RATIONAL BLOWDOWNS IN ALGEBRAIC GEOMETRY

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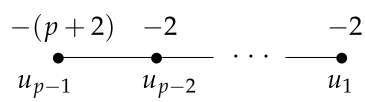
Abstract. We describe rational blowdowns in complex algebraic geometry following the work of Lee and Park. In particular, we will try to describe both the algebraic and topological perspectives of the same manifold.

1. Introduction

Rational blowdowns were introduced by Fintushel and Stern in 1995. In 1998, Symington showed that rational blowdown can be performed in the symplectic category. In this paper, we will discuss rational blowdowns in the complex category.

1.1. **Rational Blowdown.**

Recall that we can perform rational blowdowns in the smooth category. If X is a closed 4-manifold, then rational blowdown begins with $p-1$ spheres $u_1, \dots, u_{p-1}$ arranged in the following diagram.



We then plumb tubular neighborhoods of the spheres to obtain a manifold C_p. Then the boundary of C_p is the lens space $L(p^2, p-1)$, which also bounds a rational ball B_p with fundamental group $\mathbb{Z}/p\mathbb{Z}$. Then the *rational blowdown* of X is the manifold obtained by removing C_p and replacing it with B_p. This is not a priori well-defined, so we need the following lemma:

Lemma 1 (Fintushel-Stern). *Every diffeomorphism of $L(p^2, p-1)$ extends to a diffeomorphism of the rational ball B_p.*

1.2. **Overview of Techniques.**

We will construct a simply connected minimal complex surface of general type with $K^2 = 2$ and $p_g = 0$. Recall that $p_g = h^{n,0} = h^0(X, K_X)$ is the geometric genus. We begin by constructing a special rational surface Z as a blowup of a rational elliptic surface. Then we will use the rational blowdown to obtain our desired surface. Finally, we will construct the surface using methods from algebraic geometry. In particular, if we contract several chains of rational curves on the rational surface Z, we obtain a singular surface with a $\mathbb{Q}$-Gorenstein smoothing. Then we will see that the generic fiber of this smoothing is the desired surface.

2. Construction of the Rational Surface

Consider the following pencil of cubics on $\mathbb{P}^2$: Let A be a line and B be a smooth conic in $\mathbb{P}^2$. Then choose a line $L \subset \mathbb{P}^2$ such that the intersection $L \cap (A+B)$ is three distinct points. After resolving the base locus of this linear system, we obtain a rational elliptic surface $Y \to \mathbb{P}^1$. More specifically, we need to blow up B six times and A three times, obtaining exceptional curves e_i, e_i', e_i'' for $i = 1, 2, 3$. Here we will say that e_i intersects $A + B$ and e_i' intersects e_i and e_i''. Finally, we say that e_3'' intersects A.

We find that we have one IV^* singular fiber (corresponding to L, e_i, e_i') and one I_2 singular fiber (corresponding to A, B), in addition to two nodal fibers. To check that these are all the singular fibers, we can check that the sum of the Euler characteristics of the fibers is 12. Finally, we see that there are three exceptional (-1)-curves, which are sections of the fibration. A picture of the surface Y is below:

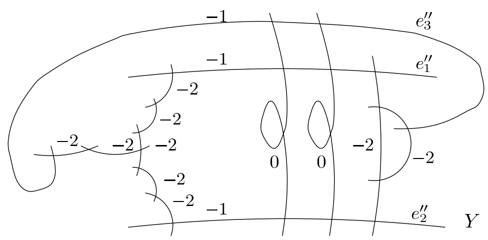


Figure 1. The elliptic surface

Now we blow up the singular points of the two nodal fibers F_1, F_2 and obtain two exceptional curves E_1, E_2. Note that each E_i is a (-1)-curve and each F_i is a (-4)-curve. To check this, note that the pullback of F under the blowup is $F + 2E$. Then if we rename the three sections S_1, S_2, S_3, we then blow up the intersection points of S_1, S_2 with F_1, F_2, B. Then the curves S_1, S_2, B all now have self-intersection -4. Then we blow up at the intersection points of S_3 with F_1, F_2, which makes S_3 a (-3)-curve. Note that now, each F_i has self-intersection -7.

Blowing up the intersection point between the strict transform of S_2 and the exceptional curve intersecting S_2 and F_1 three times, we obtain a chain of $\mathbb{P}^1$ with the following self-intersection numbers: $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$. We perform the same operation with S_2, F_2 and obtain another chain of lines with the same self-intersection numbers. Next, we can blow up at the intersection point between S_1 and the (-1)-curve meeting S_1, F_2, so the previous chain becomes

$$\overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}.$$

Finally, we blow up the right end of the previous chain at the intersection point with the exceptional curve, so we now have a chain

$$\overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ}.$$

Note that S_1 now has self-intersection -5 and S_2 now has self-intersection -10. Denote this surface by $Z \cong Y \# 18\overline{\mathbb{P}^2}$. In Figure 2, we present a picture of Z.

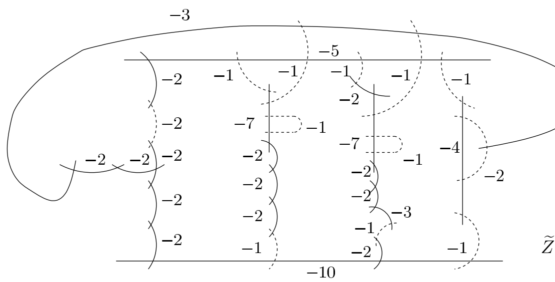


FIGURE 2. The surface Z

3. Rational Blowdown of Z

First, we will need to define a more general form of rational blowdown in the smooth category. Let $p > q$ be coprime positive integers. Then, if we have $\frac{p^2}{pq-1} = [b_k, \ldots, b_1]$ as a continued fraction, then suppose we have the following chain of embedded spheres:

$$\overset{-b_k}{\circ} - \overset{-b_{k-1}}{\circ} - \cdots - \overset{-b_2}{\circ} - \overset{-b_1}{\circ}.$$

Then we can plumb disk bundles over this configuration to obtain a simply-connected smooth 4-manifold $C_{p,q}$ with boundary the lens space $L(p^2, 1-pq)$.

Note that this lens space is also the boundary of some rational ball $B_{p,q}$. Then the *generalized rational blowdown* of a smooth 4-manifold X containing a configuration $C_{p,q}$ is the manifold obtained by replacing $C_{p,q}$ with $B_{p,q}$.

Remark 2. A version of Lemma 1 holds in this setting, so this process is well-defined.

To perform rational blowdown on Z, we need to find chains of the form $C_{p,q}$. We note that Z contains five disjoint chains[1] $C_{15,7}, C_{9,4}, C_{5,1}, C_{3,1}, C_{2,1}$. Then we blow them down to obtain a manifold $\widetilde{Z}$.

Theorem 3 (Lee-Park)**.** *The rational blowdown $\widetilde{Z}$ of the rational surface Z is a simply-connected closed symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 2$.*

4. Algebraic Construction

We will present five disjoint chains of $\mathbb{P}^1$ which will be contracted:

(1) $\overset{-2}{\circ} - \overset{-10}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$ containing S_2, S_3, and part of the IV^* singular fiber;

(2) $\overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ containing F_1;

(3) $\overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$ containing F_2;

[1] It is not explained anywhere how to find these chains.

(4) $\overset{-4}{\circ}$ given by the strict transform of B;

(5) $\overset{-5}{\circ} - \overset{-2}{\circ}$ containing S_1.

Each of these chains is the resolution graph of a special quotient singularity, so we can contract them to produce a projective surface X with five singular points. We will prove that X has a $\mathbb{Q}$-*Gorenstein smoothing*. First, we need to introduce some tools from algebraic geometry.

4.1. **Preliminaries.**

Definition 4. Let X be a normal projective surface with quotient singularities. Then let Δ be a small disk and $\mathcal{X} \to \Delta$ be a flat family of surfaces. Then $\mathcal{X} \to \Delta$ is a $\mathbb{Q}$-*Gorenstein smoothing* of X if:

(1) The general fiber is a smooth projective surface;

(2) The central fiber X_0 is isomorphic to X;

(3) The relative canonical divisor $K_{\mathcal{X}/\Delta}$ is $\mathbb{Q}$-Cartier.

Remark 5. We may also define $\mathbb{Q}$-Gorenstein smoothing locally for a quotient singularity by considering germs of quotient singularities.

We now define some terms commonly used in algebraic geometry that may not be familiar to everyone.

Definition 6. Let X be an algebraic variety over a field k. Then X is *normal* if the local ring $\mathcal{O}_{X,x}$ is integrally closed in the function field $k(X)$ for every point $x \in X$.

All smooth varieties are normal, and if a variety is normal, its singular locus has codimension at least 2. In addition, an affine variety is normal iff its coordinate ring is integrally closed.

Definition 7. Let X, Y be algebraic varieties. Then a morphism $f : X \to Y$ is *flat* if the pullback makes the stalk $\mathcal{O}_{X,x}$ a flat module over $\mathcal{O}_{Y,f(x)}$. Recall that an R-module M is flat if $- \otimes M$ is an exact functor.

Flatness is a strong property. Every fiber of a flat family of algebraic varieties has the same dimension. Even better, all of the fibers have the same Hilbert polynomial.

Definition 8. Let X be an algebraic variety (or complex analytic space). Then a *divisor* on X is a formal sum of irreducible hypersurfaces of X. A divisor D is *Cartier* if there exists a holomorphic line bundle $L \to X$ and a section $\gamma \in H^0(X, L)$ such that D is the difference

$$D = (\gamma = 0) - (\gamma = \infty).$$

Finally, a divisor D is $\mathbb{Q}$-Cartier if some positive multiple of D is Cartier.

4.2. **Smoothing of X.**

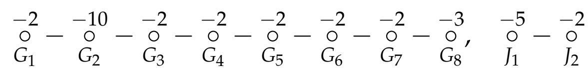
Theorem 9. *The surface X with five quotient singularities has a $\mathbb{Q}$-Gorenstein smoothing.*

We will now consider the general fiber X_t of the smoothing. First, note that because Z is a smooth rational surface and X has only mild singularities, we must have

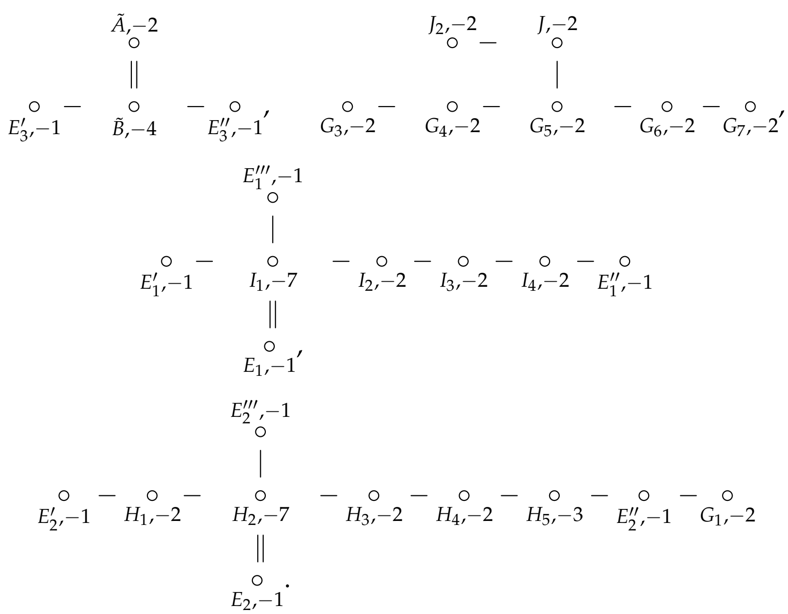
$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0.$$

By flatness of the smoothing, we must have $H^2(X_t, \mathcal{O}_{X_t}) = 0$. Then, from Hodge theory on Kähler manifolds, we have $p_g(X_t) = 0$.

Next, we will compute $K^2_{X_t}$. We will first compute K^2_X and then use flatness. Note that we have two chains with the following dual graphs:



and four special fibers of the map $Z \to \mathbb{P}^1$:



From this, if we denote the map $Z \to X$ by f, then we can compute f^*K_X explicitly as

$$\begin{aligned} f^*K_X = {} & \frac{119}{30}G_1 + \frac{14}{15}G_2 + \frac{13}{15}G_3 + \frac{12}{15}G_4 + \frac{11}{15}G_5 + \frac{10}{15}G_6 + \frac{9}{15}G_7 + \frac{8}{15}G_8 + \frac{17}{18}H_1 \\ & + \frac{7}{18}H_2 + \frac{23}{18}H_3 + \frac{39}{18}H_4 + \frac{55}{18}H_5 + \frac{3}{10}I_1 + \frac{11}{10}I_2 + \frac{19}{10}I_3 + \frac{27}{10}I_4 + \frac{1}{2}\tilde{B} \\ & + \frac{2}{3}J_1 + \frac{1}{3}J_2 + \frac{1}{2}E_1' + \frac{7}{2}E_1'' + \frac{1}{2}E_1''' + \frac{3}{2}E_2' + 7E_2'' + \frac{1}{2}E_2''' + E_3' + E_3''. \end{aligned}$$

Using this, we can then compute $K^2_X = 2$. By flatness, we then have $K^2_{X_t} = 2$.

Proposition 10 (Lee-Park). *X_t is a simply-connected minimal surface of general type with $p_g = 0$ and $K_{X_t}^2 = 2$.*

The proof that X_t is simply-connected is carried out by showing that X_t is diffeomorphic to the rational blowdown $\widetilde{Z}$ of Z.

References

[1] Lee, Yongnam, and Jongil Park. "A Simply Connected Surface of General Type with $p_g = 0$ and $K_2 = 2$." Inventiones mathematicae 170.3 (2007): 483–505.

[2] Fintushel, Ronald; Stern, Ronald J. Rational blowdowns of smooth 4-manifolds. J. Differential Geom. 46 (1997), no. 2, 181–235.