Lie Groups and Representations Spring 2021

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Disclaimer

These notes were taken during lecture using the `vimtex` package of the editor `neovim`. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the instructor. If you find any errors, please contact me at `plei@math.columbia.edu`.

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Lie algebras and algebraic groups

If G is a compact Lie group, then $\mathfrak{g} = \mathsf{Lie}(G)$ has an invariant metric $(-,-)$. If $\mathfrak{g}_1 \subset \mathfrak{g}$ is an ideal, then $\mathfrak{g}_1^\perp$ is also an ideal and we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. In particular, we have

$$\mathfrak{g} = \bigoplus \mathfrak{g}_i \qquad \mathfrak{g}_i = \begin{cases} \mathbb{R} \\ \text{simple nonabelian Lie algebra} \end{cases}.$$

The simple nonabelian Lie algebras are very interesting and very special, and there are only countably many of them. Recall that they are classified by root systems.

1.1 Solvable and Nilpotent Lie Algebras

These are built out of abelian Lie algebras. They are not very interesting, but it is easy to find them. In some sense, if we consider the moduli space of Lie algebras, most Lie algebras will be nilpotent.

Definition 1.1.1. Define the *commutant* $\mathfrak{g}'$ of a Lie algebra to be the span of $[\mathfrak{g}, \mathfrak{g}]$. Here, we have an exact sequence

$$0 \to \mathfrak{g}' \to \mathfrak{g} \to \text{abelian} \to 0.$$

By analogy, we may define G' to be the commutator subgroup of a Lie group G.

Theorem 1.1.2. *If* G *is simply connected, then* G' *is a Lie subgroup.*

Example 1.1.3. The commutant of the group of all matrices of the form

$$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

is the set of all matrices of the form

$$\begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

Now if we take

$$G = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \times \mathbb{R}/\Lambda,$$

where Λ is a lattice in the center $\mathbb{R}^2$, then G' is the image of

$$\begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix},$$

which is not necessarily a Lie subgroup.

Proof. Consider the exact sequence $0 \to \mathfrak{g}' \to \mathfrak{g} \to \mathbb{R}^k \to 0$. Because G is simply connected, we can lift to an exact sequence

$$1 \to G'' \to G \to \mathbb{R}^k \to 0.$$

We know that $G' \subseteq G''$ but then $\mathsf{Lie}(G'') = \mathfrak{g}'$ and therefore we must have $G'' = G'$. $\square$

Definition 1.1.4. A Lie algebra is called *solvable* if $(\mathfrak{g}')' \cdots = 0$. In other words, repeatedly taking the commutant eventually reaches 0. Alternatively, one should think about $\mathfrak{g}$ as an iterated extension by abelian Lie algebras.

Similarly, a group G is called *solvable* if $(G')' \ldots = 1$.

Corollary 1.1.5. *A connected Lie group* G *is solvable if and only if* $\mathsf{Lie}(G)$ *is solvable.*

Example 1.1.6. The group $B \subset GL_n$ of upper-triangular matrices is solvable. In some sense, this is an universal example.

Note that if $G_1 \subset G$ and G is solvable, then G_1 is solvable. Conversely, if $G \hookrightarrow G_2$ and G is solvable, then so is G_2. Next, if

$$1 \to G_1 \to G \to G_2 \to 1$$

is an exact sequence and G_1, G_2 are solvable, then so is G.

A stronger condition than being solvable is being *nilpotent*.

Definition 1.1.7. A Lie algebra $\mathfrak{g}$ is *nilpotent* if $[[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}], \ldots] = 0$.

There is a similar definition for Lie groups, and we have

Corollary 1.1.8. *A connected Lie group is nilpotent if and only if* $\mathsf{Lie}(G)$ *is nilpotent.*

Example 1.1.9. The group of unitriangular matrices (equivalently the Lie algebra of strictly upper-triangular matrices) is nilpotent. Again, this is in some sense a universal example.

Theorem 1.1.10 (Lie)**.** *If* $\mathfrak{g}$ *is a solvable Lie algebra over* $\mathbb{C}$ *and* $\mathfrak{g} \to \mathfrak{gl}(V)$ *is a representation, then* $\mathfrak{g}$ *maps into the set of upper-triangular matrices in some basis.*

Remark 1.1.11. $\mathbb{C}$ or any algebraically closed field of characteristic 0 is important because we need to ensure that every operator actually has an eigenvalue and therefore can be upper-triangularized. Having characteristic 0 is also important because $\mathfrak{sl}_2(\mathbb{Z}/2\mathbb{Z})$ is solvable. Here, we have $[h, e] = [h, f] = 0$, and in particular, the defining representation cannot be upper-triangularized.

Proof. The usual proof is induction. We simply need to find one common eigenvector, and first we find an eigenvector in $\mathfrak{g}'$ and then extend to $\mathfrak{g}$. We will prove the result more globally. If G has a common eigenvector v_1, then the line $\mathbb{C}v_1 \in \mathbb{P}(V)$ is fixed by G. Therefore, if G is triangular in the basis $v_1, \ldots, v_n$, this is equivalent to fixing a flag $\mathbb{C}v_1 \subset \mathbb{C}v_1 + \mathbb{C}v_2 \subset \cdots \subset V$. Then by Borel-Morozov, a fixed flag exists because the flag variety is projective. $\square$

Theorem 1.1.12 (Borel-Morozov)**.** *If* G *is a connected solvable affine algebraic group over an arbitrary algebraically closed field acting on a proper variety* X*, then* X^G *is nonempty.*

If we apply this discussion to the adjoint representation, we see that over a field of characteristic 0, $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}'$ is nilpotent.

Theorem 1.1.13 (Engel)**.** *Suppose* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *consists of nilpotent operators. Then* $\mathfrak{g}$ *is contained in the set of strictly upper-triangular matrices for some basis.*

The usual proof of this is by induction, so we skip it.

Corollary 1.1.14. *If we apply this to the adjoint representation, then* $\mathfrak{g}$ *is nilpotent is nilpotent if and only if* $\operatorname{ad} x = [x, -]$ *is nilpotent for every* x*.*

This result has a global analog, due to Kolchin (who incidentally was once a professor at Columbia).

Theorem 1.1.15 (Kolchin)**.** *Let* $G \subset GL(V)$ *be any group consistent of unipotent operators. Then* G *is contained in the set of unitriangular matrices for some basis of* V*.*

Proof. By induction, it is enough to find one common fixed vector v_1. We will assume that V is irreducible. Then we consider $\operatorname{Span}(G) \subset \operatorname{End}(V)$. On $\operatorname{End}(V)$, we have a nondegenerate pairing $(a, b) = \operatorname{tr}(ab)$, and thus if we consider

$$\operatorname{tr}(g_1 - 1) \sum c_i g_i = \sum c_i \operatorname{tr}(g_1 g_i - g_i) = 0$$

we see that for all g, $g = 1$. Therefore $\dim V = 1$ and every element is fixed. □

Proof of Borel-Morozov. Consider the exact sequence $1 \to G' \to G \to \text{abelian} \to 1$. By induction on the dimension of G' we see that $X^{G'} \neq$ is closed and thus proper. Now rename $X = X^{G'}$, so we only need to prove the result for abelian G.

Any algebraic group action on an algebraic variety has a closed orbit $\mathcal{O}$, which in this case is proper. On the other hand, $\mathcal{O} = G/\text{stabilizer}$ is an affine algebraic group and therefore must be a point.

For another proof, every affine abelian group is built out of $\mathbb{G}_a, \mathbb{G}_m$, so we simply prove the result for these two groups. For $\mathbb{G}_m$, let $x \in X$. Then we simply consider the limit as $t \to 0$ of $t \cdot x$, which exists by the valuative criterion of properness. This must be a fixed point. For $\mathbb{G}_a = \mathbb{A}^1$, we run the same argument except we consider the limit at ∞. □

In the Lie theorem, $G \subset GL(n, \mathbb{C})$ is an arbitrary connected solvable Lie group. We need to see that the Zariski closure of G is still solvable. If we write $\overline{\mathfrak{g}} = \mathsf{Lie}(\overline{G})$, then we have

Lemma 1.1.16. $[\overline{\mathfrak{g}}, \overline{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}]$.

Proof. We show that $[\overline{\mathfrak{g}}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}]$. Consider

$$\widetilde{G} = \left\{ h \mid \operatorname{Ad} h(\mathfrak{g}) = \mathfrak{g}, \operatorname{Ad} h\Big|_{\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]} = 1 \right\}.$$

In particular, we have $G \subset \overline{G} \subset \widetilde{G}$ because $\widetilde{G}$ is closed, and therefore $[\overline{\mathfrak{g}}, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]$.

Now consider the same construction but with $\mathfrak{g}$ replaced by $\overline{\mathfrak{g}}$. This implies that $[\overline{\mathfrak{g}}, \overline{\mathfrak{g}}] \subset [\mathfrak{g}, \mathfrak{g}]$. □

Corollary 1.1.17 (Borel)**.** *All maximal connected solvable subgroups* $B \subset G$ *are conjugate for any connected linear algebraic group* G*.*

Proof. The idea is that if $B_0 \subseteq G$ is connected solvable of maximal dimension, then $X = G/B_0$ is projective. Then any other B will have a fixed point $gB_0 \in X$, and so $g^{-1}Bg$ fixes B_0. This implies that $g^{-1}Bg \subset B_0$ and so they must be equal (by maximality).

Really, we will prove that $G \subseteq GL(n, k)$. Therefore we will consider the action of G on the flag variety $\mathrm{Fl}(n)$, and the stabilizer of any point is solvable. Then stabilizers of maximal dimension correspond to orbits of smallest dimension, which are closed and thus projective. Choose some maximal B_0, which stabilizes a point in a closed orbit. Then B_0 is solvable and $X = G/B_0$ is projective, so the argument above works. □

Now consider the variety $X = G/B$. This is called the *flag variety* for G.

Example 1.1.18.

1. If $G = GL(n, k)$, then G/B is the usual flag variety. Here, B is all upper-triangular matrices.

2. Let G be one of the classical groups. Suppose g preserves a flag $0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = k^n$. Then g also preserves $V_i^\perp$ and intersections of the V_i and their orthogonal complements, so we impose $V_i^\perp = V_{n-i}$. Thus we take the space of flags to be

$$X = \left\{0 \subset V_1 \subset \cdots \subset V_n = k^n \mid V_i^\perp = V_{n-i}\right\}.$$

We need to check that G acts on X transitively, so we check it up to $V_{\lfloor n/2 \rfloor}$, which is a maximal isotropic subspace.

Theorem 1.1.19. *For all* $v_1, \ldots, v_m \in k^n$*, invariants of* $G = SO$ *or* $G = Sp$ *are generated by* (v_i, v_j) *and minors like* $v_{i_1} \wedge \cdots \wedge v_{i_m}$*. But all of these vanish because the* v_i *are all orthogonal, so there are no invariants.*

Definition 1.1.20. A *linear algebraic group* G is an affine algebraic variety over k which is also a group.

Theorem 1.1.21 (Chevalley)**.** *Over any field of characteristic* 0*, any group scheme is reduced and hence smooth.*

Example 1.1.22. Consider the group $\mathbb{A}^1 = \mathbb{G}_a$, the additive group of k. Then $k[G] = k[t]$, and so the addition map $(t_1, t_2) \mapsto t_1 + t_2$ corresponds to the map $f(t) \mapsto f(t_1 + t_2)$.

If $\operatorname{char} k = p$, then $t \mapsto t^p$ is a group homomorphism. This gives us an exact sequence

$$0 \to \operatorname{Spec} k[t]/t^p \to \mathbb{G}_a \to \mathbb{G}_a \to 0.$$

Here, the first term is an affine group scheme because $\Delta t^p = t^p \otimes 1 + 1 \otimes t^p$ and therefore $k[t]/t^p$ has a well-defined coproduct.

Therefore in characteristic 0, we can simply consider algebraic varieties. Then G is smooth, and we note that the maps $m \colon G \times G \to G, i \colon G \to G, 1 \to G$ induce maps $\Delta \colon A \to A \otimes A$ (comultiplication), $S \colon A \to A$ (antipode), and $\varepsilon \colon A \to k$ (counit). Here, the comultiplication is required to be *coassociative*, and the antipode is required to satisfy the identity

$$\mu \circ (1 \otimes S) \circ \Delta = \iota \circ \varepsilon.$$

In other words, the diagram

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\varepsilon"] & A \otimes A \arrow[r, "1\otimes S"] & A \otimes A \arrow[d, "\mu"] \\ k \arrow[rr, "\iota"] & & A \end{tikzcd}$$

commutes.

Now A has two sets of tensors:

1. As a commutative algebra over k, it has $\mu\colon A \otimes A \to A$, which is dual to $\Delta\colon G \to G \times G$ and the unit $\iota\colon k \to A$. In principle, the multiplication does not need to be commutative.

2. As functions on a group, it gets $\Delta\colon A \to A \otimes A, \varepsilon\colon A \to k, S\colon A \to A$. Note that the comultiplication may not be cocommutative.

Definition 1.1.23. A *Hopf algebra* A over a field k is a bialgebra over k such that the axioms listed above for $k[G]$ are satisfied.

Note that there is no need for A to be commutative and that the set of axioms is symmetric. Therefore we can consider the dual of a Hopf algebra, where all vector spaces are replaced with their duals and all maps are replaced by the dual maps. Now we see that linear algebraic group schemes over k are equivalent to finitely generated commutative Hopf algebras over k.

Now let $f \in A$. We see that $(\Delta f)(g, h) = f(gh) = \sum c_i(g)f_i(h)$, so if $g = 1$, then f is in the span of the f_i. Also,

$$f(g_1g_2h) = \sum c_i(g_1g_2)f_i(h) = \sum c_i(g_1)f_i(g_2h)$$

so every $f \in A$ belongs to a finite-dimensional subspace that is invariant under the left regular representation. This implies that every affine algebraic group G is contained in $GL(N, k)$

Now note that if $G \xrightarrow{\varphi} H$ is a homomorphism of algebraic groups, then $\operatorname{Im}\varphi \subset H$ is closed.

Theorem 1.1.24. *For all subgroups $H \subseteq G$, there exists a morphism $G \to GL(V)$ such that $\operatorname{Im}(H)$ is contained in the stabilizer of a line.*

Proof. Let I_H be the ideal of H in $k[G]$. Then H is the stabilizer of $I_H \subset k[G]$ under the natural G-action. Here, note that $L_{h^{-1}}f(g) = f(gh)$, so if we set $g = 1$, then $f(h) = 0$ for all $h \in \operatorname{stab}(I_H)$.

Note that a tangent vector to an algebraic variety is an map in $\operatorname{Hom}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, X)$ that sends the closed point of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to $x \in X$. Therefore, we have

$$\mathsf{Lie}(G) = \left\{1 + \varepsilon\xi, \in G, \varepsilon^2 = 0\right\}.$$

Next, from last time, we know that $\dim \operatorname{Span}\{f(g^{-1})\} < \infty$, s $I_H = (f_1, \dots, f_k)$, where $f_i \in L$, a finite-dimensional G-invariant subspace. Let $L_H = I_H \cap L$. Then H is the stabilizer of L_H, so H stabilizes a point in $G(\dim L_H, \dim L)$. Now we note that $\bigwedge^k L_H \subseteq \bigwedge^k L$ is a line, as desired. □

Definition 1.1.25. We define G/H to be the orbit of the line that is stabilized in $\mathbb{P}(V)$.

Note that this definition is not necessarily independent of V and we also need to know what properties it satisfies. From now, we will assume H is smooth and $\dim \mathsf{Lie}(H) = \dim H$. Then we have an exact sequence

$$0 \to \mathsf{Lie}(H) \to \mathsf{Lie}(G) \to T_H G/H \to 0,$$

and therefore $G \to$ Orbit is *separable*.

Theorem 1.1.26. *If $X \to Y$ is dominant, separable, and generically one-to-one, then it is birational.*

Proposition 1.1.27. *Let $x \in P(V)$ be as above and let $y \in Y$, where Y is any variety with a G-action such that $H \subset G_y$. Then there exists a unique G-invariant map $G \cdot x \to G \cdot y$ such that $x \mapsto y$.*

Proof. Consider the map $g \mapsto (g \cdot x, g \cdot y)$ that sends $G \to G \cdot x \times G \cdot y$. Then the map $p_1 \colon G \cdot x \times G \cdot y \to G \cdot x$ must be separable. But then $G_y \subset G_x = H$ implies p_1 is one-to-one. This implies that p_1 is birational when restricted to the image of $g \mapsto (g \cdot x, g \cdot y)$. But this means that p_1 is an isomorphism, so we can take $p_2 \circ p_1^{-1}$ as the required map. □

Now we will study what the space G/H looks like. In the case where G is connected and B is a maximal solvable groups, then the flag variety G/B is projective. The space G/H could also be an affine variety.

Definition 1.1.28. A group G is reductive if for any $G \subset GL(V)$, we can write $V = \bigoplus V_i$, where the V_i are irreducible G-modules.

Remark 1.1.29. Over $\mathbb{C}$, this definition is equivalent to being the complexificaiton of a compact group.

Theorem 1.1.30 (Matsushita-Onishchik)**.** *If G is reductive, then G/H is affine if and only if H is reductive.*

Note that for most H, G/H is neither projective nor affine. For example, if we consider $GL(2)$ and let

$$H = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix},$$

then G/H is the orbit of a single vector, which is $\mathbb{A}^2 \setminus 0$. However, G/H is always quasiprojective, so it can be embedded in projective space.

Proposition 1.1.31. *The space G/H is quasiaffine if and only if H is the stabilizer of a point in an affine algebraic variety with a G-action. Such subgroups are called observable.*

Proposition 1.1.32. *Ths space G/H is projective if and only if $B \subset H$. Such subgroups are called parabolic.*

Proof. First, if G/H is projective, then $(G/H)^B \neq \emptyset$ and thus B is conjugate to a subgroup of H. On the other hand, □

1.2 Invariant Theory

Now note that if G acts on X, then $G \times X \to X$ is a morphism of algebraic varieties. Now we want to study the space X/G. We can consider this in some world more general than algebraic varieties (namely stacks), but this is beyond the scope of this course. Instead, we will consider the best possible approximation in the category of schemes. Here, we will consider $Y = X/G$ if for all Z (with the trivial action), G-equivariant maps $X \to Z$ factor uniquely through Y.

Our goal is to show that the GIT quotient X/G exists if X is affine and G is reductive.

Example 1.2.1. Consider the action of $GL(V)$ on $V, V \otimes V, V^*, V^* \otimes V, \ldots$. Then the first (second?) fundamental theorem of invariant theory says that all invariants of these actions come from contracting tensors. For example, if we consider $V^* \otimes V = \mathrm{Hom}(V, V)$, the invariants are generated by the coefficients of the characteristic polynomial. This means that $\mathrm{Hom}(V, V)/GL(V) = \mathbb{A}^{\dim V}$.

Remark 1.2.2. Note there are several notions of being reductive. The first is structural. The second is being linearly reductive, which means that we need something like $k \to k[G] \to k$, where the last map is some sort of invariant integration. Finally, there is the notion of being geometrically reductive. If k has characteristic 0, then all of these notions are equivalent.

Lemma 1.2.3. *Suppose we can split every module with an invariant element as $k \to M \to k$. Then all representations are linearly reductive.*

Proof. Let $M_1 \subset M$ be some submodule. We want a G-invariant map $M \to M_1$, which requires a G-equivariant map $\mathrm{Hom}(M, M_1) \to \mathrm{Hom}(M_1, M_1)$ that maps onto 1_{M_1}. But this problem is resolved by taking the transpose of a matrix acting on M that preserves M_1. □

Note that $GL(V)$ is not linearly reductive if $\operatorname{char} k = p$. In this case, consider the action of $GL(2)$ on the polynomials of degree d in x_1, x_2. Then the span of x_1^p, x_2^p does not split off.

Definition 1.2.4. A group G is *reductive* if the radical of G is a torus. Equivalently, the unipotent radical of G is trivial. Here, the radical of G is defined to be

$$\left(\bigcap_g gBg^{-1}\right)_0$$

and is the largest normal connected solvable subgroup. The unipotent radical is defined to be the largest normal unipotent connected subgroup, and is

$$\left(\bigcap_g gUg^{-1}\right)_0 \qquad U = \left\{\begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix}\right\}.$$

Definition 1.2.5. A group G is *geometrically reductive* if for any G-module M and line of G-fixed points, there exists a complementary divisor given by a G-invariant polynomial $f(m)$ that does not vanish on the line.

Example 1.2.6. Finite groups fail to be linearly reductive in positive characteristic. For example, the representation of $\mathbb{Z}/p\mathbb{Z}$ given by

$$\mathbb{Z}/p\mathbb{Z} \ni m \mapsto \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$$

is not completely reducible. On the other hand, they are geometrically reductive.

Proof. Take any f_0 such that $f_0(0) = 0$ and $f_0(x) = 1$, where x is a fixed point. Then take

$$f(m) = \prod_g f(g^{-1}m) \qquad f(x) = 1, f(0) = 0.$$

Now we can choose the Taylor series of $f(x)$ to be homogeneous of degree p. □

Theorem 1.2.7 (Haboush)**.** *Group-theoretic reductivity is equivalent to geometric reductivity.*

Corollary 1.2.8. *Let A be an algebra with a G-action and suppose $A \twoheadrightarrow B$, which also has a G-action. Then linear reductivity means the natural map $A^G \twoheadrightarrow B^G$ is surjective. Geometric reductivity means that for all $f \in B^G$, there exists $m = p^k$ such that $b^m \in \mathrm{Im}(A^G)$. In particular, B^G is integral over A^G.*

Theorem 1.2.9 (Nagata, Popov,...). *A group G is reductive if and only if for all finitely generated (commutative) algebras A, the algebra A^G of invariants is finitely generated.*

This result is extremely hard. Instead, we will prove

Theorem 1.2.10 (Hilbert). *Let X be an affine variety over a field k of characteristic 0 and G a reductive group. Then $k[X]^G$ is finitely generated.*

Proof. Let $X \subseteq V$ and $G \hookrightarrow GL(V)$. Now $k[V]^G \twoheadrightarrow k[X]^G$ by linear reductivity. Consider the ideal $I = (k[V]^G_+)$. This is finitely generated by another theorem of Hilbert (from the same paper). If $f_1, \dots, f_k$ are generators, then we will show that they also generate $k[V]^G$.

Let $F \in k[V]^G_d$ for some $d > 0$. Then $F \in I$ and we can write $F = \sum c_i f_i$. Now we will take the average over G, which is linear over invariants. Now we obtain $F = \sum \overline{c}_i f_i$, where the $\overline{c}_i$ are all invariants of degree less than d. By induction on d, we are done. $\square$

For the proof in arbitrary characteristic, there is a book on invariant theory by T. Springer.

Now consider a map $X \xrightarrow[\pi]{(f_1,\dots,f_k)} \mathbb{A}^k$, where X is an affine variety with an action of a reductive group G. Then we will show that

1. The map π takes G-invariant $X' \subseteq X$ to closed subsets.
2. If X', X'' are disjoint G-invariant closed subsets, then $\pi(X') \cap \pi(X'') = \emptyset$.
3. For any open $U \subseteq \pi(X)$, $\pi^* \mathcal{O}_U = \mathcal{O}^G_{\pi^{-1}(U)}$.

In particular, we will show that if G is reductive and $X', X'' \subseteq X$ are closed and disjoint, then there exists $f \in k[X]^G$ such that $f(X') = 0, f(X'') = 1$. To see this, we know that $I_{X'} + I_{X''} = k[X]$, so we can find $f_0 \in I_{X'}, f_1 \in I_{X''}$ such that $f_0 + f_1 = 1$. Thus $f_0(X') = 0, f_0(X'') = 1$. Then if $f_0, \dots, f_m$ span $f_0(g^{-1}-)$, then the map $X \xrightarrow{(f_0,\dots,f_m)} \mathbb{A}^{m+1}$ sends X' to $(0, \dots, 0)$ and X'' to $(1, \dots, 1)$.By geometric reductivity, there exists a polynomial $P(f_0, \dots, f_m)$ which is invariant and takes values 0 on X' and 1 on X''.

Note that if G is not reductive, then closed subsets cannot be separated by invariants. For an example, consider the action of $\mathbb{G}_a$ on $\mathbb{A}^2$ by translating the second coordinate. Then $(x, 0), (y, 0)$ cannot be separated by invariants.

Now we need to show that $\pi(X)$ is closed. If not, then if $p \in \overline{\pi(X')} \setminus \pi(X')$, then $\pi^{-1}(p)$ is closed and disjoint from X'. But this implies there exists f such that $f(X'') = f(p) = 1$ and $f(X') = f(\pi(X')) = 0$, a contradiction.

Finally, let $U \subseteq \pi(X)$ be given by $\{F_1 \neq 0, F_2 \neq 0\}$. Then

$$\mathcal{O}_U = \mathbb{k}[f_1, \dots, f_k][1/F_i] = \mathbb{k}[X]^G[1/F_i] = (\mathbb{k}[X][1/F_i])^G = \mathcal{O}^G_{\pi^{-1}(U)}.$$

This all implies that $\pi(X) = X/G$ is the categorical quotient of X under the action of G. To see this, observe that if U_i is an affine open cover of Z, then $p^{-1}(U_i)$ cover X, so $X_i = X \setminus p^{-1}(U_i)$ is closed and $\bigcap X_i = \emptyset$. Now let $V_i = Y \setminus \pi(X_i)$. These form an open cover of Y, so now write $\overline{p} \colon V_i \to U_i$. Then we have

$$\mathcal{O}_{U_i} \xrightarrow{p^*} \mathcal{O}^G_{X \setminus X_i} \hookrightarrow \mathcal{O}^G_{\pi^{-1}(V_i)} = \pi^* \mathcal{O}_{V_i} \subset \mathcal{O}_{\pi^{-1}(V_i)},$$

and this must be unique, so $\pi^* \overline{p}^* = p^*$, so $\overline{p} \colon V_i \to U_i$.

Therefore we have proved that if X is affine and G is reductive, then $Y = \operatorname{Spec} \mathbb{k}[X]^G$ is the categorical quotient. Note that this is surjective, and for $p \in Y$, $\pi^{-1}(p)$ is nonempty and contains a unique closed orbit.

Now we will discuss quotients of general varieties by algebraic groups. This is very complicated because $x \in X$ may not have a G-invariant affine neighborhood (consider the example of Hironaka). Now if we consider $X \subset \mathbb{P}(V)$ for a G-module V with $V^* = \mathcal{O}(1)$, then $\mathcal{O}(1)$ is a very ample line bundle on X with a linearzation by G. Similarly to $Y = \operatorname{Spec} \mathbb{k}[X]^G$, we may consider the affine cone $\widehat{X}$ over X and then take $Y = \operatorname{Proj} \mathbb{k}[\widehat{X}]^G$. This is covered by open sets where $\{F_i(x) \neq 0\}$, and then $\mathbb{P}(V) \setminus \{F(x) = 0\}$ is an affine G-invariant set.

Not all points have an invariant polynomial F_i such that $F_i \neq 0$. The points that do are called *semistable*.

Definition 1.2.11. The *GIT quotient* $X /\!\!/ G$ is defined to be $\operatorname{Proj} \mathbb{k}[\widehat{X}]^G = X^s/G$, where X^s is the stable locus.

The unstable points are those such that there is no invariable F_d such that $F_d(x) \neq 0$. But this implies that the closure of the orbit of x in V contains $0 \in V$.

Note that if $\chi \colon G \to \mathbb{G}_m$ is a character, then $V \mapsto V \otimes \chi$ does not change the action on $\mathbb{P}(V)$ because $S^d V^* \mapsto S^d V^* \otimes \chi^{-d}$ sends χ-covariants to invariants. Therefore, even in the affine situation, it makes sense to consider $X /\!\!/ G = \operatorname{Proj}$ covariants. For the most basic example, consider $\mathbb{P}(V) = \operatorname{Proj} \bigoplus S^d V^*$. Then $V/\mathbb{G}_m$ is a point, and $V /\!\!/ \mathbb{G}_m = \mathbb{P}(V)$. On the other hand, we have $V /\!\!/_{\chi = t} \mathbb{G}_m = \operatorname{Proj} \mathbb{C} = \emptyset$, so in both cases the map $X /\!\!/ G \to X/G$ is uninteresting.

Now we want to find generators of the algebra $\mathbb{k}[X]^G = \mathbb{k}[f_1, \dots, f_N]$. Then the affine scheme X/G is cut out by the relations among the f_i. Finding the relations is incredibly hard, so we can try to find the generators. Results of this form go under the form of the *first fundamental theorem of invariant theory*. Here, we will assume $G = GL(n), SL(n)$. These fit into the exact sequence

$$1 \to SL(V) \to GL(V) \xrightarrow{\det} GL(1) \to 1.$$

Therefore $SL(V)$-invariants are the same as $GL(V)$-covariants with respect to the determinant character. We know that $\mathbb{k}[X]$ contains a finite-dimensional G-invariant module M, which can be included in $\mathbb{k}[G]^{\oplus m}$. This implies that any X can be emdedded in some $V^{\oplus m_1} \oplus (V^*)^{\oplus m_2} =: M_{m_1,m_2}$ because there is a natural map $\mathbb{k}[\operatorname{End}(V)] \to \mathbb{k}[G]$. This gives us a map $\mathbb{k}[M_{m_1,m_2}] \to \mathbb{k}[X]$ that restricts to invariants, so we have reduced the problem of finding invariants to vector spaces.

Theorem 1.2.12 (First fundamental theorem of invariant theory)**.** *The invariants of $SL(V)$ acting on $V^{\oplus m_1} \oplus (V^*)^{\oplus m_2}$ are generated by*

1. *Contracting tensors: $(v_1, \dots, v_{m_1}, \ell_1, \dots, \ell_{m_2}) \mapsto \langle v_i, \ell_j \rangle$;*
2. *Determinants of the form $\det \begin{pmatrix} v_{i_1} & \dots & v_{i_n} \end{pmatrix}$ with weight $\det$ and dually for the ℓ_j with weight $\det^{-1}$ (weights are under the action of GL).*

Proof. Note that $M_{m_1,m_2} = \operatorname{Hom}(\mathbb{k}^{m_1}, V) \oplus \operatorname{Hom}(V, \mathbb{k}^{m_2})$. Now the two parts parts have actions by the groups $GL(m_1), GL(m_2)$ and maximal tori T_{m_1}, T_{m_2}. THen the weights record how many times we use a particular vector or covector. Now it suffices to consider functions of weight $(1^\ell, 0^k)$.

To see this, we use a polarization trick. If $\deg_{v_i} f(v_1, \dots) = d$, then we can write $v_i = \sum \lambda_i u_i$ and now we have a function of $m_1 + d - 1$ vectors $u_1, \dots, u_d, v_2, \dots, v_{m_1}$. Expanding this, we obtain a new polynomial $\widetilde{f}$ that is linear in each of the $u_1, \dots, u_d$. Then considering the polynomial $\widetilde{f}(v_1, \dots, v_1, v_2, \dots, v_m)$ gives us the desired reduction.

But now functions on M_{m_1,m_2} linear in each of the $v_1,\dots,v_m,\ell_1,\dots,\ell_{m_2}$ are just the space $(V^*)^{\otimes m_1}\otimes V^{\otimes m_2}$. We will show that

$$((V^*)^{\otimes m_1}\otimes V^{\otimes m_2})^{GL(V)}=\begin{cases}0 & m_1\neq m_2\\ \operatorname{Span}\left\{\prod_{i=1}^m\left\langle v_i,\ell_{\sigma(i)}\right\rangle\right\}_{\sigma\in S_m} & m_1=m_2=m.\end{cases}$$

The scalars $t\cdot I$ act with weights $t^{-m_1+m_2}$ so there are no invariants unless $m_1=m_2$. Now if $m_1=m_2$, we are looking for

$$\operatorname{Hom}(V^{\otimes m},V^{\otimes m})^{GL(V)}\cong \Bbbk S_m.$$

This result is known as *Schur-Weyl duality*. If we consider the natural map $GL(V)\times S_m\to \operatorname{End}(V^{\otimes m})$. In fact, each piece of the product generates the commutant of the other. We see that both images are semisimple subalgebras in $\operatorname{End}(V^{\otimes m})$. Now the desired result is equivalent to proving that $\operatorname{End}(V^{\otimes m})^{S_m}$ is the image of $GL(V)$. Then polynomials on $\operatorname{End}(V)$ of degree m are the same as $S^m\operatorname{End}(V)^*=\operatorname{End}(V^{\otimes m})^{S_m}$. Suppose that $\operatorname{End}(V^{\otimes m})^{S_m}\supsetneq GL(V)$. But then we can consider $GL(V)^\perp$ in the set of polynomials of degree m. Let P be such a polynomial. Then $P(g,\dots,g)=0$ for all $g\in GL(V)$. But then by Zariski density of $GL(V)$ in $\operatorname{End}(V)$, we see that $P=0$. This now tells us that

$$\Bbbk[V^{\oplus m_1}\otimes (V^*)^{\oplus m_2}]^{GL(V)}=\Bbbk[\langle v_i,\ell_j\rangle].$$

Now we need to compute the additional $SL(V)$-invariants, which are given by

$$\left(\Bbbk[V^{\oplus m_1}\oplus (V^*)^{\oplus m_2}]\otimes \det{}^{-1}\right)^{GL(V)}=\Bbbk[\langle v_i,\ell_j\rangle]\otimes \operatorname{Span}\det\begin{pmatrix}v_{i_1} & \cdots & v_{i_n}\end{pmatrix}.$$

We will introduce new covectors $\bar\ell_1,\dots,\bar\ell_n$ and consider the functions

$$f\cdot\det\begin{pmatrix}\bar\ell_1\\ \vdots\\ \bar\ell_n\end{pmatrix},$$

which is an invariant and thus contained in $\Bbbk[\langle v_i,\ell_j\rangle,\langle v_i,\bar\ell_j\rangle]$. Now det is multilinear and skew-symmetric, so each $\bar\ell_j$ has to be used exactly once. But now f is a product f_1,f_2 where $f_1\in\Bbbk[\langle v_i,\ell_j\rangle]$ and f_2 is contained in the antisymmetrization of $\prod\langle v_{i_k},\bar\ell_k\rangle$, so $f_2=\det(v_{ik})\cdot\det(\bar\ell_k)$. $\square$

1.2.1 Finite Subgroups of $SL(2,\mathbb{C})$

Now consider a finite group $G\subset SL(2,\mathbb{C})$. For example, G is cyclic, dihedral, etc. Now we have an exact sequence

$$1\to\{\pm1\}\to SL(2)\to SO(3)\to 1.$$

and now we can find in $SO(3)$ symmetries of the Platonic solids A_4,D_4,A_5 corresponding to tetrahedron, cube, and dodecahedron. Now if $\gamma\in SO(3)$ has order 3 with eigenvalues $1,\zeta_3,\zeta_3^2$, then we know $\tilde\gamma\in SL(2)$ has eigenvalues ζ_6,ζ_6^{-1}. Now for $G\in\tilde A_4,\tilde S_4,\tilde A_5$ and $V=\mathbb{C}^2$, we know that $SL(2)=Sp(2)$ preserves the skew pairing. We know

$$V/G=\operatorname{Spec}(S^*V^*)^G,$$

so now consider the Hilbert/Poincaré series

$$H(t) = \sum_d t^d \dim \left(S^d V\right)^G.$$

By an observation of Hilbert, this is a rational function for any finitely generated graded module over a finitely generated algebra. But now we know that $\mathbb{C}[a_1, \dots, a_m] \twoheadrightarrow A$. If a_i has degree d_i, then the free module has Hilbert series

$$H_{\text{free}}(t) = \frac{1}{\prod_i 1 - t^{d_i}}.$$

In general, a module M has a finite gree resolution

$$\cdots \to \bigoplus A_i r_i \to \bigoplus A \cdot m_i \to M \to 0.$$

This gives us

$$H_M(t) = \frac{\sum t^{m_i} - \sum t^{r_i} + \cdots}{\prod (1 - t^{d_i})}.$$

Theorem 1.2.13 (Molien)**.** *Let* G *be a reductive group over* $\mathbb{C}$ *acting on a vector space* V*. Then*

$$H_{(S^\bullet V)^G}(t) = \int_{\textit{maximal compact}} d_{\text{Haar}} g \frac{1}{\det_V(1 - tg)} \qquad , |t| < \varepsilon.$$

To do the actual computation, we can use the Weyl character formula. This is simply

$$H_{S^\bullet V}(t) = \frac{1}{|W|} \int_T d_{\text{Haar}}(s) \frac{\prod_{\alpha \neq 0}(1 - s^\alpha)}{\prod_{\text{weights } \mu}(1 - ts^\mu)},$$

and this can be computed using residues. Of course, if G is finite, then we just sum over conjugacy classes. For example, if $G = A_4$, then these are cycles of signature either $(3, 1)$ or $(2, 2)$, and therefore

$$\widetilde{A}_4 = \{\pm 1, \pm i, \pm j, \pm k\} \cup \left\{ \frac{1}{2}(\pm 1 \pm i \pm j \pm j) \right\}$$

is a group of order 24. Now the conjugacy classes are given by

$$\begin{aligned}
1 &\longrightarrow \frac{1}{(1-t)^2} \\
-1 &\longrightarrow \frac{1}{(1+t)^2} \\
i &\longrightarrow \frac{1}{(1+it)(1-it)} = \frac{1}{1+t^2} \\
\zeta_3 &\longrightarrow \frac{1}{(1-\zeta_3 t)(1-\zeta_3^{-1} t)} = \frac{1}{1+t+t^2} \\
\zeta_6 &\longrightarrow \frac{1}{1-t+t^2}.
\end{aligned}$$

Remark 1.2.14. Andrei admires the mathematicians of the past who were able to compute things by hand. Now he cannot imagine performing these computations without a computer. It is important to note that we should always use a free and open-source program to perform such computations and not something proprietary like some programs that shall not be named.[1,2]

This tells us that

$$H(t) = \frac{(1-t^{24})}{(1-t^6)(1-t^8)(1-t^{12})} = \frac{(1+t^{12})}{(1-t^6)(1-t^8)}.$$

This suggests generators of degree $6, 8, 12$ and a relation in degree 24.

Theorem 1.2.15 (E. Noether)**.** *The ring* $(S^\bullet V)^G$ *is generated in degree at most* $|G|$.

Proof. By polarization, we know $S^d V$ is spanned by v^d. But then we know that $(S^d V)^G$ is spanned by polynomials of the form

$$\sum g \cdot v^d = \sum (gv)^d = p_d(\underbrace{v, g_1 v, g_2 v, \ldots}_{|G|}),$$

and this can be expressed in elementary symmetric functions of degree at most $|G|$. □

Now we can rewrite

$$H(t) = 1 + t^6 + t^8 + 2t^{12} + t^{14} + t^{16} + 2t^{18} + \cdots$$

Let x, y, z be the generators of degree $6, 8, 12$, and then in degree 24, we have some relation

$$Ax^4 + By^3 + Cz^2 = 0.$$

There are no further relations because $\dim V/\widetilde{A_4} = 2$, so we have a map

$$V \xrightarrow{(x,y,z)} V/G \subset \mathbb{C}^3.$$

Remark 1.2.16. This classification of finite subgroups of $SL(2)$ also gives us Du Val singularities, the classification of simple Lie algebras, the McKay correspondence, and many other interesting objects in mathematics.

Now consider the action of $SL(2, \mathbb{Z})$ on the upper half-plane $\mathcal{H}$. Then we have an exact sequence

$$1 \to \Gamma(m) \to SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/m) \to 1$$

and we have an action of $SL(2, \mathbb{Z}/m)$ on $\mathcal{H}/\Gamma(m)$.

But now $\Gamma(m)$ has no torsion, so we have finitely many cusps, corresponding to the action of $\Gamma(m)$ on $\mathbb{Q}$, and $\mathcal{H}/\Gamma(m)$ is a curve of genus $g = g(\Gamma(m))$. For $m = 3, 4, 5$, we have $g = 0$ and thus $SL(2, \mathbb{Z}/m)$ acts on $\mathbb{P}^1$.

Now the tetrahedron corresponds to the standard fundamental domain for $SL(2, \mathbb{Z})$. The cube corresponds to the below:

[1] See `https://www.gnu.org/proprietary/proprietary.en.html` or `https://www.gnu.org/philosophy/why-free.en.html`

[2] Andrei says to use free software but himself uses Windows and Microsoft OneNote.

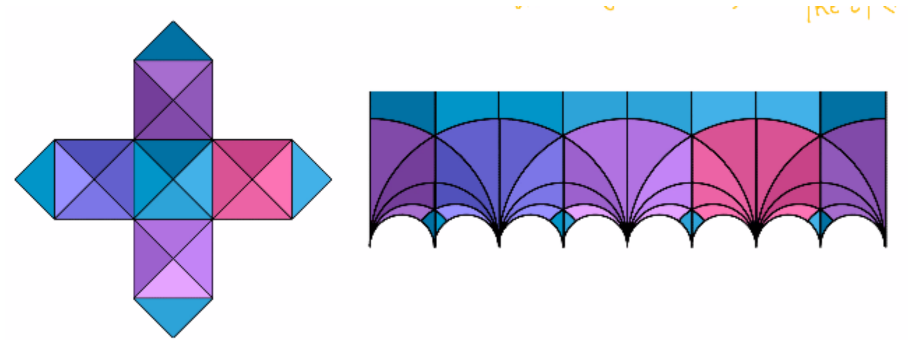

Figure 1.1: Fundamental domain subdivided

The dodecahedron and icosahedron correspond to the following:

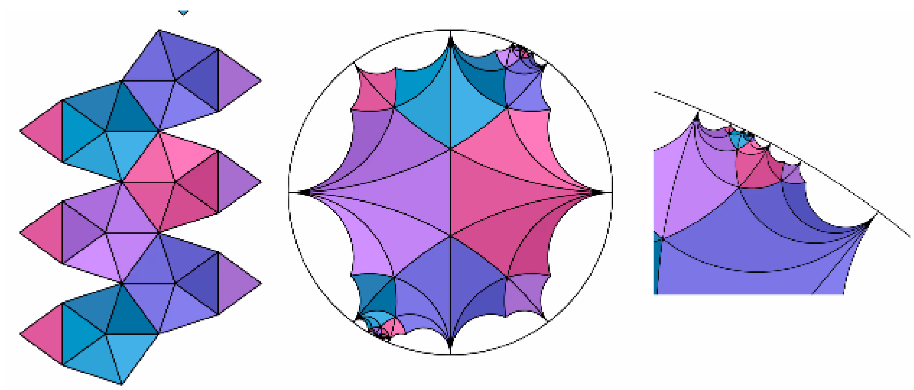

Figure 1.2: Fundamental domain for icosahedron

The cusps correspond to points of 5-fold symmetry, and correspond to $0, 1, 2, \infty, \varphi, 1/\varphi, \ldots$, and the points converging to $1/\varphi$ are given by ratios F_n/F_{n-1} of Fibonacci numbers.

Remark 1.2.17. Instead of just considering the icosahedron, we should consider an infinite strip with the given pattern in the picture.

1.3 Jordan Decomposition

Let $\Bbbk = \overline{\Bbbk}$. Then for all $g \in G = GL(V)$, we can decompose g into Jordan blocks. In particular, we can write $g = g_s g_n$, where g_s is semisimple and g_n is strictly upper triangular (in some basis). The analogous decomposition for $\xi \in \mathfrak{g}$ is $\xi = \xi_s + \xi_n$. Now if $V \subseteq \Bbbk^n$ is invariant under g, it is invariant under both g_s, g_n.

Consider the regular representation ρ of $G = GL(n, \Bbbk)$ on $\Bbbk[G]$.

Lemma 1.3.1. *For any $g \in G$, we have $\rho(g)_s = \rho(g_s)$.*

Corollary 1.3.2. *If G is an algebraic group, then $g_s, g_n \in G$ for all $g \in G$. Moreover, if $\varphi\colon G_1 \to G_2$ is a homomorphism, then $\varphi(g)_s = \varphi(g_s)$ and $\varphi(g)_n = \varphi(g_n)$.*

Proof. By Chevalley, G is the stabilizer of a subspace on $\Bbbk[GL(n)]$ and therefore because g stabilizes the subspace, so do g_s, g_n.

For the second part, we can pull back $\varphi^*\colon \Bbbk[G_2] \to \Bbbk[G_1]$ and then the desired result is obvious. $\square$

1.4 More Solvable Lie Algebras

We will return to solvable Lie algebras. Assume $\operatorname{char} \Bbbk = 0$.

Theorem 1.4.1. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra. Then $\mathfrak{g}$ is solvable if and only if $\operatorname{tr} x[y,z] = 0$ for all $x, y, z \in \mathfrak{g}$.*

Proof. One direction is clear by Borel. Here, $\mathfrak{g} \subset \mathfrak{b}$ is contained in the subalgebra of upper-triangular matrices. In the other direction, the form $\operatorname{tr} x[y,z] \in (\Omega^3\mathfrak{g})^{\mathfrak{g}}$. In particular, we see that if $\mathfrak{g} = \mathsf{Lie}(G)$, then this becomes a bi-invariant 3-form on G. This gives a class in $H^3_{dR}(G)^{G\times G}$. Now we have an exact sequence

$$1 \to \text{unipotent radical} \to G \to G_{\text{reductive}} \to 1,$$

and here the unipotent radical is homeomorphic to $\mathbb{R}^n$, while $G_{\text{reductive}}$ is a product of simple nonabelian G_i and a torus up to a finite cover. Then $\operatorname{rk}\pi_3$ is the number of simple nonabelians, and so the morphisms $SU(2) \simeq S^3 \hookrightarrow (G_i)_{\text{compact}}$ generate $\pi_3 \otimes \mathbb{Q}$.

Now suppose that $\operatorname{tr} x[y,z] = 0$. Now we will show that $\mathfrak{g}$ is solvable. It suffices to show that $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ is nilpotent. By Engel, it suffices to show that any $x \in \mathfrak{g}'$ is nilpotent. Consider the subalgebra

$$\mathfrak{gl}(V) \supset \widetilde{\mathfrak{g}} = \{\xi \mid [\xi,\mathfrak{g}] \subset [\mathfrak{g},\mathfrak{g}]\} \supset \mathfrak{g}.$$

This is the Lie algebra of an algebraic group

$$\widetilde{G} = \{h \mid \operatorname{Ad}(h)(\mathfrak{g}) = \mathfrak{g}, \operatorname{Ad}(h) \equiv 1 \mod [\mathfrak{g},\mathfrak{g}]\}.$$

But then $\operatorname{tr} x\xi = 0$ for all $x \in \mathfrak{g}', \xi \in \widetilde{\mathfrak{g}}$. Now if $x = \sum [y_i, z_i]$, then we have

$$\operatorname{tr} x\xi = \sum \operatorname{tr} y_i[z_i,\xi] = 0,$$

so if $x \in \mathfrak{g} \subset \widetilde{\mathfrak{g}}$, then $x_s \in \widetilde{\mathfrak{g}}$. Now considering $f(x_s) \in \widetilde{\mathfrak{g}}$, we will obtain some condition on $\operatorname{ad} f(x_s)$. These will have the same eigenvectors as $\operatorname{ad} x_s$. If E_{ij} is an eigenvector of $\operatorname{ad} x_s$ with eigenvalue $\lambda_i - \lambda_j$, then it has eigenvalue $f(\lambda_i) - f(\lambda_j)$ under $\operatorname{ad} f(x_s)$. If there exists ψ such that $f(\lambda_i) - f(\lambda_j)$, then $\operatorname{ad} f(x_s) = \psi(\operatorname{ad} x_s)$ and thus $f(x_s) \in \widetilde{\mathfrak{g}}$.

Now if f is linear over $\mathbb{Q}$, then $f(x_s) \in \widetilde{\mathfrak{g}}$ and $\operatorname{ad} f(x_s) = f(\operatorname{ad} x_s)$. Next we see that $\operatorname{tr} x_s f(x_s) = 0$ because we can embed $\{\lambda_i\} \subset \mathbb{C}$ and then take $f(\lambda_i) = \overline{\lambda}_i$. Then we see that $\operatorname{tr} x_s f(x_s) = \sum |\lambda_i|^2 = 0$, and then we see that all $\lambda_i = 0$. Alternatively, if $\dim_{\mathbb{Q}} \bigoplus \mathbb{Q}\lambda_i > 0$, then there exists a nonzero $f\colon \bigoplus \mathbb{Q}\lambda_i \to \mathbb{Q}$, but then $f(\sum \lambda_i f(\lambda_i)) = \sum f_i(\lambda)^2$. $\square$

Definition 1.4.2. Define the *Killing form* by

$$(x,y) \coloneqq \operatorname{tr} \operatorname{ad}(x)\operatorname{ad}(y).$$

Remark 1.4.3. Killing apparently lived a very sad life and did not get the recognition he deserved. Unfortunately, Andrei (and I) do not know more about him.

Theorem 1.4.4. *A Lie algebra $\mathfrak{g}$ is solvable if and only if $(x,[y,z]) = 0$.*

Proof. Consider the exact sequence

$$0 \to Z(\mathfrak{g}) \to \mathfrak{g} \to \operatorname{ad}\mathfrak{g} \to 0.$$

Then solvability of $\mathfrak{g}$ is equivalent to solvability of $\operatorname{ad}\mathfrak{g}$. □

Theorem 1.4.5 (Cartan Criterion)**.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate.*

Proof. Suppose $(-,-)$ is degenerate. Then $\mathfrak{g}^{\perp}$ is a solvable ideal in $\mathfrak{g}$. But then if $\mathcal{I}$ is a solvable ideal with $\mathcal{I}^{n+1} = 0$, then $\mathfrak{a} = \mathcal{I}^{n}$ is an abelian ideal. Therefore, for all x, y, we see that

$$[a, [y, [a, x]]] = 0,$$

and therefore for all $a \in \mathfrak{a}, y \in \mathfrak{g}$, we have $\operatorname{ad}(a)\operatorname{ad}(y)\operatorname{ad}(a) = 0$, so $(\operatorname{ad}(a)\operatorname{ad}(y))^2 = 0$, and thus $\operatorname{tr}\operatorname{ad}(a)\operatorname{ad}(y) = 0$. But then $a \in \mathfrak{g}^{\perp}$. □

Corollary 1.4.6. *If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is a sum of simple nonabelians.*

Proof. Suppose $\mathfrak{h} \subset \mathfrak{g}$ is an ideal. Then $\mathfrak{h} \cap \mathfrak{h}^{\perp} = 0$ (because it is a solvable ideal). Note that if $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then $(h_1, h_2)_{\mathfrak{h}} = (h_1, h_2)_{\mathfrak{g}}$. □

2

Geometric and topological aspects

2.1 Lie algebra cohomology

There are three kinds of properties:

- General abstract properties;
- Properties derived from the structure theory $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$;
- Properties derived from the calssification of root systems.

We will begin with the first. If $\mathfrak{g}$ is semisimple, then:

1. The category $\mathsf{Mod}_{\mathrm{fd}}\, \mathfrak{g}$ is semisimple. In particular, every finite-dimension $\mathfrak{g}$-module has the form $M = \bigoplus M_i$, where the M_i are simple.
2. The algebra $\mathfrak{g}$ has no *deformations*.
3. All derivations of $\mathfrak{g}$ are inner derivations. In particular, we see that $\mathfrak{g} = \mathsf{Lie}(\mathrm{Aut}(\mathfrak{g}))$. In addition, in the exact sequence
$$0 \to Z(\mathfrak{g}) \to \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{Der}\, \mathfrak{g} \to \mathrm{Out}\, \mathfrak{g} \to 0,$$
the two outside terms vanish.
4. For any Lie algebra $\mathfrak{g}_{\mathrm{any}}$, we have
$$0 \to \text{radical} \to \mathfrak{g}_{\mathfrak{any}} \to \mathfrak{g}_{\mathrm{ss}} \to 0,$$
and this exact sequence splits into $\mathfrak{g}_{\mathrm{any}} = \mathfrak{g}_{\mathrm{ss}} \ltimes \text{radical}$.

All of these pheonomena fit under the umbrella of the vanishing of some cohomology groups.

Definition 2.1.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\Bbbk$ and M be a $\mathfrak{g}$-module. We may consider the complex
$$\mathrm{Hom}_{\Bbbk}\left(\bigwedge\nolimits^n \mathfrak{g}, M\right) \ni \omega(\xi_1, \ldots, \xi_n) \qquad \xi_i \in \mathfrak{g}.$$
and define
$$d\omega\,(\xi_1, \ldots, \xi_{n+1}) = \sum_i (-1)^{i-1} \xi_i \omega(\ldots, \widehat{\xi}_i, \ldots) + \sum_{i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \ldots, \widehat{\xi}_i, \ldots, \widehat{\xi}_j, \ldots).$$

In the homework, we will show that $d^2 = 0$, so we may define the *Lie algebra cohomology* $H^n(\mathfrak{g}, M)$.

For an example in low dimension, we see that $C_0 = M \to C_1$ is given by $d\omega(\xi_1) = \xi_1(\omega)$, so $H^0(M) = M^{\mathfrak{g}}$. For a more modern definition, we see that $H^i(\mathfrak{g}, M)$ are the derived functors of $M \to M^{\mathfrak{g}}$.

We may motivate this formula in the following way from the de Rham differential. Suppose G acts on a manifold X. This gives a morphism $\mathfrak{g} \to \Gamma(X, TX)$ into the vector fields. Then if $\xi_1, \ldots, \xi_{n+1}$ are vector fields on X and ω is an n-form on X, we have

Proposition 2.1.2. *The formula for $d\omega(\xi_1, \ldots, \xi_n)$ is the same formula as in Lie algebra cohomology, where $\xi_i\omega$ is the Lie derivative of ω along ξ_i.*

Proof. Andrei's proof is way too confusing. This is also Proposition 12.19 in Lee's smooth manifolds book. The proof there is the same, but is done in a much easier-to-digest way. □

Theorem 2.1.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0.*

1. *If M is irreducible and nontrivial, then $H^\bullet(\mathfrak{g}, M) = 0$.*
2. *$H^\bullet(\mathfrak{g}, \Bbbk)$ is the free anticommutative algebra on finitely many generators of degree contained in $\{3, 5, 7, \ldots\}$. When $\Bbbk = \mathbb{C}$, this is the same as $H^\bullet(G, \mathbb{C})$. This is also the same as the cohomology of the maximal compact subgroup. For example,*

$$H^\bullet(SL(n, \mathbb{C}), \mathbb{C}) = H^\bullet(SU(n), \mathbb{C}) = \mathbb{C}\langle \omega_3, \omega_5, \ldots, \omega_{2n-1}\rangle.$$

In particular, if $\mathfrak{g}$ is semisimple, then H^1, H^2 vanish for all M. Also, we have an isomorphism $H^3(\mathfrak{g}, \Bbbk) \cong k^{\#\text{ simple factors}}$, and this is just the space of invariant bilinar forms.

Proof.

1. Suppose M is nontrivial and irreducible. Without loss of generality, assume $\mathfrak{g}$ is simple and consider $\mathfrak{g} \subseteq \mathfrak{gl}(M)$. We will show that multiplication by $\dim \mathfrak{g}$ is homotopic to 0. Consider the form $B(x, y) = \operatorname{tr}_M xy$, which is nondegenerate. Then if $\{e_1, \ldots, e_d\}, \{f_1, \ldots, f_d\}$ are dual bases of $\mathfrak{g}$, the *Casimir element* $\sum e_i f_i$ commutes with $\mathfrak{g}$ and is nonzero because $\operatorname{tr} \sum e_i f_i = \sum \operatorname{tr} e_i f_i = \dim \mathfrak{g}$. Now we will define our homotopy by

$$h\omega(\xi_1, \ldots, \xi_{n-1}) = \sum e_i \omega(f_i, \xi_1, \ldots, \xi_{n-1}).$$

Now, we compute $d \circ h + h \circ d$. We have

$$dh\omega(\xi_1, \ldots, \xi_n) = \sum_i (-1)^{i-1} \xi_i e_* \omega(f_*, \ldots, \widehat{\xi}_i, \ldots) + \sum_{i<j} (-1)^{i+j}(-1)^{i+j} e_* \omega(f_*, [\xi_i, \xi_j], \ldots)$$

Then writing

$$\begin{aligned} d_\omega(\xi_0, \xi_1, \ldots, \xi_n) = \xi_0\omega(\xi_1, \ldots, \xi_n) &+ \sum (-1)^i \xi_i \omega(\xi_0, \ldots, \widehat{\xi}_i, \ldots, \xi_n) \\ &+ \sum_i (-1)^i \omega([\xi_0, \xi_i], \ldots) + \sum_{0<i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \ldots), \end{aligned}$$

we have

$$\begin{aligned} h\,d\omega(\xi_1, \ldots, \xi_n) = e_* f_* \omega(\xi_1, \ldots, \xi_n) &+ \sum (-1)^i e_* \xi_i \omega(f_*, \ldots) \\ &+ \sum_i (-1)^i e_* \omega([f_*, \xi_i], \ldots) + \sum_{i<j} (-1)^{i+j} e_* \omega([\xi_i, \xi_j], f_*, \ldots). \end{aligned}$$

Now we may verify that all of the relevant terms cancel, so

$$(hd + dh)\omega = e_* f_* \omega + \sum_i (-1)^i ([e_*, \xi_i]\omega(f_*, \ldots) + e_* \omega([f_*, \xi_i], \ldots)).$$

The second term is given by inserting the tensor $[e_* \otimes f_*, \xi_i \otimes 1 + 1 \otimes \xi]$. But then $e_* \otimes f_*$ is an invariant tensor, so $\mathrm{ad}(\xi) e_* \otimes f_* = [\xi \otimes 1 + 1 \otimes \xi, e_* \otimes f_*] = 0$.

2. We may assume that $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$, so let G be a connected compact Lie group. We will see that $H^\bullet(\mathfrak{g}) = (\bigwedge^\bullet \mathfrak{g})^G \simeq H^\bullet(G)$. We will compute ordinary cohomology using the de Rham complex. First, we will note that $g \in G$ acts trivially on $H^\bullet(G)$. To see this, observe that $\xi \in \mathfrak{g}$ acts on forms by the Lie derivative L_ξ, and $[L_\xi \omega] = 0$ if $d\omega = 0$ by the Cartan formula.

   Now for any compact group G acting on a manifold X, the inclusion $(\Omega^i X)^G \to \Omega^i X$ induces an isomorphism on cohomology. To see this, we simply note that $\int g^* \omega \, dg$ is cohomologous to ω. Therefore, we may consider right-invariant forms in $\Omega^i G$. But then

$$(\Omega^i G)_{\text{right invariant}} \simeq \bigwedge^i \mathfrak{g}^* \simeq \bigwedge^i \mathfrak{g},$$

   and the differential on $\bigwedge^i \mathfrak{g}^*$ is the differential from Lie algebra cohomology. Now if we consider $(\Omega^i)^{G \times G} \simeq (\bigwedge^i \mathfrak{g}^*)^G$, the differential vanishes. To see this, the map $g \mapsto g^{-1}$ perserves the biinvariants but acts by $(-1)^i$ on $\bigwedge^i \mathfrak{g}^*$, and so $d \mapsto -d$, so $d = 0$. Because $\mathfrak{g}$ acts trivially on cohomology, it is also possible to see that

$$\left(\left(\bigwedge^\bullet \mathfrak{g}^*\right)^{\mathfrak{g}}, 0\right) \hookrightarrow \left(\bigwedge^\bullet \mathfrak{g}^*, d\right)$$

   is a quasi-isomorphism. □

Example 2.1.4. 1. Let $\mathfrak{g}$ be abelian. Then

$$H^\bullet(\mathfrak{g}, \mathbb{R}) = \bigwedge^\bullet \mathfrak{g}^* = H^\bullet(\mathfrak{g}/\Lambda, \mathbb{R}) = \bigwedge^\bullet H^1\left(\prod S^1, \mathbb{R}\right).$$

2. The condition that G is compact is important. Note that

$$H^\bullet(SL(2,\mathbb{R}), \mathbb{R}) = H^\bullet(S^1) \neq H^\bullet(S^3) = H^\bullet(SU(2)) = H^\bullet(\mathfrak{su}(2), \mathbb{R}).$$

Next, we will actually prove that $H^\bullet(\mathfrak{g}, \Bbbk) = \Bbbk \langle \omega_{2d_i - 1} \rangle_{i=1,\ldots,\operatorname{rank}\mathfrak{g}}$. In particular, by a theorem of Hopf, this is a Hopf algebra. Similarly, if G is a compact Lie group then $H^\bullet(G, \mathbb{R})$ is a Hopf algebra. Here are some properties of the cohomology:

- It is graded and supercommutative.
- Under the map $G \to G \times G \xrightarrow{\mu} G$ given by $g \mapsto (g, 1) \to g$, if we write

$$\Delta\omega = \sum \omega_i' \otimes \omega_i'',$$

  then $(1 \otimes \eta)\Delta\omega = \omega$ because $\Delta\omega = \omega \otimes 1 + 1 \otimes \omega + H^{>0} \otimes H^{>0}$.

Theorem 2.1.5 (Hopf). *Any finitely generated graded supercommutative Hopf algebra and the second property has the form* $\Bbbk \langle \omega_{m_i} \rangle$.

Corollary 2.1.6. *If, additionally, our Hopf algebra is assumed to be finite-dimensional, then all of the m_i are odd.*

Proof. Let ω_{m_i} be generators with $m_1 \leqslant m_2 \leqslant \cdots$. Then let $\mathcal{H}_k$ be the algebra generated by $\omega_{m_1}, \dots, \omega_{m_k}$. Then we know that $\Delta\omega_{m_k} = \omega_{m_k} \otimes 1 + 1 \otimes \omega_{m_k} + \cdots$, so each $\mathcal{H}_k$ is a sub-bialgebra. Now it suffices to show that $\mathcal{H}_k = \mathcal{H}_{k-1}\left\langle \omega_{m_k} \right\rangle$.

Suppose there is a relation $R = \sum_{i=0}^{d} c_i x^i = 0$ of degree d. But now if we consider the ideal ΔR modulo $1 \otimes \left\langle \mathcal{H}_{k-1}, x^2 \right\rangle$, which does not contain x for grading reasons, then we have

$$\begin{aligned} \Delta c_i &= c_i \otimes 1 + \cdots \\ \Delta x &= x \otimes 1 + 1 \otimes x + \cdots \\ \Delta x^n &= (\Delta x)^n = x^n \otimes 1 + n x^{n-1} \otimes x + \cdots \\ \Delta R &= R \otimes 1 + \frac{\partial}{\partial x} R \otimes x + \cdots, \end{aligned}$$

and this must be a relation of smaller degree. Therefore we have no relations beyond supercommutativity. ☐

Now an interesting problem is to compute the degrees of the generators. For example, we have

$$H^\bullet(SU(n)) = \mathbb{R}\left\langle \omega_3, \omega_5, \dots, \omega_{2n-1} \right\rangle.$$

Theorem 2.1.7 (Cartier-Kostant-Gabriel-…)**.** *If $\mathcal{H}$ is a supercommutative Hopf algebra over a field $\Bbbk$ of characteristic 0, then*

$$\mathcal{H} = \Bbbk G \ltimes \mathcal{U}(\mathfrak{g}).$$

where G is a (typically finite) group and $\mathfrak{g}$ is a Lie superalgebra over $\Bbbk$.

The elements with $\Delta g = g \otimes g$ are called *grouplike*, and the elements with $\Delta\xi = \xi \otimes 1 \pm 1 \otimes \xi$ are called *primitive*. The grouplike elements give us G, and the primitive elements give us $\mathfrak{g}$.

Now we may take the dual Hopf algebra, and this gives us another graded supercommutative Hopf algebra. These give us algebraic supergroups over $\Bbbk$. But this algebraic supergroup must be an odd vector space. Another consequence of the theorem is that a commutative Hopf algebra over $\Bbbk$ has no nilpotent elements. This implies that all group schemes over $\Bbbk$ are reduced and therefore smooth.

Recall that if G is a compact connected Lie group, then $H^\bullet(G, \mathbb{R}) = \mathbb{R}\left\langle \omega_{2d_i - 1} \right\rangle_{i=1,\dots,\mathrm{rank}\, G}$.

Theorem 2.1.8. *We have $\dim_{\mathbb{R}} H^\bullet(G, \mathbb{R}) = 2^{\mathrm{rank}\, G}$, where $\mathrm{rank}\, G$ is the dimension of the maximal torus.*

We would also like to compute the d_i. In fact, d_i are the degrees of the generators of $(S^\bullet\mathfrak{g}^*)^G$. We know that every element of $\mathfrak{g}$ is conjugate to an element of $\mathfrak{t} = \mathrm{Lie}\, T$. The normalizer of this is the Weyl group W. Now, by definition, we have

$$(S^\bullet\mathfrak{t}^*)^W = \mathbb{R}[\mathfrak{t}/W],$$

and this is free on some generators p_{d_i} of degree $\left|p_{d_i}\right| = d_i$ for $i = 1, \dots, \dim\mathfrak{t} = \mathrm{rk}\, G$.

Theorem 2.1.9. *The d_i defined in the various ways are the same numbers and are called exponents of G.*

Example 2.1.10. For $G = U(n)$, we have $\{d_i\} = \{1, 2, \dots, n\}$ and $p_d = \mathrm{tr}\, \xi^d$. Alternatively, we can use the coefficients of the characteristic polynomial. In addition, we see that

$$H^\bullet(U(n)) = \mathbb{R}\left\langle \omega_1, \omega_3, \omega_5, \dots, \omega_{2n-1} \right\rangle,$$

and $\left(\bigwedge^\bullet \mathfrak{g}^*\right)^G$ is generated by $\omega_1(\xi) = \operatorname{tr}\xi$, $\omega_3(\xi_1,\xi_2,\xi_3) = \operatorname{tr}\xi_1[\xi_2,\xi_3]$, and in general

$$\omega_d(\xi_1,\ldots,\xi_d) = \sum_{\sigma\in S(d)/(123\ldots d)} (-1)^\sigma \operatorname{tr}\prod_{i=1}^{d} \xi_{\sigma(i)}.$$

In this formula, we observe that d must be odd.

Proof of Theorem 3.2.8. Use the Molien series. We have

$$\dim V^G = \int_G \operatorname{tr}_V g\, dg = \frac{1}{|W|}\int_T \operatorname{tr}_V g \cdot \det_{\mathfrak{g}/\mathfrak{t}}(1-\operatorname{Ad}(t))d_{\mathrm{Haar}}t.$$

Setting $V = \bigwedge^\bullet \mathfrak{g}$, we see that

$$\operatorname{tr}_V g = \det_{\mathfrak{g}}(1+\operatorname{Ad}(g)) = 2^{\mathrm{rank}} \det_{\mathfrak{g}/\mathfrak{t}}(1+\operatorname{Ad}(t)).$$

This now gives us

$$2^{\mathrm{rank}}\frac{1}{W}\int_T \det_{\mathfrak{g}/\mathfrak{t}}(1-\operatorname{Ad}(t^2))d_{\mathrm{Haar}}t = 2^{\mathrm{rank}}$$

by change of variables. □

2.2 Classifying Spaces and Flag Varieties

We already know that $H^\bullet(G) = \mathbb{R}\left\langle \omega_{2d_i-1}\right\rangle$. On the other hand, we know the cohomology of the flag manifold $H^\bullet(G/T)$ is all even, and finally we have the cohomology

$$H^\bullet(BG) = H^\bullet(pt/G) = \left(S^{2\bullet}\mathfrak{g}^*\right)^G = (S^\bullet\mathfrak{t}^*)^W.$$

On the other hand, we have

$$H^{2\bullet}(G/T) = (S^\bullet\mathfrak{t}^*)/(S^\bullet\mathfrak{t}^*)^W_{>0}.$$

For $G = U(n)$, this becomes the space of polynomials divided by symmetric polynomials of positive degree and has dimension $n! = |W|$. This is the fiber over 0 of the map $\mathfrak{t} \to \mathfrak{t}/W$. To define BG, consider the category of spaces with a free action of G (equivalently principal G-bundles) for any group G. Given a commutative diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "\varphi_G"] & X \arrow[d, "\varphi_X"] \\ G' \arrow[r, hook] & X' \end{tikzcd}$$

with $\varphi(g\cdot x) = \varphi(g)\cdot\varphi(x)$, we would like to consider the possibilities for φ_X for a fixed φ_G. If we consider the graph of φ_X, this is just a section of $X \times X'/X$ because G acts freely, this is the same as a section of $(X\times X')/G \to X/G$.

Theorem 2.2.1. *If X' is contractible, then there exists a unique $\varphi_X\colon X \to X'$ compatible with φ_G.*

Proposition 2.2.2. *For any compact group G, there exists a contractible space EG with a free G-action.*

Corollary 2.2.3.

1. EG *is unique up to homotopy.*
2. EG *is functorial in* G.
3. *For any free action of* G *on* X, *the map* $X \to X/G$ *is the pullback of*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & EG \arrow[d] \\ X/G \arrow[r] & BG \end{tikzcd}$$

for some map $X/G \to BG$. *Therefore, we see that*

$$\{\textit{principal } G\textit{-bundles over } B\} = [B, BG].$$

Proof of proposition. For all G compact, there exists an embedding $G \subseteq U(n)$, so it suffices to consider $U(n)$. Consider the embedding $U(n) \hookrightarrow \mathrm{Mat}(n, N)_{\mathrm{rank}=n}$ for some $N \gg 0$. For example, we have the action of $U(1)$ on $\mathbb{C}^n \setminus 0 \simeq S^{N-1}$, so we can consider S^∞, which is contractible. Therefore we have $BU(1) = S^\infty/U(1) = \mathbb{CP}^\infty$.

In general, we can consider the action of $U(n)$ on

$$\mathrm{Mat}(n, N) \setminus \{\mathrm{rank} < n\} \supseteq \{X \mid XX^* = 1_n\},$$

and this last space becomes contractible as $N \to \infty$. To see this, it sits inside of $(S^N)^n$, and thus as $N \to \infty$, we obtain a contractible subspace of $(S^\infty)^n$. Finally, we have $BU(n) = \mathrm{Gr}(n, \infty)$. ☐

Therefore, we have a tautological $U(n)$-bundle on $\mathrm{Gr}(n, \infty)$ and a tautological $\mathbb{C}^n$-bundle where the fiber above a subspace is the subspace itself. The vector bundle is the associated bundle of the $U(n)$-bundle and the $U(n)$-bundle is the bundle of unitary operators on the vector bundle. Also, we have proved that

$$\{\text{complex vector bundles of rank } n \text{ over } B\} \longleftrightarrow [B, \mathrm{Gr}(n, \infty)].$$

The same statement holds for $O(n)$, real vector bundles, and the real Grassmannian. Explicitly over B, consider the exact sequence

$$0 \to \ker \to \mathbb{C}^N_B \to V \to 0.$$

Now the kernel defines a map $B \to \mathrm{Gr}(N - n, N) \simeq \mathrm{Gr}(n, N)$. Taking $N \to \infty$, we obtain the desired result.

Remark 2.2.4. The spaces EG and BG are naturally approximated by algebraic varieties such as $\mathrm{Gr}(n, N)$ and are therefore ind-schemes.

We want to show that $\mathbb{R}^\infty \setminus 0$ is contractible. We can consider the function T on $\mathbb{R}^\infty \setminus 0$ given by $T(x_1, x_2, \ldots, 0, \ldots) = (0, x_1, x_2, \ldots)$. Then x, Tx are never collinear for $x \neq 0$ and thus T is homotopic to the identity. But then Tx is never collinear to $e_1 = (1, 0, \ldots,)$ and therefore T is nullhomotopic. We also see that $C^\infty \setminus 0$ is contractible. Also, $S^\infty \sim \mathbb{R}^\infty \setminus 0$ is contractible and so are Stiefel manifolds

$$\{v_1, \ldots, v_n \in \mathbb{C}^\infty \mid v_i \text{ are linearly independent}\}.$$

The Stiefel manifold has a free action of $GL(n, \mathbb{C})$, so this is $EGL(n, \mathbb{C})$. We also see that $BGL(n, \mathbb{C}) = \mathrm{Gr}(n, \infty, \mathbb{C})$. We know that the maximal compact subgroup of G is homotopy equivalent to G, so $BGL(n, \mathbb{C}) = BU(n)$.

Remark 2.2.5. The high-brow way that we prove that all of these spaces are contractible is by proving that they are weakly contractible, and then smashing the remaining part that weakly contractible implies contractible for spaces with the homotopy type of a CW complex.

Now the cohomology $H^\bullet(B, G)$ is intimately connected to characteristic classes for principal G-bundles. If G acts on a manifold M, we have a map $\mathfrak{g} = \mathsf{Lie}(G) \to \mathrm{Vect}(M)$, so we may consider the Lie derivative $L_x \colon \Omega^k M \to \Omega^k M$ for $x \in \mathfrak{g}$. For the Lie derivatives, recall the identity $[L_x, \iota_y] = \iota_{[x,y]}$. This gives us a super Lie algebra $\widehat{\mathfrak{g}}$ with $\widehat{\mathfrak{g}}_1 = \mathbb{C}_d$, $\widehat{\mathfrak{g}}_0 = \mathfrak{g}$, and $\widehat{\mathfrak{g}}_{-1} \cong \mathrm{Ad}\,\mathfrak{g}$. This gives us the super-Lie bracket

$$[a, b] = ab - (-1)^{|a||b|} ba.$$

Now we see that $[\widehat{\mathfrak{g}}_1, \widehat{\mathfrak{g}}_0] = 0$, $[\widehat{\mathfrak{g}}_1, \widehat{\mathfrak{g}}_{-1}]$ is given by the Cartan formula, and $[\widehat{\mathfrak{g}}_0, \widehat{\mathfrak{g}}_{-1}]$ is the adjoint action of $\mathfrak{g}$ on itself. Therefore, if M is a manifold with a G-action, then $\Omega^\bullet M$ is a supercommutative DG algebra with an action of $\widehat{\mathfrak{g}}$.

If the action of G is free, we may choose a G-invariant metric on M. For every $v \in T_m M$, we have its projection onto $T_m Gm \simeq \mathfrak{g}$. This gives us a connection, which is a G-invariant 1-form with values in $\mathfrak{g}$ and thus gives us a map

$$\alpha \colon \mathfrak{g}^* \to \Omega^1 M \qquad [\alpha(\xi)](\iota_x) = \langle \xi, x \rangle .$$

Therefore, if a map $\widehat{\mathfrak{g}} \to \mathcal{A}^\bullet$, where $\mathcal{A}$ is a supercommutative DG algebra means that G acts on M, we would like to give an interpretation of a connection $\alpha \colon \mathfrak{g}^* \to \mathcal{A}^1$ such that $[\alpha(\xi)](\iota_x) = \langle \xi, x \rangle$.

Theorem 2.2.6. *There exists a unique acyclic supercommutative DG algebra with* $H^0(\mathcal{A}^\bullet) = \mathbb{C}, H^i(\mathcal{A}^\bullet) = 0, i > 0$. *We will denote this universal algebra by* $\mathbb{E}$.

Proof. Set $\mathcal{A}^0 = \mathbb{C}$ and $\mathcal{A}^1 = \alpha(\mathfrak{g}^*)$ with L_x acting by the coadjoint action. We set $d \colon \mathcal{A}^0 \to \mathcal{A}^1$ to be the zero map. We also set $\iota_x(\alpha(\xi)) = \langle \xi, x \rangle$. Because $d \colon \mathcal{A}^1 \to \mathcal{A}^2$ must be an isomorphism, we see that $\mathcal{A}^2 = \beta(\mathfrak{g}^*) \oplus \bigwedge^2 \mathcal{A}^1$ with the coadjoint action of $\mathfrak{g}$. Then we define i_x by

$$i_x\, d\alpha(\xi) = L_x \alpha(\xi) - d i_x \alpha(\xi) = L_x \alpha(\xi). \qquad \square$$

Note that $\mathbb{E}$ looks like $\Omega^\bullet \mathfrak{g}$, which are polynomials in x multiplied by $\bigwedge dx_i$. For $\beta \in \mathcal{A}^2$ and $\alpha \in \mathcal{A}^1$, we can define $d^*\beta(\xi) = \alpha, d^*\alpha(\xi) = 0$. Therefore, d^* is a derivation, and the Laplacian

$$[d^*, d] = d^*d + dd^*$$

is the identity on $\mathcal{A}^1$ and $\mathcal{A}^2$. This implies that multiplication by $(k + \ell)$ on $(\mathbb{E}^1)^k (\mathbb{E}^2)^\ell$ is homotopic to the identity and thus $H^\bullet(\mathbb{E}) = \mathbb{C}$.

Now we return to our manifold M and base $B = M/G$. Then the image of $H^\bullet(B) \hookrightarrow H^\bullet(M)$ are the so-called *basic forms*, which vanish on ι_x (horizontal) and are G-invariant. In particular, they are killed by both ι_x, L_x. Then the map $H^\bullet(M) \twoheadrightarrow H^\bullet(G)$ has kernel the horizontal forms, so we need to consider the horizontal forms.

Proposition 2.2.7. *Horizontal forms are generated by curvatures, which have the form*

$$\beta(\xi) + \delta\alpha(\xi),$$

where δ *is the map* $\mathfrak{g}^* \to \bigwedge^2 \mathbb{E}^1$ *given by the transpose of the Lie bracket.*

We now have

$$\iota_x(\beta(\xi) + \delta\alpha(\xi)) = \alpha(\mathrm{ad}_x^* \xi) + \alpha(\xi)[x, -] = 0$$

because $\alpha(\xi)[x,-] = -\alpha(\mathrm{ad}_x\,\xi)$. Therefore we can write

$$\mathbb{E} = \bigwedge^{\bullet}\alpha(\xi)\otimes S^{\bullet}(\text{curvatures}) = \bigwedge^{\bullet}\alpha(\xi)\otimes S^{\bullet}\beta(\xi).$$

Therefore, we have $\mathbb{E}_{\text{horizontal}} = S^{\bullet}\mathfrak{g}^*$ and $\mathbb{E}_{\text{basic}} = (S^{\bullet}\mathfrak{g}^*)^G$ with zero differential, so we have $H^{2\bullet}(\mathbb{E}_{\text{basic}}) = (S^{\bullet}\mathfrak{g}^*)^G$. Now we have a *transgression*[1] map $H^{2m}(\mathbb{E}_{\text{basic}}) \to H^{2m-1}(G)$, which is the "inverse" $\pi\circ\frac{d^*}{m}\colon \mathbb{E}^G \to \left(\bigwedge^{\bullet}\mathfrak{g}^*\right)^G$ of the differential. This vanishes on $\left((S^{\bullet}\mathfrak{g})^2_{>0}\right)^2$.

Corollary 2.2.8. $\mathbb{E}_{\textit{horizontal}} = S^{\bullet}(\textit{curvatures})$*. This is because* $\iota_\xi\colon \mathfrak{g}\otimes\mathfrak{g}^*\to\mathbb{C}$ *is a perfect pairing.*

In some sense, we have proven that

Theorem 2.2.9. $H^{\bullet(BG,\mathbb{C})} \simeq (S^{\bullet}\mathfrak{g}^*)^G$.

However, we would like to discuss $H^{\bullet}(G)$. Now we want to describe the transgression. If $p\in H^{\bullet}(BG)$, then want to compute $d^{-1}(p)\in H^{\bullet-1}(EG)$, and then we can restrict this to $H^{\bullet}(G)$ by killing the horizontal forms. Therefore, polynomial functions on $\mathfrak{g}$ map to polynomial differential forms on $\mathfrak{g}$ by d^*. This gives us something like

$$p(x)\mapsto \sum_{\partial_i} p(x)\otimes\xi^i \mapsto \text{substitute } x = -[\xi,\xi]$$

and this takes polynomials of degree m to elements of degree $2m-1$. Now in the chain

$$\left((S^m\mathfrak{g}^*)^G, 0\right)\to\cdots\to\left(\left(\bigwedge^{2m-1}\mathfrak{g}^*\right), 0\right),$$

the choice of d^{-1} does not matter. Also, this map is zero on $\left((S^{>0}\mathfrak{g}^*)^G\right)^2$ because $d^{-1}(p_1p_2) = p_1d^{-1}p_2$ if $\deg p_1, \deg p_2 > 0$, which is killed by transgression. Therefore, primitive elements of $(S^{\bullet}\mathfrak{g}^*)^G$ map isomorphically onto primitive elements of $\left(\bigwedge^{\bullet}\mathfrak{g}^*\right)^G$. Now we have explained the relationship between $H^{\bullet}(G)$ and $H^{\bullet}(BG)$.

We now want to add the third vertex of the triangle, which is $H^{\bullet}(G/T)$. Recall that any element of $\mathfrak{g}$ is conjugate to some element of $\mathfrak{t}$ and that $(S^{\bullet}\mathfrak{g}^*)^G = (S^{\bullet}\mathfrak{t}^*)^W$, which is a free algebra. Therefore,

$$H^{2\bullet}(G/T) = S^{\bullet}\mathfrak{t}^*/(S^{>0}\mathfrak{t}^*)^W.$$

We will see that this is free over $(S^{\bullet}\mathfrak{t}^*)^W$.

Example 2.2.10. For $G = SU(2)$ and $G/T = S^2$, we have $\mathrm{Lie}\,\mathfrak{t} = \mathbb{R}$ and $W = \{\pm 1\}$, and also $H^2(S^2) = \mathbb{C}[x]/(x^2)$.

Now we have a map $S^{\bullet}\mathfrak{t}^*/(S^{>0}\mathfrak{t}^*)^W \to H^{2\bullet}(G/T)$ because the image of the map $G/T\to BT\to BG$ is a point, so we have to kill all positive degree elements in $H^{\bullet}(BG)$.

Theorem 2.2.11 (Chevalley-Sheppard-Tod)**.** *Let* $\Gamma\subset GL(V)$ *be finite. Then the following are equivalent:*

1. Γ *is generated by complex reflections* r *such that* $\mathrm{rk}(r-1) = 1$.

2. $\mathbb{C}[V] = R$ *is free over the invariants* $S = \mathbb{C}[V]^{\Gamma}$.

[1] This is the same as the transgression map in the Serre spectral sequence.

3. $S = \mathbb{C}[p_1, \dots, p_{\dim V}]$.

There is a classification of all such groups generated by complex reflections, but Andrei does not remember what it is.

The key to the proof of this theorem is *divided difference operators:* If s_α is a reflection, then V^s is a hyperplane, so for all $f \in R$, we may consider $\frac{f - s_\alpha \cdot f}{\alpha}$. This vanishes on V^s, lowers the degree by 1, and commutes with x. For example, we could choose

$$\frac{f(x_1, x_2, \dots) - f(x_2, x_1, \dots)}{x_1 - x_2}.$$

Proof. Let $e_1, e_2, \dots$ be a basis of $R/S^{>0} \cdot R$. We will show that it is also a basis of R over S. Suppose there exists a relation $\sum g_i e_i = 0$, where $g_i \in S$. Then either $g_1 \in \sum_{i>0} S g_i$ or $e_1 \in RS^{>0}$. Inducting on the degree of e_1, if $\deg e_1 = 0$, then

$$g_1 = -\sum_{i>1} g_i e_i.$$

Averaging over Γ, we obtain the desired expression. If $\deg e_1 > 0$, then we can apply divided difference operators to reduce the degree. The divided differences of some function f all vanish only when $s_\alpha f = f$ for all α, which is equivalent to $f \in R^G = S$. Therefore we can assume $f \in S^{>0}$, so if $g_1 = \sum c_i g_i$, then

$$e_1, e_2 + c_2 e_1, e_3 + c_3 e_1, e_4 + c_4 e_1, \dots$$

form another basis of $R/S^{>0}$. But now we obtain a shorter relation

$$\sum g_i e_i = g_2(e_2 + c_2 e_2) + g_3(e_3 + c_3 e_1) + \cdots$$

Therefore we have proved freeness. Finally, we see that the rank of R over S is $\deg(V \to V/\Gamma) = |\Gamma|$.

Now note that $V \to V/\Gamma$ is flat. Therefore V/Γ is smooth. Recall that if X is a scheme, a point $p \in X$ is smooth if and only if a minimal resolution of $\cdots \to \mathcal{O}_X \to \mathcal{O}_p$ is finite.[2]. The most important point to check is 0. Here, have a finite resolution

$$\underbrace{\cdots\cdots}_{\text{finite}} \to S \to S/S^{>0} \to 0,$$

so after tensoring with R, flatness gives us finiteness for $R \to R/RS^{>0}$.

To prove the final leg, we will use Molien series. Recall that V/Γ is a cone with vertex 0 and is also smooth, so it must be affine space. Recall that $S = \bigoplus_{i \geq 0} S_i$, where $S_0 = \mathbb{C}$. Then

$$\begin{aligned}
\frac{1}{\prod(1 - t^{m_i})} &= \sum_i t^i S_i \\
&= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \det \frac{1}{1 - t\gamma} \\
&= \frac{1}{\Gamma} \left(\frac{1}{1-t} \dim V + \sum_{\text{reflections}} \frac{1}{(1-t)^{\dim V - 1}} \cdots + O\left(\frac{1}{(1-t)^{\dim V - 2}} \right) \right).
\end{aligned}$$

[2] You can choose your favorite singular variety and try to find a finite resolution

This implies that the number of generators is $\dim V/\Gamma = \dim V$. This implies that $|\Gamma| = \prod m_i$. For example, if $G = SU(n)$, then $|W| = n! = n(n-1)(n-2)\cdots 2$, and the exponents are $1, \ldots, n$. The next term in the expansion gives us the subgroup of all reflections fixing $\alpha = 0$ in Γ, and in fact it is cyclic and contained in the roots of unity. Denote a generator by ζ_k. Then we have

$$\begin{aligned}\frac{1}{|\Gamma|}\left(\sum_s \frac{1}{\det(1-ts)} + o(\cdots)\right) &= \sum \frac{1}{(1-t)^{r-1}} \sum_{i=1}^{k-1} \frac{1}{1-\zeta_k^i t} \\ &= \sum \frac{1}{(1-t)^{r-1}} \frac{k-1}{2} \\ &= \frac{1}{(1-t)^{r-1}} \frac{\#\text{reflections}}{2} + o(\ldots).\end{aligned}$$

This implies that $\#\text{reflections} = \sum(m_i - 1)$. For example, for $\Gamma = S_n$, we have $\binom{n}{2} = (1-1) + (2-1) + \cdots + (n-1)$. Now set $\Gamma' \subset \Gamma$ be the subgroup generated by reflections. Then $\mathbb{C}[p_i] = S \subset S' = R^{\Gamma'} = \mathbb{C}[p_i']$, where p_i' has degree m_i' and p_i has degree m_i. If we order $m_1 \leqslant \cdots \leqslant m_r$ and $m_1' \leqslant \cdots \leqslant m_r'$, then $m_i \geqslant m_i'$ because otherwise the $p_1, \ldots, p_i$ would be algebraically dependent. Thus $\sum(m_i - 1) \geqslant \sum(m_i' - 1)$ and in in fact the inequality is strict unless $m_i = m_i'$ for all i. However, both sums count reflections, which is the same for γ, γ', so in fact $m_i = m_i'$. This implies that $\prod m_i = \prod m_i'$, and thus $|\Gamma| = |\Gamma'|$, so $\Gamma = \Gamma'$. $\square$

Now we have a diagram

$$\begin{tikzcd} H^\bullet(G/T, \mathbb{C}) & & S^\bullet \mathfrak{t}^* \arrow[ll] \arrow[r, equal] \arrow[d, "\lambda \mapsto \lambda([x_1, x_2])"] & H^\bullet(BT) \\ \text{invariant differential forms} \arrow[u] & & \left(\bigwedge^\bullet (\mathfrak{g}/\mathfrak{t})^*\right)^T & \end{tikzcd}$$

which commutes. In particular, $H^\bullet(G/T, \mathbb{C})$ is the regular representation of the Weyl group, which follows from the Lefschetz fixed point formula, which states that if $f\colon M \to M$, then we have

$$\sum_{f(m)=m} \mathrm{mult}(m) = \sum_i (-1)^i \operatorname{tr}_{H^i(M)} f^*,$$

which says that the number of fixed points with multiplicity is equal to the intersection product of Δ and $\Gamma(f)$. In our case, $w \neq 1$ has no fixed points and thus its trace vanishes. For $w = 1$, all points are fixed, so we obtain $\chi(G/T) = |W|$.

Now recall that $H^{2k}(G/T) \xleftarrow{c_i} S^k \mathfrak{t}^*$. Then $\lambda \in \mathfrak{t}^*$ maps to a G-invarint 2-form that restricts to $\lambda([x, y])$ at the origin and that $H^\bullet(G/T)$ is the regular representation of W. But then we see that $H^\bullet(G/T)^W = H^0$ and thus $(S^{>0}\mathfrak{t}^*)^W \to 0$. This also follows from the fact that the composition $G/T \to BG \to BG$ maps G/T to a point. Therefore we have a map

$$S^\bullet \mathfrak{t}^* / (S^{>0}\mathfrak{t}^*)^W.$$

Both spaces have dimension W, but we want to know if this is an isomorphism. Equivalently, we want to know whether the cohomology of G/T is generated by $c_1(\text{taut})$. Both of these are 0-dimensional Gorenstein rings. Of course, we have $H^{\dim M} = \mathbb{C} \cdot [M]$ for any closed manifold M, and we call this the *socle*. This has dimension 1 as a vector space. In addition, we have a perfect pairing

$$H^k \otimes H^{\dim M - k} \ni \alpha \otimes \beta \mapsto \int_M \alpha \smile \beta,$$

and in particular $\dim H^k = \dim H^{\dim M - k}$.

Another important example is a 0-dimensional complete intersection $Z \subset \mathbb{A}^d$. This has

$$\mathcal{O}_Z = \Bbbk[x_1, \dots, x_d]/(f_1, \dots, f_d),$$

where $f_1, \dots, f_d$ is a regular sequence. For example, a dimension 0 subscheme of the plane generated by a monomial ideal is Gorenstein if and only if it is generated by two elements. Equivalently, its Young diagram is a rectangle. Then the socle of $\mathcal{O}_Z$ is spanned by $\Bbbk \det\left(\frac{\partial f_i}{\partial x_j}\right)$. In the case of a monomial ideal $(x_1^{d_1}, x_2^{d_2})$, this is spanned by $x_1^{d_1-1}x_2^{d_2-1}$. Now a map between Gorenstein rings is injective if and only if it preserves the socle. Then we know that $S^\bullet \mathfrak{t}^W = \mathbb{C}[p_1, \dots, p_r]$, where r is the rank, so $I = (p_1, \dots, p_r)$. Therefore $S^\bullet \mathfrak{t}^*/(S^{>0}\mathfrak{t}^*)^W$ is a complete intersection. It has socle given by

$$\mathbb{C} \det\left(\frac{\partial p_i}{\partial x_j}\right) = \mathbb{C} \prod \text{roots}$$

and is the first anti-invariant J. This means $s_\alpha J = -J$ and thus the socle is the sign representation of W. But then any f with $s_\alpha f = -f$ has to vanish along $\alpha = 0$, so $\alpha \mid f$. In particular, $\prod \alpha \mid J$. We conclude that the socle of $S^\bullet \mathfrak{t}^*/(S^{>0}\mathfrak{t}^*)^W$ is $\mathbb{C} \cdot \prod \alpha$ with

$$(f, g) \to \frac{\text{antisymmetrize}(fg)}{\prod \alpha}.$$

Now it suffices to check where $\prod \alpha$ goes in $H^\bullet(G/T)$. We know that $\prod \alpha$ is the volume form on $\mathfrak{g}/\mathfrak{t}$, so it goes to $H^{\text{top}}(G/T)^G$, as desired.

Remark 2.2.12. There are other approaches to proving this result. Let $M = \mathrm{Gr}(k, n)$ and consider the locus

$$\text{diag} = \{L_1, L_2 \subset \mathbb{C}^n \mid \dim L_1 = k, L_1 \to \mathbb{C}^n \to \mathbb{C}^n/L_2 = 0\}.$$

This is the zero locus of a section of $\mathrm{Hom}(L_1, \mathbb{C}^n/L_2)$, where L_1, L_2 are the tautological bundles. This has rank $k(n-k)$ and thus $\mathrm{Gr}(k,n) = U(n)/U(k) \times U(n-k)$. Now in $H^\bullet(M \times M)$, the class [diag] is given by the characteristic classes of L_1, L_2. Like in Lefschetz, this acts by the identity operator on $H^\bullet(M)$ and thus $H^\bullet(M)$ is spanned by the characteristic classes of L_1.

It remains to discuss the relationship between $H^\bullet(G)$ and $H^\bullet(G/T)$. In principle, we can consider $T \hookrightarrow G \to G/T$, but studying this requires spectral sequences. Alternatively, we will study this using the Weyl integration formula. Recall that the map

$$G/T \times T \to G \qquad (g, t) \mapsto gtg^{-1}$$

is generically $|W|$-to-1. Therefore we have a map

$$G/T \times T \to G/T \times_W T \to G,$$

and because the action $(g, t) \mapsto (gw^{-1}, wtw^{-1})$ is free, the middle term is a smooth manifold. This gives us a map $H^\bullet(G) \to H(G/T \times T)^W$. This is a map between Gorenstein rings and is injective on the socle by inspection. Thus it remains to compute the dimension. By the Künneth formula, we have

$$H^\bullet(G/T \times T) = H^\bullet(G/T) \otimes H^\bullet(T).$$

The first term is the regular representation of W and the second term is $\bigwedge^\bullet \mathfrak{t}^*$ with the natural W action. But then we see that $H(G/T \times T)^W$ has dimension $\dim \bigwedge^\bullet \mathfrak{t}^* = 2^r$. Chasing equivalences, we have

$$(\Omega^\bullet \mathfrak{t})^W = \left(\bigwedge^\bullet \mathfrak{t}^* \otimes S^\bullet \mathfrak{t}^*\right)^W = \bigwedge\nolimits_{\Bbbk}[dp_1, \dots, dp_r].$$

Here, all elements of $S^\bullet \mathfrak{t}^*$ have their degrees doubled. In particular, $\deg dp_i = 2m_i - 1$.

Remarks 2.2.13. Recall that Lie algebra cohomology give the derived functors of $M \to M^{\mathfrak{g}} = \mathrm{Hom}(\Bbbk, M)$. This is computed by resolving, taking invariants, and then taking the cohomology. This is a general principle in homological algebra, and in fact we can apply this to topological spaces. If G acts freely on M, then M/G is nice (a smooth manifold) and therefore is nice. Otherwise, it is better to consider $(M \times EG)/G$ and the fibration $M \hookrightarrow (M \times EG)/G \to BG$. Then we can define the *equivariant cohomology*

$$H^\bullet_G(M) = H^\bullet((M \times EG)/G),$$

and this is a module over $H^\bullet_G(pt) = H^{bullet}(BG)$. Then we can view $\operatorname{Spec} H^\bullet_G(M)$ (here the Spec is taken as a superscheme) as a sheaf over $\mathfrak{t}^*/W$. If G acts on M freely, then $(M \times EG)/G$ is homotopy equivalent to M/G. The module structure over $H^\bullet(EG)$ via $H^\bullet(BG) \to H^\bullet(EG) = H^0$. Now we obtain the skyscraper sheaf over $\mathfrak{t}^*/W$ with stalk $H^\bullet(M)$ at the origin.

In this language, we have $H^\bullet_T(G/T) = S^\bullet \mathfrak{t}^* \otimes_{(S^\bullet \mathfrak{t}^*)^W} S^\bullet \mathfrak{t}^*$, and this has a map to $H^\bullet(G/T)$ killing the positive degree part of the first factor of the tensor product. This recovers the cohomology of $H^\bullet(G/T)$ that we computed before.

Now let V be a rank r vector bundle on X. Then V is given by a map $X \to BU(r) = \mathrm{Gr}(r, \infty)$. This induces a map $H^*(BU(r), \mathbb{Z}) \to H^*(X, \mathbb{Z})$. Now there is a cell decomposition $\mathrm{Gr}(r, N)$ into *Schubert cells*, which is given by the row reduced echelon form of a matrix in $\mathrm{Gr}(r, N)$. Now this gives a basis

$$H^*(\mathrm{Gr}(r, N), \mathbb{Z}) = \bigoplus \mathbb{Z}[\Sigma],$$

where Σ ranges over the Schubert cells. Then the character

$$\sum_{\sigma \in S(r)} (-1)^\sigma x^{\sigma \cdot (\lambda + \rho) - \rho} / \prod (x_i - x_j)$$

is the character of an irreducible representation of $U(r)$. Now in terms of the characteristic classes, pulling back $\mathbb{Z}[e_1, \dots, e_r]$ gives us classes $c_k(V) \in H^{2k}(X)$, where e_i are the elementary symmetric polynomials.

Now recall that $\mathrm{Gr}(N, r)$ parameterizes surjections $\mathbb{C}^N \to L^r$, and if $X \xrightarrow{\varphi} [\mathfrak{S}]$, then we know $\mathfrak{S}$ is a locus where a section of L vanishes, and $V = \varphi^* L$. Then $C_r(V) = C_{top}(V)$. Thus if X is a complex manifold, TX has rank $\dim X$, so $c_{top}(TX)$ is the locus where a vector field vanishes, so

$$\int_{[X]} c_{top}(TX) = \chi(X)$$

Now c_1 parameterizes the locus where r generic sections have rank $r - 1$, or equivalently $\det V = 0$. Therefore $c_1(V) = c_1\left(\bigwedge^r V\right)$. Then c_k describes the locus where $r - k + 1$ sections have rank $r - k$. Then the splitting principle tells us that we can write $\sum c_k(V) = \prod (1 + x_i)$, where x_i are the Chern roots. This is because if $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence, then $c(V) = c(V_1)c(V_2)$.

Now we will consider the case of a line bundle $V = \mathcal{L}$. Then the curvature of $\mathcal{L}$ is a class in $H^2(X, \mathbb{R})$. Then we can trivialize away from the zero locus of a section $s = 0$, If we draw a loop in the total space of $\mathcal{L}$ over $X \setminus x$, then this may not be trivial. If the loop around D_2 trivial, then

$\int_{D_2} \text{curv} = 0$, but if there is a zero in the loop, then the integral is the total angle of rotation 2π times the order of vanishing. Thus $c_1 = \frac{\text{curv}}{2\pi}$ and $x_1 = \frac{t_1}{2\pi}$.

Now if $\pi\colon \widehat{X} \to X$ is the flag bundle of V, then $\pi^* V = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$. Then

$$\pi^*(c(V)) = \prod_{i=1}^{r} (1 + c_1(\mathcal{L}_i)).$$

Now we can use this to prove that

$$\left[\mathfrak{S}_{\ydiagram{1,1,1,1}}\right] = e_k(x_1, \ldots, x_r).$$

To see this, note that every symmetric polynomial of degree k is determined uniquely by its values on $x = (x_1, \ldots, x_k, 0, \ldots, 0)$. It is enough to consider vector bundles $V = V' \oplus \mathbb{C}^{r-k}$, where $V' = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$ is a sum of line bundles. Then if $s_1, \ldots, s_k$ are sections of $\mathcal{L}_1, \ldots, \mathcal{L}_k$, then all $s_1, \ldots, s_k$ must vanish, so $e_k(x) = x_1 x_2 \cdots x_k$.

Now we want to see that $[\mathfrak{S}_\lambda] = s_\lambda(x)$ using equivariant cohomology. Consider the action of $GL(N)$ on $\mathrm{Gr}(r, N)$. It is easy to see that the Schubert classes are preserved by the Borel subgroup. Now there is no difference if we take the Borel or the maximal torus A, so we have a map

$$\operatorname{Spec} H^\bullet_A(\mathrm{Gr}) \to \operatorname{Lie} A = \operatorname{Spec} H^\bullet_A(\mathrm{pt}) = H^\bullet(BA).$$

This is flat of length $\binom{N}{r}$ because we are taking the Chern roots $x_1, \ldots, x_r$ up to permutation. If $\mathbb{C}^n \twoheadrightarrow L$, then $c(L) \mid c(\mathbb{C}^n)$. Because $c(L) = \prod(1 + t x_i)$ while $c(\mathbb{C}^n) = \prod(1 + t a_i)$. Now if we consider the vector bundle

$$\mathbb{C}^n \hookrightarrow (\mathbb{C}^n \times EA)/A \to BA,$$

we see that $a_i \in H^\bullet(BA)$. But then we must have $x_i = a_j$ for all i and some j.

By flatness, we obtain a result called *equivariant formality*. If $a \in \operatorname{Lie}(A)$, the fiber above a is simply $H^\bullet(\mathrm{Gr}(r, N)^a)$. If $a = 0$, then we obtain $H^\bullet(\mathrm{Gr}(r, N))$, and if a is generic, then $\mathrm{Gr}(r, N)^a$ is a set of coordinate subspaces and is thus a disjoint union of $\binom{N}{r}$ points.

Now if $[\mathfrak{S}_\lambda] \in \mathbb{Z}[a_1, \ldots, a_N][x_1, \ldots, x_r]^{S_r}/\sim$, this has degree $|\lambda|$ in $x_1, \ldots, x_r$ and misses many fixed points. For example, on $\mathbb{P}^N$, the class $(1, *, \ldots, *)$ hits everything, while the class $(\underbrace{0, \ldots, 0}_{k}, 1, *, \ldots)$ misses anything with a 1 in the first k components. Thus we have a polynomial of degree k in x that vanishes on $x = a_1, \ldots, a_k$, so we obtain $(x - a_1) \cdots (x - a_k) =: p_k(x)$. Now we want a symmetric version of this, which is a Schur function in the

$$\frac{\det\big(p_{\lambda_i + N - i}(x_j)\big)}{\prod(x_i - x_j)}.$$

By an interpolation argument, this is exactly $[\mathfrak{S}_\lambda] \in H^\bullet_A(G)$. In ordinary cohomology, we set $a = 0$ and obtain the Schur function.

2.3 Minimizing norms

Let G be a compact Lie group. This is associated to a complex Lie algebra $\mathfrak{g}$. Then there exists a simply connected $\widetilde{G}$ such that $\mathsf{Lie}\,\widetilde{G} = \mathfrak{g}$. By the Lie theorem, we have a central extension

$$\begin{tikzcd}
\text{central} \arrow[r] \arrow[dr] & \widetilde{K} \arrow[r] \arrow[d] & K \arrow[d] \\
& \widetilde{G} \arrow[r] & G.
\end{tikzcd}$$

Therefore complex representations of G are the same as complex representations of K. In particular, G is linearly reductive.

On the other hand, by Peter-Weyl, $L^2(K)$ has a dense subset given by $\bigoplus_V \operatorname{End}(V)$, where V ranges over all irreducible representations. This is a finitely generated commutative Hopf algebra generated by matrix elements of a faithful representation. Therefore it is $\mathbb{C}[G]$ for some linear algebraic group with $\mathsf{Rep}\,G = \mathsf{Rep}\,K$. This also works when K is not connected. For example, if $K = (S^1)^n$ is abelian, then $G = (\mathbb{C}^\times)^n$.

Now we will prove that if G is a complex reductive group, there exists a compact $K \subset G$ such that $\mathsf{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C} = \mathsf{Lie}(G)$. In other words, every complex reductive group is a complexification of a compact Lie group. Now if G is a complex reductive group, then there exists an exact sequence

$$1 \to Z(G) \to G \to \operatorname{Ad}(G) \to 1.$$

Then $\operatorname{Ad}(G)$ is open in $\operatorname{Aut}(\mathfrak{g})$, and thus everything is algebraic. Therefore we have an embedding $G \hookrightarrow GL(\mathbb{C}^N) \supset U(N)$. Now all of the maximal compact subgroups are conjugate because any compact group preserves some Hermitian metric, so we hope for $K = G \cap U(n)$.

What we want to do is to minimize some norm. Consider $X = GL/G = \operatorname{Spec} \mathbb{C}[GL]^G$. This is a finitely generated algebra with a linear action of GL, so if we take a GL-invariant subspace that contains the generators, then there is a closed embedding $X \hookrightarrow V$ into a finite-dimensional GL-module V as a closed orbit. Now choose a U-invariant norm $\|-\|^2$ on V and minimize it on X. Set

$$X_{\min} = \{x \in X, \|x\| \text{ minimal }\}.$$

Proposition 2.3.1. $\dim_{\mathbb{R}} X_{\min} \leqslant \dim_{\mathbb{C}} X$.

Assuming this, choose $x = gG \in X_{\min}$, where $g \in GL$. Then

$$\dim_{\mathbb{R}} Ux \leqslant \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} GL - \dim_{\mathbb{C}} G$$

and therefore $\dim_{\mathbb{C}} G \leqslant \dim_{\mathbb{R}} g^{-1}Ug \cap G$. In fact, the real dimension of a compact subgroup is at most the complex dimension of G, and in fact any compact subgroup is *totally real* in G. Here a submanifold Y of a complex manifold X is totally real if $T_yT \cap iT_yT = 0$ for all $y \in Y$. In particular, we have $\dim_{\mathbb{R}} X_{\min} = \dim_{\mathbb{C}} X$ and $\dim_{\mathbb{R}} G \cap g^{-1}Ug = \dim_{\mathbb{C}} G$. Therefore it remains to prove the inequality.

First, note that $\|-\|^2$ is a *plurisubharmonic* or J-*convex* function.

Definition 2.3.2. Let X be a complex manifold and $f\colon X \to \mathbb{R}$ be a real function. Then f is *plurisubharmonic* if $(\bar{\partial}_i \partial_j f)$ is positive semidefinite and *strictly plurisubharmonic* if $(\bar{\partial}_i \partial_j f)$ is positive definite.

This is equivalent to the function $U \to X \xrightarrow{f} \mathbb{R}$ being subharmonic for all open $U \subset \mathbb{C}$, which is equivalent to the Laplacian being nonnegative (or strictly positive). To see this, note that

$$\overline{\partial}_w \partial_w (f \circ w) = \sum_{ij} \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j} \partial_w z_i \overline{\partial}_w \overline{z}_j$$

and so we have the product of the Hessian and the norm of $\frac{\partial z_i}{\partial w}$. Therefore the restriction of a plurisubharmonic function to a complex submanifold remains plurisubharmonic. Thus $\|-\|^2$ is clearly plurisubharmonic because its Hessian is the standard Hermitian metric.

Proposition 2.3.3. *The minima of any strictly plurisubharmonic function are totally real.*

Proof. Note that $T_x X_{\min}$ is in the kernel of the Hessian of f. Therefore it cannot contain any complex lines and is thus totally real. □

There are some variations. Let $X \subset \mathbb{C}^N$ be an affine variety (or a Stein manifold) and consider $f(x) = \|x - p\|^2$ for some fixed p. This is a Morse function for generic p. Then the negative index of $f(x)$ at any critical point x_0 is at most $\dim_{\mathbb{C}} X$, so the tangent space contains no complex lines. This implies that X has the homotopy type of a CW complex of real dimension at most $\dim_{\mathbb{C}} X$. This can be found in a 1959 paper of Andreoti and Frankel about the Lefschetz hyperplane theorem.

Theorem 2.3.4 (Lefschetz hyperplane theorem)**.** *Let Z be a smooth projective manifold of dimension n. Let $D = \mathcal{O}(1)$ be a hyperplane section. Then the restriction map $H^i(Z) \to H^i(D)$ is an isomorphism for $i \leqslant n-2$ and injective for $i = n-1$.*

Now let G be a complex reductive group and let $G \to GL(V)$ be a representation. Then we know that $\operatorname{Spec} \mathbb{C}[V]^G$ parameterizes closed G-orbits. For example, under the action of $\mathbb{C}^\times$ on $\mathbb{A}^2$ by (t, t^{-1}), the closed orbits have the form $x_1 x_2 = c \neq 0$ and the origin. Here, the only closed orbit in $x_1 x_2 = 0$ is the origin. Therefore, we have $\operatorname{Spec} \mathbb{C}[x_1, x_2]^G = \operatorname{Spec} \mathbb{C}[x_1 x_2]$. For any closed orbit we can look for minima of a K-invariant Hermitian metric $\|-\|^2$, where K is a compact real form of G. We may assume that $\|-\|^2$ has the form $c_1\|x_1\|^2 + c_2\|x_2\|^2$. We would like to prove the following result:

Theorem 2.3.5 (Kempf-Ness; Matsushita-Onishchik)**.** *The orbit $G \cdot v$ is closed if and only $\|-\|^2$ attains a minimum. When this is the case, the minima form a single K-orbit and there are no other critical points of $\|-\|^2$. Finally, the stabilizer of v is reductive.*[3]

Corollary 2.3.6. *We can identify V/G with the quotient of the critical loci of $\|-\|^2$ by K. Equivalently, the moment map $\mu \colon V \to \mathsf{Lie}(K)^*$ vanishes.*

Returning to our example, the critical locus of $\|-\|^2$ is the set $\left\{c_1|x_1|^2 - c_2|x_2|^2 = 0\right\}$, which is the union of two lines. This intersects each orbit exactly once, as desired.

It is clear that if the orbit is closed, the minimum of the norm function is attained. Now we will assume Kempf-Ness and prove Matsushita-Onishchik. Write $X_{\min}$ for the locus where the minimum of the norm is attained. Then we know that $\dim_{\mathbb{R}} X_{\min} \leqslant \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} H$, where H is the stabilizer of a point $v \in X_{\min}$. However, we can extend this to the inequality

$$\dim_{\mathbb{R}} K - \dim_{\mathbb{R}} K \cap H \leqslant \dim_{\mathbb{R}} X_{\min} \leqslant \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} H.$$

[3]Everything but the last statement is Kempf-Ness, the last statement is Matsushita-Onishchik

This implies that $\dim_{\mathbb{C}} H \leqslant \dim_{\mathbb{R}} K \cap H$. Of course $K \cap H$ is totally real, so the reverse inequality holds and thus H is the complexification of $K \cap H$, so it is reductive. In fact, all inequalities are equalities, so $\dim_{\mathbb{R}} X_{\min}$ is the dimension of any K-orbit contained in $X_{\min}$. This implies that $X_{\min}$ is smooth. We will also see later that $X_{\min}$ is connected, so it forms a single orbit.

Now for $g \in GL(n)$, write $gg^* \in GL(n)/U(n)$. This is some Hermitian metric, so $GL(n)/U(n)$ can be identified with the space of Hermitian metrics on $\mathbb{C}^n$. Then we know that

$$\mathsf{Lie}\, GL(n) = \mathsf{Lie}\, U(n) \oplus i\, \mathsf{Lie}\, U(n).$$

Recall that for $G \subset GL(n)$ reductive, there exists some metric $\|-\|^2$ such that $G \cap U(n)$ is a compact real form K. Then we have a morphism $G/K \to GL(n)/U(n)$, and the image is $\exp(i\, \mathsf{Lie}\, K)$. Then we know every element of $\mathsf{Lie}(K)$ is conjugate to an element of $\mathsf{Lie}(T)$ for a maximal torus $T \subset K$, Therefore we can represent G/K as kAk^{-1} for some K, where $A = \exp(i\, \mathsf{Lie}\, T) \subset G$. In particular, we have the decomposition $G = KAK$. In our previous example, $G = \mathbb{C}^*, K = U(1), A = \mathbb{R}_{>0}$.

This means we can write $g \cdot v = k_1 a k_2 v$, and only a changes $\|-\|^2$. Now if $\alpha \in \mathsf{Lie}\, A$, we know

$$e^{t\alpha} \sim \begin{pmatrix} e^{t\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\alpha_n} \end{pmatrix},$$

and therefore we can write

$$\left\|e^{t\alpha}x\right\|^2 = \sum e^{2t\alpha_i}|x_i|^2.$$

This is a convex nonnegative function. When the α_i are not all of the same sign, we have a unique maximum, but not necessarily when all $\alpha_i > 0$. We can consider the weights of the T-action on V in the characters of T. In the first case, the Newton polytope contains the origin, so our function is strictly convex and bounded below. In the second case, the Newton polytope does not contain the origin, so the closure of Av contains some other vector v' and thus is not closed. This implies $KAKv$ is not closed. There is a third case in which 0 is contained on the boundary on the Newton polytope. In this case, we have a semistable point.

In conclusion, of $\|-\|^2$ has a local minimum, then the orbit is closed and there are no other critical points. In addition, for any maximal torus $T \subset K$, the Newton polytope of the weights of the T-action on V contains the origin. This is called the **Hilbert-Mumford criterion.**[4]

Now consider the action of K on coadjoint orbits in $\mathsf{Lie}(K)^*$. Then every $\xi \in \mathsf{Lie}(K)$ defines a function which is the Hamiltonian for the vector field $\mathrm{ad}^*(\xi)$. Recall that the coadjoint orbits are Poisson manifolds. For a basic example of a Hamiltonian, the function $H = \frac{1}{2}(p^2 + q^2)$ gives us the flow which is rotation with velocity 1. Changing coordinatse, we see that $H = \frac{1}{2}\|z\|^2$ generates e^{it} and $H = \frac{m}{2}\|z\|^2$ generates e^{imt}. If we write $im = \xi \in \mathsf{Lie}(U(1))$, then $\alpha = \frac{\xi}{i} \in \mathsf{Lie}\, A$. Of course, we have $\left\|e^{t\alpha}z\right\|^2 = 2^{2mt}\|z\|^2$ and thus

$$H = \frac{1}{4}\,\frac{\partial}{\partial t}\left\|e^{\frac{\xi}{i}t}z\right\|^2\bigg|_{t=0}.$$

We obtain the same formula when ξ is a larger diagonal matrix, where $H = \frac{1}{2}\sum m_i|z_i|^2$. In this case, we have

$$\langle \mu(z), \xi \rangle = \frac{1}{4}\,\frac{\partial}{\partial t}\left\|e^{\frac{\xi}{i}t}z\right\|^2\bigg|_{t=0}.$$

[4]To see how this is related to the usual statement found in any text on geometric invariant theory, look at the lecture of Nicolás in my GIT notes at `https://math.columbia.edu/~plei/GIT.pdf`.

This implies that critical points of $\|-\|^2$ are the same as the zeroes of the moment map.

Now we can generalize this in several directions:

1. We can consider GIT quotients. For example, if we have a character $\xi\colon G \to \mathbb{C}^*$, then we can replace V/G by
$$V /\!\!/_{\chi} G = \operatorname{Proj} \bigoplus_{n \geqslant 0} (\mathbb{C}[V] \otimes \chi^n)^G.$$
Instead of $\mu = 0$, we can consider $\mu = \pm\, d\chi$.

2. Many moduli problems are fomally quotients by infinite-dimensional groups. Then the moment map equations (or minimization of $\|-\|^2$) are very useful and important PDEs. A very classical example of this is the work of Hitchin-Kobayashi-Donaldson-Uhlenbeck-Yau-... who studied stable holomorphic vector bundles. In fact, stable holomorphic bundles are precisely those with Hermitian Yang-Mills connection. These minimize $\|\text{curvature}\|^2_{L^2}$.

   For example, if we consider a curve C and a line bundle of degree 0, this line bundle lives in $\mathrm{Jac}_0(C)$. Then we are looking for flat unitary line bundles, which have the form $(\widetilde{C} \times \mathbb{C})/\pi_1(C)$ under a map $\pi_1(C) \to U(1)$. Therefore our line bundles are parameterized by
$$\operatorname{Hom}(\pi_1(C), U(1)) = \operatorname{Hom}(H_1(C, \mathbb{Z}), U(1)) = U(1)^{2g}.$$
   In fact, this is isomorphic to the Jacobian as a smooth manifold. More recently, there is the work of Chen-Donaldson-Sun in higher dimension.

2.4 Symmetric spaces

Now let $G \subset GL(V)$ be a complex semisimple group and K be a maximal compact. Then K is fixed under $\sigma(g) = (g^*)^{-1}$, which is an antiholomorphic automorphism of G. Now consider a G orbit of $\|-\|^2$, which is thet set $\{gg^* \mid g \in G\}$ of positive self-adjoint elements of G. Now $T_1(G) = \mathfrak{k} + i\mathfrak{k}$. Also, G/K has a G-invariant metric, and this is unique up to multiple if K is simple. Also, σ preserves the metric, so it acts as an isometry of G/K, which is a symmetry about $1 \in G/K$.

Definition 2.4.1. Let M be a Riemannian manifold. We say M is a *Riemannian symmetric space* if for all m, there exists some isometry σ_m fixing m such that σ_m acts by -1 on $T_m M$.

In particular, G/K is an example of a Riemannian symmetric space. There is a structure theory of Riemannian symmetric spaces, and the classification is essentially the classification of all real Lie groups. Here are some properties of symmetric spaces. If m, m' lie on a geodesic, then $\sigma_{m'}\sigma_m$ is translation by the distance between m', m. In particular, M is complete. In addition, the isometry group G of M acts transitively on M.

This implies that $M = G/H$ where H is the stabilizer of some point $m \in M$. In particular, H is compact because it is a closed subgroup of $O(T_m M)$. We also know that H commutes with σ_m, so $\sigma_m \in \mathrm{Aut}(G)$ is an inner automorphism. Now we have
$$T_1 G = \mathsf{Lie}(H) \oplus T_m M,$$
where $\mathsf{Lie}(H)$ are transformations in $O(T_m M)$ and $T_m M$ moves m. We see that $\mathsf{Lie}(H)$ is the $+1$ eigenspace and $T_m M$ is the -1 eigenspace. Therefore a Riemannian symmetric space M is the same as a Lie group G with a choice of automorphism σ and compact subgroup $(G^\sigma)_1 \subset H \subset G^\sigma$, where $M = G/H$.

Now consider the symmetric space S^n. If we choose $(x_1, \dots, x_n, x_{n+1}) = (0, \dots, 0, 1)$, the involution fixing the north pole is the map

$$(x_1, \dots, x_n, x_{n+1}) \mapsto (-x_1, \dots, -x_n, x_{n+1}),$$

which has two fixed points. This means that σ_m does not have to have a unique fixed point. Now if we conjugate $O(n+1)$ by $\sigma = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$, then the fixed locus is

$$O(n+1)^\sigma = \begin{pmatrix} O(n) & \\ & \pm 1 \end{pmatrix}.$$

Now if we take $H = O(n)$, we can remove the ± 1 if we take $S^n / \pm 1 = \mathbb{RP}^n$. Now the classification of Riemannian symmetric spaces is as follows:

1. Spaces of the form G/K, where G is a real Lie group and K is a maximal compact group. These have negative curvature.
2. Spaces of the form $K \times K/\Delta$, where K is a compact Lie group. Thus we have $M = K$ with the action of K by conjugation.
3. There are compact summetric spaces with positive curvature, and these are dual to the negative curvature case.

An important example of a symmetric space is the *hyperbolic space* $\mathbb{H}^n = SO(n,1)^+/SO(n)$. This is dual to $S^n = SO(n+1)/SO(n)$. The classification reduces again to the classification of root systems, which sit inside the class of finite reflection groups. Of course, finite reflection groups sit inside discrete reflection groups. In fact, the maximal symmetric spaces have $H = O(T_m M)$, and these are classified by

$$\mathsf{Lie}\, G = \mathfrak{so}(n) \oplus \mathbb{R}^n.$$

We also must have $[\mathfrak{so}(n), \mathbb{R}^n] \subset \mathbb{R}^n$ and $[\mathbb{R}^n, \mathbb{R}^n] \subset \mathfrak{so}(n)$ because $\mathfrak{so}(n)$ is the positive eigenspace of some involution and $\mathbb{R}^n$ is the negative eigenspace. This gives us an $\mathfrak{so}(n)$-equivariant map $\bigwedge^2 \mathbb{R}^n \to \mathfrak{so}(n)$. These are both irreducible and isomorphic, so these are multiplication by a real constant c. If $c > 0$, then $\mathfrak{g} = \mathfrak{so}(n+1)$ and $G/H = S^n$. If $c = 0$, then $\mathfrak{g} = \mathfrak{so}(n) \ltimes \mathbb{R}^n$ and $G/H = \mathbb{R}^n$, and if $c < 0$, then $\mathfrak{g} = \mathfrak{so}(n,1)$ and $G/H = \mathbb{H}^n$.

3

Semisimple Lie algebras

3.1 Reflection groups and root systems

Now we want to consider discrete isometry groups $\Gamma \subset \mathrm{Iso}(M)$ where M is one of $S^n, \mathbb{R}^n, \mathbb{H}^n$. This is in fact equivalent to the fact that for all $x \in M$, $\Gamma x \subset M$ is discrete. Therefore the subsets $\{M^\gamma\}_{\gamma \in \Gamma}$ are also discrete in M. To see this, consider any point m. Then the distance $d(m, \gamma m) \leqslant 2d(m, M^\gamma)$. Thus if M^γ is not discrete, the distances $d(m, \gamma m)$ accumulate. If Γ is generated by reflections, then the set of reflecting hyperplanes is discrete.

Now we will consider the *Dirichlet fundamental domain*

$$\mathcal{D} = \{y \mid d(y, x) \leqslant d(y, \gamma x) \text{ for all } \gamma \in \Gamma\}.$$

This is a locally finite intersection of half-spaces. We are mainly interested in the case where M is compact, so we will have a globally finite intersection. In the compact case, we may have to subdivide the boundaries $\partial\mathcal{D}$ such that each facet F corresponds to a particular $\gamma_F \in \Gamma$:

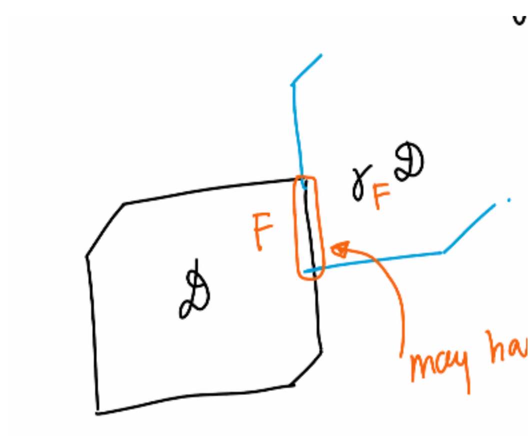

Figure 3.1: Subdivision of facet

In the globally finite case, we obtain $\gamma_{F_i} \in \Gamma$. These generate Γ and the relations are as follows: Note that $F' \coloneqq \gamma_F^{-1}(F)$ is a facet of $\mathcal{D}$. It is also easy to see that $\gamma_{F'} = \gamma_F^{-1}$. In the case of a reflection group, then $F = F'$ and $\Gamma_F^2 = 1$. In other words, $\{\gamma_F\}$ is a symmetric set of generators. We also have more complicated relations coming from codimension 2 strata. If the fixed hyperplanes of r_1, r_2 have angle $\frac{pi}{p}m$, then r_1r_2 is rotation by 2θ, so we obtain the relation $(r_1r_2)^m = 1$. In fact, if we have the square lattice, then $\gamma_1\gamma_2 = \gamma_2\gamma_1$.

Theorem 3.1.1. *The relations above are all relations between the γ_F. This means that*

$$\Gamma = \langle \gamma_F \mid \textit{above relations} \rangle .$$

Proof. Consider the exact sequence $1 \to \ker \to \widetilde{\Gamma} = \langle \Gamma_F \mid \text{relations} \rangle \to \Gamma \to 1$. Now consider

$$M \setminus (\text{codim} \geqslant 3) = \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{D} \setminus \text{codim} \geqslant 3).$$

But now we have a covering

$$\bigcup_{\gamma \in \widetilde{\Gamma}} (\gamma, \mathcal{D} \setminus \text{codim} \geqslant 3)/\text{relations} \to M \setminus \text{codim} \geqslant 3.$$

However, $M \setminus \text{codim} \geqslant 3$ is 1-connected, so this covering is an isomorphism. ☐

Therefore it remains to classify possible $\mathcal{D}$. First, however, we will consider a very classical example.

Example 3.1.2. Consider a triangle with angles $\frac{\pi}{m_{12}}, \frac{\pi}{m_{13}}, \frac{\pi}{m_{23}}$. Then we can consider the discrete reflection group $\Gamma = \left\langle s_1, s_2, s_3 \mid s_i^2 = 1, (s_is_j)^{m_{ij}} = 1 \right\rangle$. We have the following cases for where the triangle lives:

$$\frac{1}{m_{12}} + \frac{1}{m_{13}} + \frac{1}{m_{23}} \begin{cases} > 1 & S^2 \\ = 1 & \mathbb{R}^2 \\ < 1 & \mathbb{H}^2. \end{cases}$$

Thus in the first case Γ is finite, in the second case it has polynomial growth, and in the final case it has exponential growth. To define the growth, we need to consider the length function on Γ. Here, we count the number of hyperplanes separating $\mathcal{D}, \gamma\mathcal{D}$, and this is the *length* $\ell(\gamma)$, which is also the length of the shortest $\gamma = s_{i_1} \cdots s_{i_\ell}$. Some nice properties are that $\ell(\gamma) = \ell(\gamma^{-1})$ and that $\ell(\gamma s_i) = \ell(s_i\gamma) = \ell(\gamma) \pm 1$. Then we consider the function that takes x to the number of words with length less than x, and consider the growth of this function.

There are finitely many solutions to $\frac{1}{m_{12}} + \frac{1}{m_{23}} + \frac{1}{m_{12}} \geqslant 1$, and these correspond to platonic solids (spherical) or regular tesselations of the plane (Euclidean).

Now we are ready to classify discrete reflection groups acting on S^n or $\mathbb{R}^n$. We can embed $S^n \subset \mathbb{R}^{n+1}$, so the two cases are really the same. We want to impose that the normal vectors to the bounding hyperplanes span $(\mathbb{R}^n)^*$. We also assume that there is no partition of the normal vectors into two mutually perpendicular sets. We will carry out the classification using the connection to complex semisimple Lie algebras (or compact semisimple real Lie algebras). Consider the outer normals e_i and consider the *Coxeter* matrix $(e_i, e_j)_{ij}$. Now we know

$$Q = (e_i, e_j) = \begin{cases} 1 & i = j \\ -\cos\left(\frac{\pi}{m_{ij}}\right) & i \neq j \end{cases}.$$

Because this is a Gram matrix, it is positive semidefinite. Also, the diagonal emenets are all positive and all off-diagonal elements are nonpositive.

Lemma 3.1.3. *Together with indecomposability, the two properties imply that either the e_i are linearly independent (in which case $\mathcal{D}$ is a cone over a tetrahedron, which is the image of $\mathbb{R}^n_{\geqslant 0}$ under a linear map) or that* $\ker Q = \mathbb{R} \cdot v$*, where v has all positive coordinates.*

In the first case, $\mathcal{D}$ is the cone over a simplex and Γ is finite. In the second case, $\mathcal{D}$ is a simplex, so it is the image of $\{x_i \geqslant 0 \mid \sum x_i = 1\}$. In this case, we obtain an irreducible reflection group in $\mathrm{Iso}(\mathbb{R}^n)$.

Proof. Consider the matrix Q as a quadratic form. Then the kernal $\ker Q = \{v \mid Q(v,v) = 0\}$. For $v \in (v_1, \dots, v_n)$, denote $|v| = (|v_1|, \dots, |v_n|)$. Thus $Q(|v|,|v|) \leqslant Q(v,v)$. Thus if $Q(v,v) = 0$, we also have $Q(|v|,|v|) = 0$. Now suppose that some $v_i = 0$ and $Qv = 0$. Then $(Qv)_i = \sum_{j \neq i} q_{ij} v_j$. If there exist $i \neq j$ such that $q_{ij} \neq 0$, then $Q(|v|,|v|) < Q(v,v)$. This is impossible if Q is indecomposable, so no v_i can vanish for $v \in \ker Q$. Thus $\dim_{\mathbb{R}} \ker \leqslant 1$. Moreover, the kernel is closed under $v \to |v|$, so it must be spanned by a vector with $v_i > 0$. $\square$

This implies that the classification of reflection groups in $S^n, \mathbb{R}^n$ is the same as the classification of indecomposable positive-semidefinite Coxeter matrices. This is explicit and fun, and involves solving inequalities.

Now let $\mathfrak{k} = \mathsf{Lie}(K)$ for a compact group K with maximal torus T. Then write $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{t} = \mathsf{Lie}(T) \otimes_{\mathbb{R}} \mathbb{C}$. Then we have a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathrm{char}(T)} \mathfrak{g}_\alpha,$$

and we know $\mathfrak{g}_\alpha$ is 1-dimensional if and only if α is a root. Next, consider $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$. Then we write $h_\alpha = [e_\alpha, f_\alpha] \in \mathfrak{t}$, and this is unique. Then we have $(h_\alpha, h) = \alpha(h)$ and $[h_\alpha, e_\alpha] = 2e_\alpha$ and $[h_\alpha, f_\alpha] = -2e_\alpha$. Therefore we have an embedding $\mathfrak{sl}(2)_\alpha \subset \mathfrak{g}$, so an $SL(2)_\alpha \subset G$. Next, consider the map

$$\beta \mapsto \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha = s_\alpha(\beta).$$

We need $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ so that $s_\alpha(\beta)$ is in the lattice spanned by the roots. The reflections in the roots α generate a finite group W of reflections, which also preserves the root lattice. Now we may consider the Cartan matrix $\left(2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right)$. This is an integer matrix with 2 on the diagonal and nonpositive off diagonal entries. The Cartan matrix must be positive definite.

Now each reflecting hyperplane is the zero set of a linear function (which is a root). Now we can choose a fundamental domain, and this gives us a partition of roots into positive and negative roots. After reflection by some s_i, if $\beta \neq \alpha_i$ is a positive root, then $\beta - \alpha_i$ is still positive after α_i. Therefore every positive root has a sum of the form $\beta = \sum m_i \alpha_i$, where the α_i are the simple positive roots (corresponding the the faces). Moreover, this expression is unique (because the simple roots are independent as affine linear functions).

Now recall that for a semisimple Lie algebra $\mathfrak{g}$, we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha,$$

where α ranges over the roots. We know that $[h, e_\alpha] = \alpha(h)e_\alpha$, that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$, that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$, and that $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ if $\alpha + \beta \neq 0$. Also, there exist $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ such that $(e_\alpha, e_{-\alpha}) = 1$. If we write $h_\alpha \coloneqq [e_\alpha, e_{-\alpha}]$, then $(h_\alpha, h) = \alpha(h)$. Now our goal is go from root systems to Lie algebras.

Because every root is either positive or negative, and every positive root can be written uniquely as a sum of simple roots, we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- \qquad \mathfrak{n}_\pm = \bigoplus_{\alpha>0} \mathfrak{g}_{\pm\alpha}.$$

Note that if $\mathfrak{g}$ is finite-dimensional, then $\mathfrak{n}_+$ is nilpotent.

Proposition 3.1.4. *The nilpotent Lie algebra $\mathfrak{n}_+$ is generated by $e_i \colon e_{\alpha_i}$, where $\{\alpha_1, \ldots, \alpha_r\}$ are the simple positive roots.*

Proof. We show that if $\alpha+\beta$ is a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Consider the action of $(\mathfrak{sl}_2)_\alpha$ on $\mathfrak{g}_{n\alpha+\beta}$. Now each $\mathfrak{g}_{n_\alpha+\beta}$ is one-dimensional. Now recall that the weights of irreducible representations with respect to h_α are $\lambda, \lambda-2, \ldots, 2-\lambda, -\lambda$. But now recall that f_α decreases the weight by 2 and e_α increases the weight by 2, and so the action of $e_\alpha \colon \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+\alpha}$ is nonzero, as desired. $\square$

Corollary 3.1.5. *The Lie algebra $\mathfrak{g} = \langle h_i, e_i, f_i \rangle$ /relations, where $e_i = e_{\alpha_i}, f_i = e_{-\alpha_i}, h_i = h_{\alpha_i}$, where the α_i are the simple roots.*

The relations are given by

$$[h_i, e_j] = \alpha_j(h_{\alpha_i})e_j, \qquad \alpha_j(h_i) = 2\frac{(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}.$$

We will write $a_{ij} \coloneqq \alpha_i(h_i)$. We also have $[h_i, f_j] = -a_{ij}f_j$ and $[e_i, f_j] = \delta_{ij}h_i$.

Lemma 3.1.6 (Chevalley-Serre)**.** *For all i, j, we have $\mathrm{ad}(e_i)^{1-a_{ij}}e_j = 0$.*

For example, for the A_2 root system with Cartan matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, we have $\mathrm{ad}(e_1)^2 e_2 = \mathrm{ad}(e_2)^2 e_1 = 0$.

Proof. Write $\square = \mathrm{ad}(e_i)^{1-a_{ij}}e_j$. We will prove that if $[f_k, \square] = 0$, then $\mathrm{ad}(\mathfrak{g})\square$ is invariant under $\mathfrak{g}$. Then $\square$ is a lowest weight vector, so $\mathrm{ad}(\mathfrak{g})\square \subset \mathfrak{n}_+$ would be a proper submodule, contracticting the simplicity of $\mathfrak{g}$.

Now we need to prove that $[f_k, \square] = 0$. This is clear for $k \neq i, j$, so we need to check it for $k = i$ and $k = j$. For $k = i$, we have weights $a_{ij}, a_{ij}+2, \ldots, -a_{ij}$ for h_i (by the action of $(\mathfrak{sl}_2)_i$ on e_j), so $\square$ corresponds to the weight $-a_{ij}+2$, so we must have $\square = 0$ (because otherwise it would generate an infinite-dimensional module). Now we consider the case when $k = j$. Here, we have

$$\begin{aligned} [f_j, (\mathrm{ad}\, e_i)^{1-a_{ij}}e_j] &= -(\mathrm{ad}\, e_i)^{1-a_{ij}}h_j \\ &= (\mathrm{ad}\, e_i)^{-a_{ij}}[e_i, h_j] \\ &= (\mathrm{ad}\, e_i)^{-a_{ij}}(-a_{ij}e_i). \end{aligned}$$

Now if $a_{ij} < 0$, we see $[e_i, e_i] = 0$ and if $a_{ij} = 0$, we also have vanishing. $\square$

Theorem 3.1.7. *For a simple Lie algebra $\mathfrak{g}$, we have $\mathfrak{g} = \langle h_i, f_i, e_i \rangle$ /relations with the Cartan matrix $C = (a_{ij}) = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$. Moreover, for any matrix C such that $a_{ii} = 2$, $a_{ij} \leqslant 0$, and C is symmetrizable, then the matrix $\mathfrak{g}$ is simple and finite-dimensional if and only if C corresponds to a finite root system.*

Remark 3.1.8. Note that the conditions that $a_{ii} = 2, a_{ij} \leqslant 0$, and C is symmetrizable can be relaxed in principle.

3.2 Kac-Moody Lie algebras

Let $C = (a_{ij})$ be a matrix such that $a_{ii} = 2$, $a_{ij} \leqslant 0$, and $a_{ij} = 0$ if and only if $a_{ij} = 0$. Note this is weaker than being symmetrizable. Now we define the Lie algebra

$$\widetilde{\mathfrak{g}} = \langle e_i, f_i, h_i \rangle \Bigg/ \left(\begin{matrix} [h_i, e_i] = a_{ij} e_j \\ [h_j, f_j] = -a_{ij} f_j \\ [e_i, f_j] = \delta_{ij} h_i \end{matrix} \right).$$

Now we have the decomposition $\widetilde{\mathfrak{g}} = \mathfrak{h} \oplus \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_-$, where $\widetilde{\mathfrak{n}}_+$ is the free Lie algebra generalized by the e_i, whose universal enveloping algebra is a free associative algebra. Note if $C = (2)$, then $\widetilde{\mathfrak{g}} = \mathfrak{sl}_2$. Also, we have

$$\widetilde{\mathfrak{n}}_+ = \bigoplus_{\alpha = \sum m_i \alpha_i \geqslant 0} \mathfrak{g}_\alpha,$$

so in fact all roots are already either positive or negative. Here, note that

$$[f_i, \mathfrak{n}_+] \subset \mathbb{C} h_i \oplus \mathfrak{n}_+,$$

and we also know that

$$[f_1, [e_1, e_2]] = -[h_1, e_2] = -a_{12} e_2.$$

The previous argument for $\square = \mathrm{ad}(e_i)^{1-a_{ij}} e_j$ shows that it is a lowest weight vector in $\mathfrak{n}_+$. Now we know that $\mathfrak{n}_+ \supseteq \mathrm{ad}(\widetilde{\mathfrak{g}})\square$. More generally, we can consider the maximal submodule of $\mathrm{ad}(\widetilde{\mathfrak{g}})$ (which cannot contain the e_i).

Definition 3.2.1. We define the *Kac-Moody algebra* $\mathfrak{g}_{KM}$ to be the quotient of $\widetilde{\mathfrak{g}}$ by the maximal submodules in $\mathfrak{n}_+$ and $\mathfrak{n}_-$.

Theorem 3.2.2 (Gabber-Kac)**.** *If C is symmetrizable, then the maximal submodules in $\mathfrak{n}_+, \mathfrak{n}_-$ are generated by the Chevalley-Serre relations.*

Theorem 3.2.3. *If $\dim \mathfrak{g}_{KM} < \infty$, then C corresponds to a finite reflection group.*

Remark 3.2.4. There are even larger generalizations of this, for example Borcherds-Kac-Moody Lie algebras and other classes of Lie algebras.

Now Chevalley-Serre tells us that for large enough M, $\mathrm{ad}(e_i)^M$ applied to any generator vanishes. Thus for any $x \in \mathfrak{g}_{KM}$, there exists $M = M(x)$ such that $\mathrm{ad}(e_i)^M x = \mathrm{ad}(f_i)^M x = 0$. I particular, any $x \in \mathfrak{g}_{KM}$ is contained in a finite-dimensional $\mathfrak{sl}(2)_i = \langle e_i, f_i, h_i \rangle$-module.

By definition, a $\mathfrak{g}_{KM}$-module is *integrable* if for all $\mathfrak{sl}(2)_i$, the module decomposes as a sum of finite-dimensional modules. Thus, in this language, the adjoint representation of $\mathfrak{g}_{KM}$ is integrable. Now we can integrate each $\mathfrak{sl}(2)_i$-module to $SL(2)_i$, and the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ acts by reflection r_i. Then we see that

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i,$$

and the r_i generate the Weyl group of $\mathfrak{g}_{KM}$. In particular, it permutes the set of roots.

Recall that when we had an invariant quadratic form, we had the identity

$$(h_i, h_j) = (h_i, [e_j, f_j]) = ([h_i, e_j], f_j) = a_{ij}(e_j, f_j).$$

Therefore there exists an invariant quadratic form if and only if there exists a diagonal D such that CD is symmetric. Now note that $\widetilde{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, and $\mathfrak{n}_+, \mathfrak{n}_-$ are dual with respect to the quadratic form. In fact, we obtain expressions of the form

$$(\mathrm{ad}(e_i)e_j, f_k) = -(e_j, \mathrm{ad}(e_i)f_k).$$

Now if we consider the ideal of relations $\mathfrak{r}_+ \subset \mathfrak{n}_+$, then we note that

$$(r, \mathrm{ad}(f_i)*) = -(\mathrm{ad}(f_i)r, *) = 0.$$

Now note that any invariant bilinear form on $\mathfrak{g}$ is nondegenerate because $\mathfrak{g}^\perp$ is an ideal which does not intersect $\mathfrak{h}$, and thus vanishes. If $a_{ii} \neq 0$, then we can assume $a_{ii} = 2$, and therefore we have a subalgebra $\mathfrak{sl}(2)_i = \langle e_i, f_i, h_i \rangle$, then for $j \neq i$, e_j is the lowest weight vector for $\mathfrak{sl}(2)_i$ with weight a_{ij}. If $a_{ij} \in 0, -1, -2, \ldots$, then it is possible that the $\mathfrak{sl}(2)_i$-module is finite-dimensional. Kac-Moody theory is about *integrable modules*. Also, recall that $\mathrm{ad}(e_i)^{1-a_{ij}}e_j = 0 \in \mathfrak{g}$, and by Gabber-Kac this is the complete list of relations. Also recall the adjoint representation is integrable.

Example 3.2.5. Consider $C = \left(\begin{smallmatrix} 2 & a_{12} \\ a_{21} & 2 \end{smallmatrix}\right)$. Note that if we want $\det C > 0$, we can only have $(a_{12}, a_{21}) = (-1,-1), (-2,-1), (-3,-1)$. Now the three cases correspond to the Lie algebras $\mathfrak{sl}_3, \mathfrak{sp}(4) = \mathfrak{so}(5), G_2$. In the case where $\det C = 0$, or $(a_{12}, a_{21}) = (-2,-2), (-4,-1)$, then we need to enlarge $\mathfrak{h}$ to have $\alpha_j(h_i) = a_{ij}$. The extra elements in $\mathfrak{h}$ will be central, and we will need to consider a central extension of $\mathfrak{sl}(2) \otimes \mathbb{C}[t^{\pm 1}]$. Here, we will have

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \qquad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}.$$

Now we have

$$\mathfrak{h} = \mathbb{C}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \oplus \mathbb{C} \cdot \mathrm{central} \oplus \mathbb{C} \cdot t\frac{d}{dt}.$$

These are called affine Lie algebras, and these correspond to reflection groups in $\mathbb{R}^n$. Now consider the case when $\det C < 0$. This is much more complex, and suppose $a_{12} = a_{21} = -m$, where $m > 4$. Then the quadratic form $\|(x,y)\|^2 = 2x^2 - 2mxy + 2y^2$ has signature $(1,-1)$. Now after we kill the Serre relations, we have our Weyl group $W = \langle r_1, r_2 \rangle$, where

$$r_1 = \begin{pmatrix} -1 & m \\ 0 & 1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & 0 \\ m & -1 \end{pmatrix}.$$

It is easy to see that $r_2(1,0) = (1,m), r_2(0,1) = (0,-1), r_1(0,1) = (m,1), r_1(1,0) = (-1,0)$. Now the vectors with $\|-\|^2 = 2$ form a hyperbola, and it is easy to see that r_1, r_2 are translations on the hyperbola. Note that these are solutions to $x^2 - mxy + y^2 = 1$ and are units in $\mathbb{Q}(\sqrt{m^2-4})$. Also, note that

$$\frac{m + \sqrt{m^2-4}}{2} = m - 1 + \frac{1}{1 + \frac{1}{m-2+\cdots}}$$

is periodic and the coefficients are $1, m-2$.

The fact that there are no roots of $\widetilde{\mathfrak{g}}$ in $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\leqslant 0} \setminus \{(0,1),(1,0),(0,0)\}$ and the W-action implies that there are no roots with $\|-\|^2 > 2$. Now for $\beta = m_1\alpha_1 + m_2\alpha_2$, we would like to minimize the height $m_1 + m_2$. Now if $(r_i, \beta) = \beta - (\alpha_i, \beta)\alpha_i$, then we can decrease the height unless $\beta = \alpha_i$. Now in every W-orbit of a root β, either there is a simple root α_i, so $\|\beta\|^2 = 2$, or there is a root with $(\beta, \alpha_i) > 0$ for all i. Now roots are either real (which means $\dim \mathfrak{g}_\beta = 1$) or imaginary (which implies exponential growth in the cone bounded by lines of slope $m/2, 2/m$). In general, if the signature of our matrix $(n-1,1)$, there is a similar picture, where we have a light cone (notation stolen from physics, all hyperboloids will asymptotically converge to this), and inside we have the funramental domain. Unfortunately, there is only a compact fundamental domain in low dimension, so the theory fails in high dimension. If the signature is worse, then the theory gets worse. However, every root is either real or belongs to the orbit of a fundamental domain (which is the set of vectors such that $(\beta, \alpha_i) \leqslant 0$ for all but finitely many roots).

It should be clear that $\dim \mathfrak{g} < \infty$ if and only if $C > 0$. This explains why we end up with infinite-dimensional Lie algebras when we attempt classification.

3.3 Integrable representations of Kac-Moody Lie algebras

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ be a symmetrizable Kac-Moody Lie algebra and $C = (a_{ij})$ be the Cartan matrix. Then the Weyl group $W = \langle r_i \rangle$ acts on $\mathfrak{h}, \mathfrak{h}^*$.

Now if M is a module and $v \in M$, by definition v is a *highest weight vector* if $h_i v = \lambda(h_i) v$ for some $\lambda \in \mathfrak{h}^*$ and $e_i v = 0$ for all i. Some examples of highest weight modules are *Verma modules* M_λ, which is the free module generated by a highest weight vector v_λ. Equivalently, we can write

$$M_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda.$$

This is not even close to being integrable (just note we can act infinitely many times by f_i). To make M_λ integrable, we need $\lambda(h_i) = 0, 1, \dots$ to be a nonnegative integer and $f_i^{\lambda(h_i)+1} v = 0$. This gives us a candidate for an irreducible integrable module L_λ with highest weight[1] λ. We will see later that this is indeed irreducible by the classification of irreducible finite-dimensional modules. We will also compute the character using the Weyl-Kac formula.

We will now consider the Casimir element. First suppose that $\dim \mathfrak{g} < \infty$ and let x_i, y_i be dual bases for the Killing form $(-,-)$. Then the Casimir element is

$$C = \sum x_i y_i \in \mathcal{U}(\mathfrak{g}).$$

Now because $\mathfrak{g}_\alpha$ is dual to $\mathfrak{g}_{-\alpha}$, we have

$$C = \sum h_i^2 + \sum_{\alpha > 0} (e_\alpha f_\alpha + f_\alpha e_\alpha).$$

This acts by a scalar in M_λ because it commutes with all f_i, and so

$$Cv = \sum \lambda(h_i)^2 + \sum_{\alpha > 0} \lambda(h_\alpha).$$

For any $h \in \mathfrak{h}$, we have $(h, h_\alpha) = \alpha(h)(e_\alpha, f_\alpha)$. Now consider

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \in \mathfrak{h}^*.$$

Lemma 3.3.1. *For all* r, $r_i(\rho) = \rho - \alpha_i$. *In other words,* $\rho(h_i) = 1$.

Proof. Note that r_i permutes the set of positive roots distinct from α_i. Therefore $r_i(\alpha_i) = -\alpha_i$, and the desired result follows by splitting the expression for ρ. □

[1] Allegedly there is a physical interpretation for the existence of a highest weight vector.

Remark 3.3.2. Some analog of this also makes sense if $\dim \mathfrak{g} = \infty$.

We can take any other function of the set $\{\alpha > 0 \mid \alpha \neq \alpha_i\}$, such as $x \mapsto 1 - e^{-x}$. Then the product

$$e^{\rho} \prod_{\alpha>0} (1 - e^{-\alpha})$$

is a Laurent series in $e^{-\alpha_i}$ which is often analytically convergent. Under the action of r_i, we obtain

$$e^{\rho - \alpha_i}(1 - e^{\alpha_i}) \prod_{\alpha \neq \alpha_i} (1 - e^{-\alpha}) = -e^{-\rho} \prod (1 - e^{-\alpha}).$$

Therefore this product is W-anti-invariant. Note that the product $\prod_{\alpha>0} \frac{1}{1-e^{-\alpha}}$ is the character of $h^{\mathfrak{h}}$ acting on $\mathcal{U}(\mathfrak{n}_-)$. Here, the operators $1, f_\alpha, \dots$ have weight $1, e^{-\alpha}.e^{-2\alpha}, \dots$, and taking the sum, we obtain the expression. Therefore, the character of a Verma module is

$$\frac{e^{\lambda}}{\prod_{\alpha>0}(1 - e^{-\alpha})}.$$

Theorem 3.3.3 (Weyl-Kac)**.** *The character of the integrable representation* L_λ *is given by*

$$\sum_{w \in W} w \cdot \left(\frac{e^{\lambda}}{\prod_{\alpha>0}(1 - e^{-\alpha})} \right) = \frac{1}{\prod_{\alpha>0}(1 - e^{-\alpha})} \sum_{w} (-1)^{w} e^{w(\lambda+\rho)-\rho}.$$

Returning to the Casimir element, we now write

$$C = \sum h_i h^i + \sum_{\alpha>0} h_\alpha + 2 \sum_{\alpha>0} e_\alpha f_\alpha.$$

For infinite-dimensional modules with highest weight, this expression makes sense because the final term in the sum is locally finite. Here, h^i is the dual basis to the h_i. This still commutes with the f_i. Therefore C acts on M_λ by $(\lambda, \lambda) + 2(\lambda, \rho) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho)$.

To prove this, consider the exact sequence

$$0 \to M \to M_\lambda \to L_\lambda \to 0.$$

Here, M is some highest-weight module, so it has a surjection from $\bigoplus M_{\mu_i}$. Thus there exists a resolution of L_λ by Verma modules of the form

$$\cdots \to \bigoplus M_{\nu_j} \to \bigoplus M_{\mu_i} \to M_\lambda \to L_\lambda \to 0.$$

By the action of the Casimir element, we have $\|\mu_i + \rho\|^2 = \|\lambda + \rho\|^2$. In addition, the character of L_λ is W-invariant by integrability. Together, we obtain that the character of L_λ is a sum of the form

$$\sum_{\mu} c_\mu \mathrm{char}(M_\mu),$$

where $\lambda - \mu$ is either a sum of positive roots or zero, $c_\lambda = 1$, $\|\mu + \rho\|^2 = \|\lambda + \rho\|^2$, c_μ is anti-invariant under the action of W by $w \cdot \mu = w(\mu + \rho) - \rho$.

Lemma 3.3.4. *Every* μ *such that* $\lambda - \mu$ *is a sum of roots can be brought by the* $W\cdot$ *action to the positive cone (where* $(\mu, \alpha_i) \geqslant 0$*). The positive cone is also called the dominant cone.*

Proof. Choose w such that in the expression $\lambda+\rho-w(\mu+\rho)=\sum m_i\alpha_i$, $\sum m_i$ is minimal. If $(\mu,\alpha_i)<0$, then we can apply r_i and decrease m_i. $\square$

Lemma 3.3.5. *If* $(\lambda,\alpha_i)>0,(\mu,\alpha_i)\geqslant 0$, $\lambda\geqslant\mu$, *and* $(\lambda,\lambda)=(\mu,mu)$, *then* $\lambda=\mu$.

Proof of this result is left as an exercise to the reader.
As a consequence, we have:

1. If $c_\mu\neq 0$, then the intersection of $W\cdot\mu$ with the dominant cone is simply λ, and thus $c_\mu(-1)^w\delta_{W\cdot\lambda}$.
2. In the exact sequence $\bigoplus\mathcal{U}(\mathfrak{n}_-)f_i^{\lambda(h_i)+1}v\to M_\lambda\to\text{coker}\to 0$, the first term is simply $\bigoplus_{r_i}M_{r_i\cdot\lambda}$. Now the cokernel is irreducible because it is integrable and therefore there is no room for other singular vectors. In fact, this can be continued to a Bernstein-Gelfand-Gelfand resolution of L_λ by
$$\cdots\to\bigoplus_{\ell(w)=k}M_{w\cdot\lambda}\to\cdots$$

As a corollary of Weyl-Kac we obtain

Corollary 3.3.6 (Denominator identity)**.** *Apply the formula to* $L_0=\mathbb{C}$ *to obtain*

$$1=\frac{e^{-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})}\sum_w(-1)^w e^{w\rho}$$

and rearrange to obtain

$$\sum_w(-1)^w e^{w\rho}=e^\rho\prod_{\alpha>0}(1-e^{-\alpha}).$$

For example, when $\mathfrak{g}=\mathfrak{sl}(n)$, the positive roots are of the form $(0,1,\ldots,-1,0)$, so $e^\alpha=\frac{x_i}{x_j}$. Then we have $\rho=((n-1)/2,\ldots,(1-n)/2)$, and therefore we have

$$x^{\cdots}\prod_{i>j}\left(1-\frac{x_j}{x_i}\right)=\sum_{s\in S_n}(-1)^s s(x^{\cdots}).$$

This has the more familiar form

$$\prod_{i>j}(x_i-x_j)=\sum_{s\in S_n}(-1)^s s\cdot(x_1^{n-1}\cdots x_n^0).$$

For $\widehat{\mathfrak{sl}}_2$, this gives us the *Jacobi triple product identity* for theta functions

$$\vartheta(x;q)=\sum_n(-1)^n x^n q^{n^2/2}=\prod_{n>0}(1-q^n)\prod_{n>0}(1-xq^n)\prod_{n\geqslant 0}(1-x^{-1}q^n).$$

3.4 Affine Lie Algebras

Let $\Gamma=\langle r_i\rangle$ be a group of isometries in $\mathbb{R}^n$. For example, if r_1,r_2,r_3 are reflections along the sides of an equilateral triangle, we have the $\widehat{A}_2$ group. Now isometries of $\mathbb{R}^n$ embed in isometries of $\mathbb{H}^{n+1}$ because we can embed $\mathbb{R}^n$ as a horocycle. Now consider the group $\widehat{A}_1$ generated by r_1,r_2 with r_1r_2 of infinite order. If we consider spheres S^n in $\mathbb{H}^{n+1}$, we have an action of $SO(n)$ on

S^n which commutes with the action of $SO(n,1)$ on $\mathbb{H}^{n+1}$. Note that $SO(n)$ stabilizes a vector of negative norm. In the limit as the radius approaches ∞, the curvature approaches 0, so we obtain $\mathbb{R}^n$.

Now the semidirect product $SO(n-1) \ltimes \mathbb{R}^n$ stabilizes a null vector, so it acts on $\mathbb{R}^n \subset \mathbb{H}^{n+1}$. Now if we fix that $SO(n,1)$ preserves the form $x_0x_n + x_1^2 + \cdots + x_{n-1}^2$, then if we consider the light cone where $\|x'\|^2 + x_0x_n = 0$, then our isometry group acts in a way such that orbits are like parabolas. In the case of $\widehat{A}_1$, the orbit of a single point in $\mathbb{R}^1$ will lie on a parabola when we embed in $\mathbb{R}^{2,1}$.

Now our reflection group $\widehat{A}_1$ could have Cartan matrix $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 2 & -4 \\ -1 & 2 \end{smallmatrix}\right)$. The first corresponds to a central extension $\widehat{\mathfrak{sl}}(2)$ of $\mathfrak{sl}(2) \otimes \mathbb{C}[t,t^{-1}]$ while the second corresponds to a twisted loop algebra. Here, if $\mathfrak{g}$ is a Lie algebra and σ is an automorphism of order m, then we may consider the subalgebra

$$\{f(t) \mid f(t\zeta_m) = \sigma f(t)\} \subset \mathfrak{g}[t,t^{-1}].$$

For example, note that $\mathfrak{sl}(2)[t^{\pm 1}]$ is generated by the matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_0 = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, f_0 = \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}.$$

Then $[e_1,f_1] = h_1 = -[e_0,f_0]$. If we define $h_0 := [e_0,f_0]$, then for $C = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$, $h_0 + h_1$ is contained in the kernel of the map

$$0 \to \mathbb{C}(h_0 + h_1) \to \langle h_1, e_0, f_0, h_1, f_1, e_1 \rangle \mapsto \mathfrak{sl}(2)[t^{\pm 1}] \to 0.$$

Now note that if we set $\deg e_0 = (1,0), \deg e_1 = (0,1)$, then the Lie algebra $\langle h_0, e_0, f_0, h_1, f_1, e_1 \rangle$ is graded by $\mathbb{Z}^2$. Because C is degenerate, only one of these two gradings is internal. To restore the grading, we introduce the element $D := t\frac{d}{dt}$, where $De_0 = e_0$ and $Df_0 = -f_0$. Now we set

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha, \qquad C' = \begin{pmatrix} 2 & -2 \\ -2 & 2 \\ 1 & 0 \end{pmatrix}$$

where we set $h_2 = D$. Now we have a map

$$0 \to \mathbb{C}(h_0 + h_1) \to \langle h_0, h_1, h_2, e_0, f_0, e_1, f_1 \rangle \to \mathbb{C}t\frac{d}{dt} \ltimes \mathfrak{sl}(2)[t^{\pm 1}] \to 0.$$

Now because there are three simple roots, the matrix of the invariant bilinear form is

$$\begin{pmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and this has signature $(2,1)$. In the basis of $C = (h_0 + h_1), h_1, D$, this takes the form

$$\begin{pmatrix} & & 1 \\ & 2 & \\ 1 & & \end{pmatrix},$$

which is precisely of the form $x_0x_{n+1} + x_1^2 + \cdots + x_n^2$.

Now we would like to describe the central extension in terms of matrices. Here, we have

$$[e_0, f_0] = h_0 = C - h_1 = \left[\left(\begin{smallmatrix} 0 & 0 \\ t & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & t^{-1} \\ 0 & 0 \end{smallmatrix}\right)\right] + C \cdot 1,$$

then we can write $1 = \operatorname{Res}\operatorname{tr} P'(t)Q(t) = \operatorname{Res}\left(\frac{d}{dt}P(t), Q(t)\right)$ if we set $e_1 = P(t), f_1 = Q(t)$. Using integration by parts, this is skew-symmetric. Now we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\begin{pmatrix} 0 & t^n \\ 0 & 0 \end{pmatrix} \oplus \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\begin{pmatrix} 0 & 0 \\ t^n & 0 \end{pmatrix} \oplus \bigoplus_{n\neq 0} \mathbb{C}\begin{pmatrix} t^n & \\ & -t^n \end{pmatrix}.$$

Now the eigenvalues of (C, h_1, D) on each factor are $(0, 2, n), (0, -2, n), (0, 0, n)$. The values $x + n (n \geqslant 0), -x + n (n \geqslant 1), n (n \geqslant 1)$ are positive for $x \in (0, 1)$. Now if we define

$$\Delta \coloneqq \prod_{\alpha>0} (1 - e^{-\alpha}),$$

then if we set $\exp(0, 0, -1) = q, \exp(0, 1, 0) = x$, we obtain

$$\Delta = (1 - x^{-1}) \prod_{n>0} (1 - q^n x^{-1})(1 - q^n)(1 - q^n x).$$

This is simply the genus 1 theta function $\vartheta(x)$, which has the property that $\vartheta(1) = 0$ and that

$$\vartheta(qx) = (1 - q^{-1}x^{-1}) \prod_{n>0} (1 - q^{n-1}x^{-1})(1 - q^n)(1 - q^{n+1}x) = \vartheta(x)\frac{1 - q^{-1}x^{-1}}{1 - qx} = -q^{-1}x^{-1}\vartheta(x).$$

Therefore ϑ is a section of a degree 1 line bundle over $\mathbb{E} \coloneqq \mathbb{C}^\times/q^{\mathbb{Z}}$. Note that Δ converges analytically if $|q| < 1$. Because $-q^{-1}x^{-1}$ is invertible, we may use it as a clutching function. By Riemann-Roch, ϑ only has a single zero at $x = 1$.

Another way to write solutions of the equation $\vartheta(qx) = -q^{-1}x^{-1}\vartheta(x)$ is using the form $\sum x^m q^{m^2/2}$. Here we want to think of $\frac{m^2}{2}$ as $\binom{m}{2}$, so if we apply $x \mapsto qx$, we have

$$\sum x^m q^{\frac{m(m-1)}{2}} \mapsto \sum_m x^m q^{\frac{m(m+1)}{2}} = x^{-1} \sum_m x^m q^{\frac{m(m-1)}{2}},$$

and thus we have $\widetilde{\vartheta}(qx) = x^{-1}\widetilde{(\vartheta)}(x)$. Therefore $\vartheta(x) \approx \text{const}\widetilde{\vartheta}(-qx)$, and thus everything is unique up to a multiple section of a line bundle on $\mathbb{E}$. Classically, this is the Jacobi product formula. As a sum, we really have the sum in the Weyl-Kac formula, because

$$\Delta = \prod_{\substack{\alpha>0 \\ n\geqslant 0}} (1 - q^n x^\alpha) \prod_{\substack{\alpha<0 \\ n>0}} (1 - q^n x^\alpha) \prod_{n>0} (1 - q^n)^{\operatorname{rank} \mathfrak{g}}.$$

This also satisfies some q-difference equation, so is a section of a Line bundle of $\mathbb{E}^{\operatorname{rank} \mathfrak{g}}$. In general, we have

$$\Delta = \sum_{W_{\text{finite}} \ltimes \mathbb{Z}^r} (-1)^{\text{sign}} q^{\text{quadratic}} x^{\text{something}},$$

and this can be rewritten as an expression in the characters of $\mathfrak{g}_{\text{finite}}$ is we sum over W first and as a theta function if we sum over the lattice first.