Duality and projectives in category $\mathcal{O}$

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This talk is based on Chapter 3 of Humphreys' *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*.

1 Introduction

In Fan's talk, we saw that category $\mathcal{O}$ is an abelian category with many nice properties. In this talk, we will discuss two important pieces of structure on $\mathcal{O}$. The first is duality: in $\mathcal{O}$, we can take duals of representations, even infinite-dimensional ones. The second is the existence of enough projectives, allowing us to do homological algebra in $\mathcal{O}$.

2 Duality in $\mathcal{O}$

A finite-dimensional representation M of a semisimple Lie algebra $\mathfrak{g}$ has a dual representation $M^\vee$ with $\mathfrak{g}$-action

$$(xf)(v) = -f(xv).$$

Since $M^{\vee\vee} \cong M$, this defines a contravariant self-equivalence on the category of finite-dimensional representations. Unfortunately, duals, as defined above, aren't as nice for infinite-dimensional representations, e.g. $M^{\vee\vee} \ncong M$. Moreover, we want duals of representations in $\mathcal{O}$ to be in $\mathcal{O}$; the naive duals may not satisfy the required finiteness conditions.

It turns out that we can still define duals in $\mathcal{O}$, if we use more structure of $\mathfrak{g}$. Recall that any $\mathfrak{g}$ has an anti-involution τ (the **transpose**) defined as follows. If we present $\mathfrak{g}$ as the Lie algebra generated by $x_\alpha, y_\alpha, h_\alpha$ with the Weyl and Serre relations, we get an anti-involution

$$\begin{aligned} \tau : \mathfrak{g} &\to \mathfrak{g} \\ x_\alpha &\mapsto y_\alpha \\ y_\alpha &\mapsto x_\alpha \\ h_\alpha &\mapsto h_\alpha \end{aligned}$$

because the Weyl and Serre relations are symmetric in the x's and y's. We can then define the dual of $M = \bigoplus_\lambda M_\lambda \in \mathcal{O}$. The underlying vector space is $M^\vee := \bigoplus_\lambda M_\lambda^\vee$. The action is defined using τ:

$$(xf)(v) = f(\tau(x)v)$$

for a weight vector $f \in M_\lambda^\vee$. Since the weight spaces M_λ are finite-dimensional, $M^{\vee\vee} \cong M$. Thus, the duality functor on $\mathcal{O}$ is a contravariant self-equivalence.

Since the sizes of the weight spaces are preserved by duality, duality preserves formal characters. Moreover, duality takes submodules to quotients and vice versa, so it preserves simple modules. Thus, $L(\lambda)^\vee \cong L(\lambda)$.

Finally, it can be shown that τ preserves $Z(\mathfrak{g})$. Thus, duality preserves central characters:

$$(M^\vee)^\chi \cong (M^\chi)^\vee$$

for any central character χ.

3 Dominant and antidominant weights

We now move on to the second part of this talk, which is about projectives. To prove that $\mathcal{O}$ has enough projectives, we first need to introduce dominant and antidominant weights, which are important in their own right. Recall that when we restrict to integral weights and consider W-orbits (say, under the dotted action), we can always pick a weight $\lambda_{\max}$ that has maximal inner products with the positive roots ($\lambda_{\max} + \rho$ is dominant) and weight that has minimal inner products with the positive roots ($\lambda_{\max} + \rho$ is antidominant). Unfortunately, when we consider general weights in $\mathfrak{h}^*$, it is not clear how to pick representatives of the W-orbits, so we need to do things a bit differently. Instead of considering W-orbits, for each $\lambda \in \mathfrak{h}^*$, we consider a subgroup $W_{[\lambda]} \subset W$ and the corresponding orbit $W_{[\lambda]} \cdot \lambda$, for which we can pick optimal representatives.

For each $\lambda \in \mathfrak{h}^*$, define

$$W_{[\lambda]} := \{w \in W | w\lambda - \lambda \in \Lambda\}.$$

The $W_{[\lambda]}$-orbits $W_{[\lambda]} \cdot \lambda$ are refinements of the linkage classes from Chapter 1. For instance, the composition factors $L(\mu)$ of $M(\lambda)$ must satisfy $\mu \in W_{[\lambda]} \cdot \lambda$. Eventually (in Chapter 4), we will see that the $W_{[\lambda]} \cdot \lambda$ are the blocks of $\mathcal{O}$. It can be shown that $W_{[\lambda]}$ is the Weyl group of the root system

$$\Phi_{[\lambda]} := \{\alpha \in \Phi | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\},$$

so it enjoys all the nice properties of Weyl groups, which we will use shortly.

Within each $W_{[\lambda]} \cdot \lambda$, there are unique dominant and antidominant weights. We must, of course, define these terms for general weights $\lambda \in \mathfrak{h}^*$. λ is **dominant** if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in \Phi^+$, and λ is **antidominant** if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi^+$. For integral weights, this does not match the usual notion of dominance/antidominance but does when we shift the weights by ρ.

Proposition 1. For a weight $\lambda \in \mathfrak{h}^*$, let $W_{[\lambda]}$ and $\Phi_{[\lambda]}$ be as above, and let $\Delta_{[\lambda]} \subset \Phi_{[\lambda]}$ be the simple roots with respect to the choice of positive roots $\Phi^+_{[\lambda]} = \Phi^+ \cap \Phi_{[\lambda]}$. Then the following are equivalent:

(a) λ is dominant.

(b) $\langle \lambda + \rho, \alpha^\vee \rangle \geqslant 0$ for $\alpha \in \Delta_{[\lambda]}$.

(c) $\lambda \geqslant s_\alpha \cdot \lambda$ for all $\alpha \in \Delta_{[\lambda]}$.

(d) $\lambda \geqslant w \cdot \lambda$ for all $w \in W_{[\lambda]}$.

Proof. (a) $\Leftrightarrow$ (b): (a) $\implies$ (b) by definition. (b) $\implies$ (a) because the positive roots $\Phi^+_{[\lambda]}$ are positive integer combinations of the $\Delta_{[\lambda]}$.

(b) $\Leftrightarrow$ (c): This follows from
$$s_\alpha \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha.$$

(c) $\Leftrightarrow$ (d): (d) $\implies$ (c) automatically, so it suffices to show that (c) $\implies$ (d). We show this by inducting on $\ell(w)$. If $\ell(w) = 0$, then $w = 1$, and there is nothing to prove. If $\ell(w) = 1$, then $w = s_\alpha$ for some $\alpha \in \Delta_{[\lambda]}$, and the conclusion follows from (c). Now suppose $\ell(w) > 1$. Then $w = w' s_\alpha$ in $\Delta_{[\lambda]}$ with $\ell(w') = \ell(w) - 1$. We have $w\alpha < 0$ by standard facts about length. We write
$$\lambda - w \cdot \lambda = (\lambda - w' \cdot \lambda) + w' \cdot (\lambda - s_\alpha \cdot \lambda).$$
By induction, $\lambda - w' \cdot \lambda \geqslant 0$. By (c), $\lambda - s_\alpha \cdot \lambda$ is a positive integer multiple of α, and since $w'(\alpha) > 0$, we have $\lambda - s_\alpha \cdot \lambda > 0$. Thus, $\lambda - w \cdot \lambda > 0$, as desired.

□

Corollary 1. $W_{[\lambda]} \cdot \lambda$ contains a unique dominant weight and a unique antidominant weight.

Proof. The dominant weight thing follows from (a) $\Leftrightarrow$ (d) above. The antidominant weight thing follows from an identical argument.

□

4 $\mathcal{O}$ has enough projectives

Theorem 1. (a) If $\lambda \in \mathfrak{h}^*$ is dominant, then $M(\lambda)$ is projective.

(b) If $P \in \mathcal{O}$ is projective and $L \in \mathcal{O}$ is finite-dimensional, then $P \otimes L$ is projective.

(c) $\mathcal{O}$ has enough projectives.

Proof. (a) Let $M \to N$ be a surjection, and let $M(\lambda) \to N$ be a map hitting the maximal vector $v \in N$. We may assume that $M, N \in \mathcal{O}_\chi$, where χ is the central character of λ. It suffices to show that v lifts to a maximal vector in M. Suppose not. Then the $\mathfrak{n}$-submodule generated by any lift $u \in M$ of v (that is a weight vector) contains some maximal vector with weight linked with and greater than λ (since we are in $\mathcal{O}_\chi$). No such weight exists because λ is dominant, so u must be maximal.

(b) For any $M \in \mathcal{O}$, $\mathrm{Hom}_{\mathcal{O}}(P \otimes L, M) \cong \mathrm{Hom}_{\mathcal{O}}(P, \mathrm{Hom}(L, M)) \cong \mathrm{Hom}_{\mathcal{O}}(P, L^* \otimes M)$ (note that this is the naive dual L^*, which exists because L is finite-dimensional). Since $\mathrm{Hom}_{\mathcal{O}}(P, L' \otimes -)$ is exact (by the projectivity of P), $P \otimes L$ is projective.

(c) We first find a projective mapping onto $L(\lambda)$ for every $\lambda \in \mathfrak{h}^*$. For each λ, $\lambda + n\rho$ is dominant for sufficiently large n. Then by (a), $M(\lambda + n\rho)$ is projective. By (b), $M(\lambda + n\rho) \otimes L(n\rho)$ is projective.

We claim that $M(\lambda + n\rho) \otimes L(n\rho)$ has a quotient isomorphic to $M(\lambda)$. To prove this, we use the following **tensor identity**: for M a $U(\mathfrak{g})$-module and L a $U(\mathfrak{b})$-module,
$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L) \otimes M \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (L \otimes M).$$
Recall that $M(\lambda + n\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda + n\rho}$. Thus, the tensor identity implies that $M(\lambda + n\rho) \otimes L(n\rho) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_{\lambda+n\rho} \otimes L(n\rho))$. Since $L(n\rho)$ has $-n\rho$ as a weight, we see that

$M(\lambda + n\rho) \otimes L(n\rho)$ has $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_{\lambda+n\rho} \otimes \mathbb{C}_{-n\rho}) \cong M(\lambda)$ as a quotient, as claimed. Thus, there exists a projective mapping onto $L(\lambda)$.

Let $M \in \mathcal{O}$ be aribtrary. We induct on the length of M. It remains only to consider M with length > 1. In this case, there exists a short exact sequence

$$0 \to L(\lambda) \to M \to N \to 0.$$

By induction, there exists a surjection $P \to N$ from a projective P. We can lift this to a map $P \to M$ by projectivity. If the image of $P \to M$ intersects $L(\lambda)$, then it contains $L(\lambda)$ by simplicity; thus, $P \to M$ is surjective, and we are done. Otherwise, the lift $P \to M$ induces a splitting $N \to M$, so we can cover $M \cong N \oplus L(\lambda)$ by some $P \oplus Q$.

□

5 Projective covers

It remains to characterize the projectives in $\mathcal{O}$. Because $\mathcal{O}$ is Artinian, standard homological algebra shows that each $M \in \mathcal{O}$ admits a **projective cover** $P_M \to M$, which is a surjection from a projective such that no proper submodule of P_M surjects onto M. Such a projective cover is unique up to isomorphism. Let $P(\lambda)$ denote the projective cover of $L(\lambda)$. Since the $L(\lambda)$ are the simple objects, the $P(\lambda)$ are precisely the indecomposable projectives in $\mathcal{O}$, i.e. every projective is a direct sum of $P(\lambda)$. Moreover, the $P(\lambda)$ have unique maximal submodules given by $\ker(P(\lambda) \to L(\lambda))$.

Theorem 2. Every projective module in $\mathcal{O}$ has a **standard filtration**, i.e. a filtration with subquotients isomorphic to Verma modules. In such a standard filtration, the multiplicity $(P(\lambda) : M(\mu))$ is nonzero only if $\mu \geqslant \lambda$, and $(P(\lambda) : M(\lambda)) = 1$.

Proof. By the proof of the earlier proposition, $P(\lambda)$ is a direct summand of a projective of the form $M(\lambda + n\rho) \otimes L(n\rho)$ with $\lambda + n\rho$ dominant and n large. This $M(\lambda + n\rho) \otimes L(n\rho)$ has a standard filtration induced by a filtration of $L(n\rho)$ as a $U(\mathfrak{b})$-module with 1-dimensional subquotients. Since $P(\lambda)$ is a direct summand of $M(\lambda + n\rho) \otimes L(n\rho)$, $P(\lambda)$ has a standard filtration (by a standard fact about standard filtrations).

The facts about the multiplicities follow from the multiplicities of $L(n\rho)$.

□

Corollary 2. A projective is determined by its formal character.

Theorem 3 (BGG Reciprocity). $(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$

Proof. Since $M(\mu)$ and $M(\mu)^\vee$ have the same composition factors, we can rewrite this as $(P(\lambda) : M(\mu)) = [M(\mu)^\vee : L(\lambda)]$. The proof has two steps.

(a) $\dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^\vee) = [M(\mu)^\vee : L(\lambda)]$: We claim that $\dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$ more generally for $M \in \mathcal{O}$. This follows from the additivity of both sides for M lying in short exact sequences and the equality of both sides for M a simple module.

(b) $(P(\lambda) : M(\mu)) = \dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^\vee)$: We claim that $(M : M(\mu)) = \dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^\vee)$ for M with a standard filtration. This is proven with the same argument as above, except that we have to use the fact that $\operatorname{Ext}^1_{\mathcal{O}}(M(\mu'), M(\mu)^\vee) \cong 0$ for all μ, μ', which we will not prove.

□