Review of semisimple Lie algebras and introduction to category $\mathcal{O}$

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September 29, 2021

This talk is based on Chapter 0 and Sections 1.1-1.5 of Humphreys' *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*.

Everything is over $\mathbb{C}$.

1 Structure theory of semisimple Lie algebras

The first part of this talk is a review of the structure of semisimple Lie algebras and their representations.

A Lie algebra $\mathfrak{g}$ is **semisimple** if any of the following equivalent conditions hold:

- $\mathfrak{g}$ is a direct sum of simple Lie algebras (Lie algebras with no proper nonzero ideals).
- The Killing form $\kappa(x, y) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y))$ is nondegenerate.
- The radical (maximal solvable ideal) of $\mathfrak{g}$ is 0.

Examples include $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$.

We can decompose $\mathfrak{g}$ as a vector space according to the adjoint action of a **Cartan subalgebra** $\mathfrak{h}$. The operators $\operatorname{ad}(\mathfrak{h}) \subset \mathfrak{gl}(\mathfrak{g})$ commute and are all semisimple, so they are simultaneously diagonalizable. Thus, we have a decomposition into finitely many **root spaces**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* - \{0\}} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}.$$

Several important facts about the root spaces include the following:

- $\dim \mathfrak{g}_\alpha = 1$ for all α that appear.
- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ (here, $\mathfrak{g}_0 = \mathfrak{h}$, and $\mathfrak{g}_{\alpha+\beta} = 0$ if $\alpha + \beta$ is not a root). Moreover, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ if $\alpha + \beta$ is a root or 0.
- The α come in pairs $\alpha, -\alpha$.

For each positive root α, we can pick $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha = [x_\alpha, y_\alpha] \in \mathfrak{h}$ such that $\alpha(h_\alpha) = 2$. This can be thought of as an inclusion of Lie algebras $\mathfrak{sl}_2 \to \mathfrak{g}$ sending

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto x_\alpha$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto y_\alpha$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h_\alpha.$$

The x_α, y_α (for $\alpha \in \Phi^+$) and h_{α_i} (for $\alpha_i \in \Delta$) form a **standard basis** of $\mathfrak{g}$.

It can be shown that the α form a reduced **root system**, denoted Φ. Here, the $\mathbb{R}$-vector space is the $\mathbb{R}$-span of the roots, and the inner product is the restriction of the Killing form. We denote a choice of positive roots by Φ^+ and the simple roots by $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. We also have a **dual root system** $\Phi^\vee$ whose roots are $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$.

The root system determines $\mathfrak{g}$, as the commutators between the elements of the standard basis can be read off the data of the root system:

$$\begin{aligned}
[h_{\alpha_i}, h_{\alpha_j}] &= 0 \\
[x_\alpha, y_\beta] &= \begin{cases} h_\alpha & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \\
[h_\alpha, x_\beta] &= \langle \beta, \alpha^\vee \rangle x_\beta \\
[h_\alpha, y_\beta] &= -\langle \beta, \alpha^\vee \rangle y_\beta \\
\mathrm{ad}(x_{\alpha_i})^{1-\langle \alpha_j, \alpha_i^\vee \rangle}(x_{\alpha_j}) &= 0 \text{ if } i \neq j \\
\mathrm{ad}(y_{\alpha_i})^{1-\langle \alpha_j, \alpha_i^\vee \rangle}(y_{\alpha_j}) &= 0 \text{ if } i \neq j.
\end{aligned}$$

It is not obvious, but these relations uniquely determine $\mathfrak{g}$. Thus, any two semisimple Lie algebras with isomorphic root systems are isomorphic. Moreover, it can be shown that any reduced root system gives rise to a semisimple Lie algebra (take the Lie algebra generated by $x_{\alpha_i}, y_{\alpha_i}, h_{\alpha_i}$ with the above relations). Thus, there is a bijective correspondence between reduced root systems and semisimple Lie algebras.

Under this correspondence, irreducible reduced root systems correspond to simple Lie algebras, which semisimple Lie algebras are direct sums of. This is really nice because we have a full classification of irreducible reduced root systems:

A_n	$\mathfrak{sl}_{n+1}$
B_n	$\mathfrak{so}_{2n+1}$
C_n	$\mathfrak{sp}_{2n}$
D_n	$\mathfrak{so}_{2n}$
E_6, E_7, E_8, F_4, G_2	exceptional Lie algebras

2 Finite-dimensional representation theory of semisimple Lie algebras

Finite-dimensional representations are particularly nice because of **Weyl's complete reducibility theorem**, which asserts that finite-dimensional $\mathfrak{g}$-representations are direct sums of simple representations. Hence, it suffices to describe the simple finite-dimensional $\mathfrak{g}$-representations. Just as with the structure theory of $\mathfrak{g}$, the Cartan subalgebra $\mathfrak{h}$ is important. The finite-dimensional $\mathfrak{g}$-representations all decompose into **weight spaces** according to the action of $\mathfrak{h}$ (the λ are called **weights**):

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda.$$

Because M is finite-dimensional, the λ must be **integral**, which means that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$ (it suffices to check for $\alpha_i \in \Delta$). $\mathfrak{h}$ acts on M by scaling the weight spaces separately. The root spaces $\mathfrak{g}_\alpha$ act on M by raising and lowering weights: $\mathfrak{g}_\alpha M_\lambda \subset M_{\lambda+\alpha}$. We partially order the weights by $\lambda \geq \mu \Leftrightarrow \lambda - \mu \in \mathbb{Z}_{\geq 0}\Phi^+$. When $\alpha \in \Phi^+$, x_α raises the weight, and y_α lowers the weight.

The simple finite-dimensional representations are parametrized by **dominant integral weights**, which are integral weights λ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $\alpha_i \in \Delta$. The simple finite-dimensional representations are all **highest weight representations**, meaning that they are generated as $U(\mathfrak{g})$-modules by a single vector in the highest weight space. More precisely, we say that M is a highest weight representation if there is a **maximal** vector $v \in M_\lambda$ generating M as a $U(\mathfrak{g})$-module, i.e. $\mathfrak{n}v = 0$ (here, $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$). For a highest weight representation M, $\lambda \geq \mu$ for any weight μ of M, and $\dim M_\lambda = 1$, as the $\mathfrak{g}$-action can only lower the weight. Moreover, λ must be dominant, as we will see in a second. We will obtain a clearer picture of the simple modules when we discuss Verma modules later.

The finite-dimensional representation theory of a semisimple Lie algebra $\mathfrak{g}$ is really controlled by its **Weyl group** W, which is the group generated by reflections in the root system. W acts on the weight space $\mathfrak{h}^*$ and the **integral weight lattice** Λ. The significance of the Weyl group to the representation theory of $\mathfrak{g}$ is that the set of weights appearing in M must be invariant under W, as W permutes the weights through its action on $\mathfrak{h}^*$. Thus, any highest weight λ must be dominant, since otherwise, there would exist some simple reflection s_i for which $s_i\lambda > \lambda$. In the other direction, each W-orbit of Λ contains exactly one dominant weight λ. This dominant weight satisfies $\lambda \geq w\lambda$ for all $w \in W$, and the weights that can appear in M_λ are those in the convex hull of $W\lambda$. Thus, the simple finite-dimensional modules are parametrized by the dominant integral weights, which are in bijection with the W-orbits of Λ.

3 Introduction to category $\mathcal{O}$

Unlike the finite-dimensional situation, infinite-dimensional $\mathfrak{g}$-representations are not necessarily completely reducible. The **BGG category $\mathcal{O}$** enlarges the category of finite-dimensional representations to include infinite-dimensional representations that retain several desirable finiteness properties seen in the finite-dimensional setting. Category $\mathcal{O}$ is defined to be the full subcategory of $U(\mathfrak{g})$-modules satisfying the following conditions:

(1) M is a finitely generated $U(\mathfrak{g})$-module.

(2) M is $\mathfrak{h}$-semisimple: $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.

(3) M is $\mathfrak{n}$-finite: The $U(\mathfrak{n})$-submodule generated by any $v \in M$ is finite-dimensional.

The following additional properties follow from the definition:

- All M_λ are finite-dimensional.
- The set of weights of M is contained in the union of finitely many sets of the form $\lambda - \mathbb{Z}_{\geq 0}\Phi^+$.

Theorem 1. $\mathcal{O}$ is a Noetherian abelian category.

Proof. Noetherianness follows from the fact that $U(\mathfrak{g})$ is a Noetherian ring (with the natural filtration, it has Noetherian associated graded $\mathrm{Sym}^*(\mathfrak{g})$).

To prove abelianness, it suffices to show that $\mathcal{O}$ is closed under taking submodules, quotients, and direct sums. The only property here that I'll explain is that submodules are finitely generated. However, this follows from the Noetherianness of $\mathcal{O}$.

□

Since $\mathcal{O}$ consists of weight modules, it makes sense to talk about **highest weight modules**, just like in the finite-dimensional case. A $U(\mathfrak{g})$-module M is a highest weight module if there is a maximal vector $v \in M_\lambda$ generating M as a $U(\mathfrak{g})$-module. Just as in the finite-dimensional case, the set of weights of M is contained in $\lambda - \mathbb{Z}_{\geq 0}\Phi^+$, and $\dim M_\lambda = 1$.

Theorem 2. A highest weight module M has a unique maximal $U(\mathfrak{g})$-submodule and thus a unique simple quotient.

Proof. Every proper submodule $N \subset M$ is a weight module because submodules of M are in $\mathcal{O}$. Since M is generated by M_λ, N cannot contain M_λ, and neither can any sum of proper submodules N. Thus, M has a unique maximal submodule given by the sum of the proper submodules.

□

Theorem 3. There is a unique simple highest weight module of weight λ. Moreover, we have Schur's lemma: $\mathrm{End}_{U(\mathfrak{g})}\, \lambda = \mathbb{C}$.

Proof. Suppose M_1 and M_2 are highest weight modules of weight λ with maximal vectors v_1 and v_2, respectively. Then (v_1, v_2) is a maximal vector in $M_1 \oplus M_2$, so the submodule N that it generates is a highest weight module. We have surjections $N \twoheadrightarrow M_1$ and $N \twoheadrightarrow M_2$. The previous theorem says that N has a unique simple quotient, so we get an isomorphism $M_1 \cong M_2$.

We prove the second part. If M is a simple highest weight module with maximal vector v, then any map $M \to M$ sends v to a multiple of itself. This determines the entire map because v generates M. Thus, $\mathrm{End}_{U(\mathfrak{g})}\, \lambda = \mathbb{C}$.

□

Theorem 4. Every $M \in \mathcal{O}$ has a filtration whose subquotients are highest weight modules.

Proof. By $\mathfrak{n}$-finiteness, we can find a maximal vector $v \in M$. Take the $U(\mathfrak{g})$-submodule it generates and quotient by it. Repeat until we stop (by Noetherianness).

□

We finish by introducing **Verma modules**, which are the universal highest weight modules. For every λ, we define the Verma module $M(\lambda) \coloneqq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathfrak{n}$ acts on $\mathbb{C}_\lambda$ by 0 and $\mathfrak{h}$ acts on $\mathbb{C}_\lambda$ by λ. $M(\lambda)$ is universal because it maps uniquely (up to scalar) to any highest weight module of weight λ.

The Verma modules for $\mathfrak{g} = \mathfrak{sl}_2$ are in bijection with $\mathbb{C}$ (once we fix the standard basis x, y, h). If the highest weight is $\lambda \in \mathbb{C}$, the weights of $M(\lambda)$ are $\lambda, \lambda - 2, \lambda - 4, \ldots$. $M(\lambda)$ is simple iff $\lambda \notin \mathbb{Z}_{\geq 0}$. If $\lambda \in \mathbb{Z}_{\geq 0}$, then we get the usual simple finite-dimensional representations as quotients.