots, c_n, \ldots]$. From the multiplication on U, we obtain a coproduct structure

$$\Delta(c_n) = \sum_{k=0}^{n} c_k \otimes c_{n-k}$$

turning $H^*(BU)$ into a Hopf algebra. By the universal coefficient theorem,

$$H_*(BU) = \mathbb{Z}[a_1, \ldots, a_n, \ldots]$$

is the dual Hopf algebra.

Because the formulas for the action of $\mathcal{A}_p$ on cohomology are very complicated, we will consider the action of the dual Steenrod operations $Sq_*^n \in \mathcal{A}_2, P_*^n \in \mathcal{A}_p^*$ on homology.

Lemma 3.2. *In* $H_*(BU; \mathbb{Z}/2)$*, we have* $Sq_*^{2k-1} = 0$ *and* $Sq_*^{2k}(a_n) = \binom{n-k}{k} a_{n-k}$*. For* p *an odd prime,* $P_*^k(a_n) = \binom{n-k(p-1)}{k} a_{n-k(p-1)}$.

Proof. We have $a_k = (c_1^k)^*$, so

$$Sq^{2k}(c_1^{n-k}) = \binom{n-k}{k} c_1^n$$

and

$$P^k(c_1^{n-k(p-1)}) = \binom{n-k(p-1)}{k} c_1^n$$

on $\mathbb{CP}^\infty$. □

First note that $MU(1) \simeq \mathbb{CP}^\infty$, with homotopy equivalence given by the zero section $\mathbb{CP}^\infty = BU(1) \to M(\gamma \to BU(1))$, where γ is the tautological bundle. Here, note that $S(\gamma) = S^\infty$ which is contractible, so $D(\gamma) \to M(\gamma)$ is a homotopy equivalence, and thus the zero section of $M(\gamma)$ is a homotopy equivalence.

Before giving an expression for $H^*(MU)$, we need a notion from the theory of Hopf algebras.

Definition 3.3. Let H be a Hopf algebra and C be a comodule over H with structure morphism $\psi\colon C \to H \otimes C$. Then $x \in C$ is *primitive* if $\psi(x) = 1 \otimes x$. We will denote the vector space of primitive elements in C by PC.

Theorem 3.4. *As $\mathbb{Z}/p$-algebras and $\mathcal{A}_p^*$-comodules, we have*

$$H_*(MU, \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1^2, \dots, \xi_n^2, \dots] \otimes PH_*(MU, \mathbb{Z}/2)$$

and

$$H_*(MU, \mathbb{Z}/p) \cong \mathbb{Z}/p[\xi_1, \dots, \xi_n, \dots] \otimes PH_*(MU, \mathbb{Z}/p)$$

for odd primes p, where

$$PH_*(MU; \mathbb{Z}/p) = \mathbb{Z}/p[y_k \mid k \geqslant 1, k \neq p^t - 1]$$

for all primes p and $|y_k| = 2k$.

In the proof of this result, we use the following result for $\mathbb{CP}^\infty$.

Proposition 3.5. *Recall that $H_*(\mathbb{CP}^\infty; \mathbb{Z}/p) = \mathbb{Z}/p\{1, a_1, \dots, a_k, \dots\}$. The coaction $\psi\colon H_*(\mathbb{CP}^\infty; \mathbb{Z}/p) \to \mathcal{A}_p^* \otimes H_*(\mathbb{CP}^\infty; \mathbb{Z}/p)$ satisfies*

$$\psi(H_*(\mathbb{CP}^\infty; \mathbb{Z}/2)) \subset \mathbb{Z}/2[\xi_1^2, \dots, \xi_n^2, \dots] \otimes H_*(\mathbb{CP}^\infty; \mathbb{Z}/2)$$

and

$$\psi(H_*(\mathbb{CP}^\infty; \mathbb{Z}/p)) \subset \mathbb{Z}/2[\xi_1, \dots, \xi_n, \dots] \otimes H_*(\mathbb{CP}^\infty; \mathbb{Z}/p)$$

for p odd. Furthermore, the component of $\psi(a_k)$ in $(\mathcal{A}_p^*)_{k-2} \otimes H_2(\mathbb{CP}^\infty; \mathbb{Z}/p)$ is

$$\begin{cases} \xi_n^2 \otimes a_1 & p = 2, k = 2^n \\ \xi_n \otimes a_1 & 2 \nmid p, k = p^n \\ 0 & \text{otherwise.} \end{cases}$$

4. The complex cobordism ring

We begin this section by stating the Adams spectral sequence, which computes the homotopy groups of sufficiently nice spectra. After this, we will state some results about comodules over coalgebras, and finally we will use the Adams spectral sequence to compute $\pi_*(MU)$.

4.1. **Adams spectral sequence.**

Definition 4.1. A spectrum E is *connective* if $\pi_{-n}(E) = 0$ for all $n > 0$.

Definition 4.2. A spectrum E is *finite type* if $\pi_n(E)$ is a finitely generated abelian group for all n.

Now let X, Y be connective spectra of finite type.

Theorem 4.3 (Adams spectral sequence)**.** *Let p be a prime. Then there exists a spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}_p}(H^*(X, \mathbb{Z}/p), H^*(Y, \mathbb{Z}/p))_t \Rightarrow [Y, X]_t \otimes \mathbb{Z}_p,$$

where d_r has degree $(r, -1)$. Furthermore, if X is a ring spectrum and $Y = S$ is the sphere spectrum, we have

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}_p}(H^*(X, \mathbb{Z}/p), \mathbb{Z}/p)_t \Rightarrow \pi_t(X) \otimes \mathbb{Z}_p$$

and the following multiplicative structure:

(1) If μ is the multiplication on X, then E_2 has algebra structure induced from $\mu^*\colon H^*(X, \mathbb{Z}/p) \to H^*(X, \mathbb{Z}/p) \otimes H^*(X, \mathbb{Z}/p)$. If μ is homotopy commutative, then the product on E_2 is commutative.

(2) For all r, d_r is a derivation.

(3) The algebra structure on E_{r+1} is induced from that on E_r.

(4) The algebra structure on E_∞ agrees with the algebra structure on $\pi_*(X)$.

4.2. **Comodules over coalgebras.** Let k be a field, A be a k-coalgebra, M be a right comodule, and N be a left comodule. Consider structure morphisms $\varphi\colon M \to M \otimes A, \psi\colon N \to A \otimes N$.

Definition 4.4. The *cotensor product* $M \square_A N$ is defined to be

$$M \square_A N = \ker(\varphi \otimes 1 - 1 \otimes \psi\colon M \otimes N \to M \otimes A \otimes N).$$

We have an isomorphism $M \square_A N \cong (M^* \otimes_{A^*} N^*)^*$.

Similar to the identities for the tensor product, we have $M = M \square_A A$ and $N = A \square_A N$.

Definition 4.5. If M is a right A-comodule, a *free coresolution* $F_\bullet$ of M is a long exact sequence

$$0 \to M \to F_0 \to \cdots \to F_n \to \cdots$$

where each F_i is a direct sum of copies of A.

Definition 4.6. Let M be a right A-comodule and N be a left A-comodule. Then define

$$\mathrm{Cotor}_A(M, N) = H_*(F_\bullet \square_A N),$$

where $F_\bullet$ is a free coresolution of M.

Lemma 4.7. *We have the identity* $\mathrm{Cotor}_A(M, k) = \mathrm{Ext}_{A^*}(M^*, k)$.

Now in the Adams spectral sequence, because X is a connective spectrum of finite type, we may dualize $H^*(X, \mathbb{Z}/p) = H_*(X, \mathbb{Z}/p)^*$. Now using the lemma, if $Y = \mathbb{S}$, then we have

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}_p}(H^*(X, \mathbb{Z}/p), \mathbb{Z}/p) = \mathrm{Cotor}^s_{\mathcal{A}_p^*}(H_*(X, \mathbb{Z}/p), \mathbb{Z}/p).$$

We want to be able to compute these Cotors, so we further state some results in this direction. Let A be a Hopf algebra, B be a sub-Hopf algebra of A, and B^+ be the set of positive degrees of A. If $AB^+ = B^+A$, then we call B a *normal* subalgebra of A. Then the Hopf algebra structure on A induces a Hopf algebra structure on

$$A /\!\!/ B = A/(AB^+) = k \otimes_B A.$$

If $\pi\colon A \to A /\!\!/ B$ is the projection map, then

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi \otimes 1} A /\!\!/ B \otimes A$$

makes A a $A /\!\!/ B$-comodule. Also we have $k \square_{A /\!\!/ B} A = B$.

Proposition 4.8. *Let* A *be a Hopf algebra over* k *and* B *is a normal sub-Hopf algebra, then*

$$\mathrm{Cotor}_A(B, k) = \mathrm{Cotor}_{A /\!\!/ B}(k, k).$$

Proposition 4.9. *Let* I *be a sequence of integers and* $E = \bigwedge_k [x_i \mid i \in I]$. *Suppose that if* $i < j$, *then* $|x_i| \leqslant |x_j|$ *and that there are only finitely many* x_i *of every degree. Then*

$$\mathrm{Cotor}_E(k, k) = k[y_i \mid i \in I],$$

where $|y_i| = (1, |x_i|)$.

We will use this result when we compute the homotopy groups $\pi_*(MU)$.

5. Homotopy of MU

We are now ready to compute the homotopy groups $\pi_*(MU)$. Write Ω^U_* for Ω^{BU}_*, the bordism ring for the (B, f)-structure BU.

Theorem 5.1 (Milnor)**.** *There exist* $y_n \in \Omega^U_{2n}$ *such that* $\Omega^U_* = \mathbb{Z}[y_1, \ldots, y_n, \ldots]$.

Proof. By the Pontryagin-Thom theorem, this is equivalent to computing $\pi_*(MU)$. Now we will consider the Adams spectral sequence. To simplify our notation, we will write

$$\mathcal{A}'_p = \begin{cases} \mathbb{Z}/2[\xi_1^2, \ldots, \xi_n^2, \ldots] & p = 2 \\ \mathbb{Z}/p[\xi_1, \ldots, \xi_n, \ldots] & 2 \nmid p \end{cases}$$

and

$$S_p = \mathbb{Z}/p[Y_n \mid n \geqslant 1, n \neq p^t - 1 | Y_n| = 2n]$$

for the set of primitive elements. Recall that $H_*(MU, \mathbb{Z}/p) = \mathcal{A}_p' \otimes S_p$ by Theorem 3.4, so now we may compute

$$\begin{aligned} \mathrm{Cotor}_{\mathcal{A}_p^*}(H_*(MU, \mathbb{Z}/p), \mathbb{Z}/p) &= \mathrm{Cotor}_{\mathcal{A}_p^*}(\mathcal{A}_p' \otimes S_p, \mathbb{Z}/p) \\ &= \mathrm{Cotor}_{\mathcal{A}_p^*}(\mathcal{A}_p', \mathbb{Z}/p) \otimes S_p \\ &= \mathrm{Cotor}_{\mathcal{A}_p^* /\!/ \mathcal{A}_p'}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes S_p \end{aligned}$$

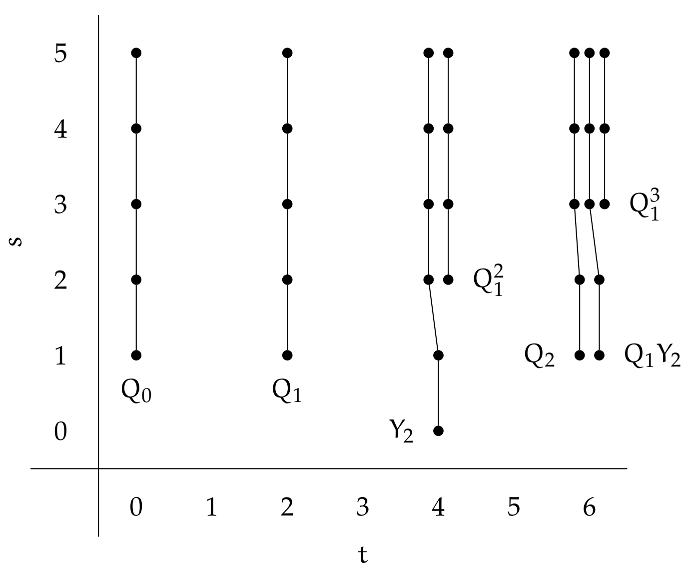
because S_p is a trivial comodule. Next, we observe that

$$\mathcal{A}_p^* /\!/ \mathcal{A}_p' = \begin{cases} \bigwedge[\xi_1, \ldots, \xi_n, \ldots] & p = 2 \\ \bigwedge[\tau_0, \ldots, \tau_n, \ldots] & 2 \nmid p. \end{cases}$$

Now we apply Proposition 4.9 and obtain

$$\begin{aligned} E_2 &= \mathrm{Cotor}_{\mathcal{A}_p^*}(H_*(MU, \mathbb{Z}/p), \mathbb{Z}/p) \\ &= \mathrm{Cotor}_{\mathcal{A}_p^* /\!/ \mathcal{A}_p'}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes S_p \\ &= \mathbb{Z}/p[Q_0, \ldots, Q_n, \ldots] \otimes \mathbb{Z}/p[Y_n \mid n \geqslant 1, n \neq p^t - 1, |Y_n| = 2n], \end{aligned}$$

where $|Q_n| = (1, 2p^n - 2)$. By degree reasons, $E_2 = E_\infty$, so $\Omega_*^U \otimes \mathbb{Z}_p$ is a free abelian group. Because MU is of finite type, Ω_*^U is also a free abelian group. For example, when $p = 2$, the Adams spectral sequence for $\pi_*(MU)$ looks like



Now write $(\Omega_*^U)^+$ for the elements of positive degree and note that by the Adams spectral sequence, the set of indecomposable elements

$$I_* \coloneqq (\Omega_*^U)^+ / ((\Omega_*^U)^+)^{\times 2}$$

is a free abelian group with a single generator in each degree. Choose $y_n \in \Omega_{2n}^U$ representing I_{2n} and write

$$\alpha \colon R \coloneqq \mathbb{Z}[x_1, \ldots, x_n, \ldots] \to \Omega_*^U \qquad x_n \mapsto y_n.$$

Now define R_0, R_1 by $x_n \in R_0$ if $n \neq p^t - 1$ for all t and $x_n \in R_1$ if $n = p^t - 1$ for some t. Then the induced map $\operatorname{gr}\alpha\colon \operatorname{gr} R \to E_\infty$ is an isomorphism, so α is injective. It remains to prove surjectivity.

First, note that $\Omega_0^U = \mathbb{Z}$. Now suppose $\Omega_t^U \in \operatorname{Im}\alpha$ for all $t < 2n$. For all $y \in \Omega_{2n}^U$, if $y = ky_n$ in I_{2n}, then we know $x - ky_n \in \left((\Omega_*^U)^+\right)^{\times 2}$ is a proeuct of things of lower degree, and therefore $x \in \operatorname{Im}\alpha$ by induction. $\square$

References

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