

*Commutative Algebra*  
*Fall 2020*

*Notes by Patrick Lei*

Lectures by Eric Urban



## **Disclaimer**

These notes were taken during lecture using the vimtex package of the editor neovim. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the instructor. If you find any errors, please contact me at [plei@math.columbia.edu](mailto:plei@math.columbia.edu).

Any similarities to Matsumura's *Commutative Algebra* are due to the instructor.

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## Basic Notions

The references we will use in this course are Matsumura's *Commutative Algebra* and Serre's *Algèbre Locale, Multiplicités*. There is an English translation of Serre. We will begin with general results on rings and modules. We will assume all rings are commutative and unital. Recall that an ideal  $I$  of a ring  $A$  is prime if and only if  $A/I$  is a domain, and  $I$  is maximal if and only if  $A/I$  is a field.

### 1.1 Basics of Ideals

**Definition 1.1.1.** Let  $I \subset A$  be an ideal. Then the *radical*  $\sqrt{I}$  of  $I$  is the set

$$\sqrt{I} := \{x \in A \mid x^a \in I \text{ for some } a \in \mathbb{N}\}.$$

**Definition 1.1.2.** An ideal  $I \subset A$  is *primary* if  $I \neq A$  and the zero divisors in  $A/I$  are nilpotent. Thus if  $xy \in I$  and  $x \notin I$ , then  $y^n \in I$  for some  $n$ .

**Proposition 1.1.3.** If  $Q \subset A$  is primary, then  $\sqrt{Q}$  is a prime ideal.

*Proof.* If  $xy \in \sqrt{Q}$ , then  $x^n y^n \in Q$ . If  $x^n \notin Q$ , then  $y \in \sqrt{Q}$  because  $(y^n)^a \in Q$ .  $\square$

*Remark 1.1.4.* The converse to Proposition 1.1.3 is false in general.

**Definition 1.1.5.** Let  $A$  be a ring. Then the *spectrum*  $\text{Spec } A$  of  $A$  as a set is the set of prime ideals of  $A$ . We may place the Zariski topology on this set, where the basis of open sets is given by  $D_f = \text{Spec } A \setminus V_f$ , where  $V_f$  is the set of prime ideals containing  $f$ .

If  $\varphi : A \rightarrow B$  is a morphism of rings, the morphism  $\varphi^* : \text{Spec } B \rightarrow \text{Spec } A$  is continuous in the Zariski topology.

**Exercise 1.1.6.** In particular, if  $\pi : A \rightarrow A/I$ , then  $\pi^*$  is an embedding.

**Exercise 1.1.7.** Let  $I \subset A$  be an ideal. Then let  $P_1, \dots, P_r$  be ideals of  $A$  that are all prime except possibly two of them. Show that if  $I \not\subset P_i$  for all  $i$ , then  $I \not\subset \bigcup_i P_i$ .

**Exercise 1.1.8.** Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_r$  be ideals of  $A$  such that  $\mathfrak{a}_i + \mathfrak{a}_j = A$ . Then

1.  $\bigcap_i \mathfrak{a}_i = \mathfrak{a}_1 \cdots \mathfrak{a}_r$
2. There is an isomorphism of rings  $A / \bigcap_i \mathfrak{a}_i \cong \prod_i A / \mathfrak{a}_i$ .

## 1.2 Localization

Let  $S \subset A$  be a multiplicative subset. The main examples are  $S_f = \{1, f, f^2, \dots\}$  and  $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$  for a prime ideal  $\mathfrak{p}$ . Then if  $0 \notin S$ , there is at least one ideal  $\mathfrak{a}$  such that  $\mathfrak{a} \cap S = \emptyset$ . Denote the set of such  $\mathfrak{a}$  by  $\mathcal{M}_S$ . Then any maximal element of  $\mathcal{M}_S$  is a prime ideal in  $A$ . Existence of a maximal element is seen using Zorn's lemma.

To see that maximal elements of  $\mathcal{M}_S$  are prime ideals, note that  $(x) + P$  is not in  $\mathcal{M}_S$ , so if  $x, y \notin P$ , there exist  $a, b \in A$  and  $s, s' \in S$  such that  $ax \equiv s \pmod{P}$  and  $by \equiv s' \pmod{P}$ . Therefore  $abxy \notin P$ , so  $xy$  is not in  $P$ .

**Lemma 1.2.1.** *Let  $\text{nil } A$  be the set of all nilpotent elements. Then*

$$\text{nil } A = \bigcap_{\substack{P \subset A \\ P \text{ prime}}} P.$$

*Proof.* One direction is easy, so let  $x$  be contained in all prime ideals. Then consider the set  $S_x$ . If  $0 \notin S_x$ , then  $\mathcal{M}_{S_x}$  is nonempty, so it has a maximal element. This is a prime ideal, which implies  $x$  is not contained in some prime.  $\square$

**Corollary 1.2.2.** *Let  $Q$  be an ideal of  $A$ . Then  $\sqrt{Q}$  is the intersection of all prime ideals containing  $Q$ .*

Now fix a multiplicative subset  $S$ . Then we will define an equivalence relation on  $A \times S$ . We write

$$(a, s) \sim (b, r)$$

if there exists  $t \in S$  such that  $t(ar - bs) = 0$ . If  $A$  is a domain, then this says that  $\frac{a}{s} = \frac{b}{r}$ . Now we will define the *localization*  $S^{-1}A$  to be the set of equivalence classes for this relation. Note there is a natural morphism  $A \rightarrow S^{-1}A$  that sends  $a \mapsto \frac{a}{1}$ .

Note that the localization has a universal property: If  $\varphi : A \rightarrow B$  is a morphism such that  $\varphi(S) \subset B^\times$ , then  $\varphi$  factors uniquely through  $S^{-1}A$ .

Localization gives a map  $\text{Spec } S^{-1}A \rightarrow \text{Spec } A$ , and in particular, if  $S = \{1, f, f^2, \dots\}$ , we recover the set  $D_f = \text{Spec } A_f$ .

## 1.3 Modules

Let  $A$  be a ring. Then an  $A$ -module  $M$  is an abelian group with an action of  $A$ . If  $M$  is an  $A$ -module and  $S \subset A$  is a multiplicative set, then  $S^{-1}M$  is the set of equivalence classes for  $(m, s) \sim (m', s')$  if there exists  $t \in S$  such that  $t(s'm - sm') = 0$ . This is an  $S^{-1}A$ -module.

**Lemma 1.3.1.** *Let  $M$  be an  $A$ -module. Then the map*

$$M \rightarrow \prod_{\substack{\mathfrak{p} \subset A \\ \mathfrak{p} \text{ maximal}}} M_{\mathfrak{p}}$$

*is injective.*

*Proof.* Let  $x \in M$  be nonzero. Then the annihilator of  $x$  is a proper ideal of  $A$ , so it is contained in a maximal ideal. This implies that  $x_{\mathfrak{p}} \in M_{\mathfrak{p}}$  is nonzero.  $\square$

**Corollary 1.3.2.** *Let  $A$  be a domain. Then  $A = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$ , where this intersection makes sense inside the fraction field of  $A$ .*

*Proof.* Apply the previous lemma to  $M = K/A$ .  $\square$

**Definition 1.3.3.** Let  $M$  be an  $A$ -module and  $x \in A$ . Then  $x$  is  $M$ -regular if the morphism  $m \mapsto xm$  is injective. Additionally, if  $x$  is  $A$ -regular, then it is called *regular*.

The set  $S_0$  of all regular elements in  $A$  is multiplicative, and the ring  $S_0^{-1}A$  is called the *total ring of fractions*. If  $A$  is a domain, then  $S_0 = A \setminus \{0\}$ , and  $S_0^{-1}A$  is the field of fractions.

**Definition 1.3.4.** A ring  $A$  is a *local ring* if  $A$  has only one maximal ideal. In this case, all elements not in the maximal ideal are units.

*Remark 1.3.5.* If  $I \subset A$  is an ideal such that  $A \setminus I = A^\times$ , then  $A$  is a local ring and  $I$  its maximal ideal.

**Example 1.3.6.** Now let  $A$  be a general ring and  $\mathfrak{p} \in \text{Spec } A$ . Then  $A_{\mathfrak{p}}$  is a local ring with maximal ideal  $\mathfrak{p}A_{\mathfrak{p}}$ .

**Definition 1.3.7.** Now suppose  $A, B$  are local rings. Then a morphism  $\varphi : A \rightarrow B$  of rings is *local* if  $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$ . This means we have a commutative diagram

$$(1.1) \quad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ k_A & \longrightarrow & k_B \end{array}$$

where  $k_A = A/\mathfrak{m}_A$  is the residue field of  $A$ .

Recall that the nilradical is the set of all nilpotent elements, or equivalently the intersection of all prime ideals. Then the *Jacobson radical*  $\text{rad } A$  is defined to be the intersection of all maximal ideals.

**Proposition 1.3.8.** Let  $x \in A$ . Then  $x \in \text{rad } A$  if and only if  $1 + xa$  is a unit for any  $a \in A$ .

*Proof.* If  $(1+x)A \neq A$ , then  $1+x$  is contained in some maximal ideal  $\mathfrak{m}$ , which implies  $1 \in \mathfrak{m}$ . In the other direction, suppose there exists some maximal ideal  $\mathfrak{m}$  such that  $x \notin \mathfrak{m}$ . Then  $x$  is nonzero in  $A/\mathfrak{m}$ . Thus there exists  $b$  such that  $1 - xb \in \mathfrak{m}$ , which contradicts the assumption that  $1 + xa$  is a unit for any  $a$ .  $\square$

**Lemma 1.3.9** (Nakayama's Lemma). Let  $M$  be a finitely generated  $A$ -module. Then let  $I$  be an ideal such that  $IM = M$ . Then there exists  $x \in I$  such that  $(1+x)M = 0$ . In particular, if  $I \subseteq \text{rad } A$ , then  $M = 0$ .

*Proof.* We will induct on the number of generators. If  $M = A.m$ , then  $m = xm$  for some  $x \in I$ , and thus  $(1-x)m = 0$ . Now suppose  $M = Am_1 + \cdots + Am_r$ . Let  $M' = M/Am_r$ . By the inductive hypothesis,  $(1+x)M' = 0$  for some  $x \in I$ . Therefore  $(1+x)M \subset Am_r$ , so  $(1+x)IM = (1+x)M \subset Im_r$ . Therefore  $(1+x)m_r = ym_r$  for some  $y \in I$ , and thus  $(1+x-y)m_r = 0$ . Thus  $(1+x)(1+x-y)M \subset (1+x-y)Am_r = 0$ .  $\square$

**Corollary 1.3.10.** Let  $N, N' \subset M$  and  $I \subset A$  such that  $M = N + IN'$ . Then if either

1.  $I$  is nilpotent;
2.  $I \subset \text{rad } A$  and  $N'$  is finitely generated,

then  $M = N$ .

*Proof.* 1. Suppose  $I$  is nilpotent. Then

$$\begin{aligned} M &= N + IN' = N + IM \\ &= N + I(N + IM) \\ &= N + I^2M \\ &\vdots \\ &= N + I^nM \\ &= N \end{aligned}$$

because  $I$  is nilpotent.

2. Let  $I \subseteq \text{rad } A$  and  $N'$  be finitely generated. Then set  $M_0 = M/N = IN'_0$ , where  $N'_0$  is the image of  $N'$  inside  $M_0$ . Because  $N'_0$  is finitely generated, so is  $M_0$ . Therefore  $M_0 = IM_0 = 0$ , so  $M = N$ . □

*Remark 1.3.11.* Most of the time, we apply this result when  $A$  is local and  $I$  is the maximal ideal of  $A$ . In this case,  $M/\mathfrak{m}M$  is a finite-dimensional vector space over  $A/\mathfrak{m}$ .

## 1.4 Artinian and Noetherian Rings

**Definition 1.4.1.** We say that an  $A$ -module  $M$  satisfies the *ascending chain condition* if any ascending chain of submodules of  $M$  becomes stationary. Similarly,  $M$  satisfies the *descending chain condition* if any descending chain of submodules becomes stationary. If  $M$  satisfies the ascending chain condition, it is called *Noetherian*, and if  $M$  satisfies the descending chain condition, it is *Artinian*.

**Proposition 1.4.2.** Assume we have a short exact sequence of  $A$  modules

$$(1.2) \quad 0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0.$$

Then  $M$  is Noetherian (resp. Artinian) if and only if  $N$  and  $P$  are.

*Proof.* Proving that if  $M$  is Noetherian, then  $N$  and  $P$  are is left to the reader. Now consider a chain

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$$

Then let  $P_i$  be the image of the  $M_i$  in  $P$  and  $N_i = N \cap M_i$ . Then we have an exact sequence

$$0 \rightarrow N_i \rightarrow M_i \rightarrow P_i \rightarrow 0.$$

Because  $(N_i)$  and  $(P_i)$  stabilize, so must  $M_i$  from the exact sequence. □

**Corollary 1.4.3.** If  $A$  is Noetherian (resp. Artinian), then any finitely generated  $A$ -module is Noetherian (resp. Artinian).

**Corollary 1.4.4.** Assume  $A$  is Noetherian. Then any finitely generated  $A$ -module  $M$  has a projective resolution by finite free  $A$ -modules. In other words, there exists an exact sequence

$$\cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

such that each  $F_i = A^{m_i}$ .



*Proof.* Suppose  $M$  is finitely generated. Then  $M = Am_1 + \cdots + Am_r$ , so we have a sequence

$$A^r \xrightarrow{\varphi_0} M \rightarrow 0.$$

Then  $\ker \varphi_0 = N_0$  and  $F_0 = A^r$ . Then we repeat this process with  $N_0$  taking the role of  $M$ .  $\square$

**Proposition 1.4.5.** *An  $A$ -module  $M$  is noetherian if and only if any submodule of  $M$  is finitely generated.*

*Proof.* Let  $N \subseteq M$ . Then choose  $n_1 \in N$ . Then if  $An_1 \neq N$ , choose  $n_2 \in N \setminus An_1$ . This process will stop because  $M$  is Noetherian, so  $N$  is finitely generated.

Now suppose any submodule is finitely generated. Given a chain

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots,$$

set  $N = \bigcup_i M_i$ . This is finitely generated and is also equal to the first  $M_i$  that contains all of the generators.  $\square$

This means that a ring  $A$  is noetherian if and only if all ideals of  $A$  are finitely generated. In particular, fields and principal ideal domains are Noetherian.

**Proposition 1.4.6.** *Let  $M$  be Noetherian and suppose  $S$  is a multiplicative subset of  $A$ . Then  $S^{-1}M$  is Noetherian.*

*Proof.* Consider the morphism  $M \rightarrow S^{-1}M$ . Then let  $N_i$  be a chain of  $S^{-1}A$ -modules in  $S^{-1}M$ . Their preimages  $M_i$  form a chain, and they are stationary, so  $N_i$  is also stationary.  $\square$

**Theorem 1.4.7.** *Let  $A$  be a Noetherian ring. Then  $A[X]$  is Noetherian.*

*Proof.* Let  $I \subset A[X]$ . Then  $\mathfrak{A}_n \subset A$  be generated by the dominant coefficients of polynomials in  $I$  of degree at most  $n$ . Then we can write  $a \in \mathfrak{A}_n$  as  $a = \sum \alpha_i \beta_i$  where  $\alpha_i \in A$  and  $\beta_i$  a dominant coefficient of a polynomial of degree at most  $n$  in  $I$ . Thus the  $\mathfrak{A}_n$  form a chain of ideals of  $A$  that stabilize for  $n \geq N$ . Then  $\mathfrak{A}_N = (\beta_1, \dots, \beta_r)$ . Set  $Q_i = \beta_i X^N + \cdots \in I$ . If  $P \in I$ , then there exists  $S$  such that  $P = QS + R$  such that  $Q \in AQ_1 + \cdots + AQ_r$  and  $\deg R < N$ .

Therefore  $P \in (Q_1, \dots, Q_r) + A[X]_{N-1} \cap I$ , so  $I \subset (Q_1, \dots, Q_r) + A[X]_{N-1} \cap I$  and is thus finitely generated.  $\square$

**Corollary 1.4.8.** *Let  $B$  be a finitely-generated  $A$ -algebra. Then if  $A$  is Noetherian,  $B$  is also Noetherian.*

**Corollary 1.4.9.** *Any finitely generated algebra over a field is Noetherian.*

*Remark 1.4.10.* Suppose  $A$  is Noetherian and  $M$  an  $A$ -module. If  $M$  is finitely generated, then  $M$  is Noetherian, but submodules are not necessarily Noetherian. However, they are finitely generated.

Suppose  $A \subset B$  is an inclusion of rings. Then we say that  $x \in B$  is *integral over  $A$*  if there exists a monic polynomial  $Q \in A[t]$  such that  $Q(x) = 0$ .

**Proposition 1.4.11.** *The following are equivalent:*

1.  $x \in B$  is integral over  $A$ ;
2.  $A[x]$  is a finitely-generated  $A$ -module;
3. There exists  $A[x] \subset C \subset B$  such that  $C$  is a finitely-generated  $A$ -module.
4. There exists a faithful  $A[x]$ -module  $M$  which is finitely generated over  $A$ .

*Proof.* **1 implies 2** Note that  $A[x]$  is generated by  $1, x, x^2, \dots, x^m$ , where  $Q$  has degree  $m$ .

**2 implies 3** Set  $C = A[x]$ .

**3 implies 4** Choose  $M = C$ .

**4 implies 1** Write  $M = Am_1 + \dots + Am_r$ .  $M$  is an  $A[x]$ -module, so we can consider  $x.M \subset M$ . Then for all  $i$ , we have  $xm_i = \sum a_{ij}m_j$ , so if we write consider the matrix  $T = (a_{ij})$ , then this matrix represents the map given by multiplication by  $x$ . Therefore we have

$$\det(T - xI_r) \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = 0,$$

so set  $Q = \det(T - xI_r)$ . This is monic and  $Q(x).m_i = 0$  for all  $i$ , and therefore  $Q(x)$  acts by 0 on  $M$ . Because  $M$  is a faithful  $A[x]$ -module, we have  $Q(x) = 0$ . □

**Exercise 1.4.12.** Let  $x, y \in B \supset A$ . Show that if  $x$  and  $y$  are integral over  $A$  then so are  $x + y, xy$ .

**Proposition 1.4.13.** Let  $A \subset B \subset C$ . Assume that  $A$  is Noetherian and that  $C$  is a finitely-generated  $A$ -algebra. If  $C$  is a finitely-generated  $B$ -module, then  $B$  is a finitely-generated  $A$ -algebra.

*Proof.* Write  $C = Bc_1 + \dots + Bc_r$ . Also, we can write  $C = A[x_1, \dots, x_m]$  for some  $x_i \in C$ . Then we can write  $x_i = \sum b_{ij}c_j$  and  $c_i c_j = \sum b_{ijk}c_k$  for  $b_{ij}, b_{ijk} \in B$ . Then  $B_0 = A[b_{ij}, b_{ijk}]$  is a finitely-generated  $A$ -algebra. Any element of  $C$  is a polynomial in the  $x_i$  with coefficients in  $A$ , so  $C$  is a finitely-generated  $B_0$ -module. In particular,  $B_0$  is Noetherian. Because  $B \subset C$ , this implies that  $B$  is a finitely generated  $B_0$ -module, so it is a finitely-generated  $A$ -algebra. □

**Corollary 1.4.14.** Let  $k$  be a field and  $E$  a finitely-generated  $k$ -algebra. If  $E$  is a field, then  $E$  is a finite extension of  $k$ .

*Proof.* Let  $E$  be a finitely-generated  $k$ -algebra. Then there exist  $x_1, \dots, x_r \in E$  that are algebraically independent over  $k$ . Then  $E$  is algebraic over  $k(x_1, \dots, x_r)$ , which is the field of fractions of  $k[x_1, \dots, x_r]$ . However, this gives an inclusion  $k \subset F \subset E$ , where  $F$  is a finitely-generated  $k$ -algebra and  $E$  is algebraic over  $F$ .

By the proposition,  $F$  is a finitely-generated  $k$ -algebra. Therefore, we can write  $F = h[y_1, \dots, y_s]$ , where  $y_i = \frac{f_i}{g_i}$ . Because  $k[x_1, \dots, x_n]$  is a UFD, then we can write

$$h = \prod_{i=1}^s g_i + 1 \in k[x_1, \dots, x_n].$$

$h$  is relatively prime to all of the  $g_i$ , so  $\frac{1}{h} \notin k[y_1, \dots, y_s]$ . This gives a contradiction, so  $E$  must be algebraic over  $k$ . □

### 1.4.1 Primary Decomposition in Noetherian Rings

**Definition 1.4.15.** An ideal  $\mathfrak{a} \subset A$  is *irreducible* if for any decomposition  $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ , then either  $\mathfrak{a} = \mathfrak{b}$  or  $\mathfrak{a} = \mathfrak{c}$ .

**Example 1.4.16.** If  $\mathfrak{a}$  is a prime ideal, then  $\mathfrak{a}$  is irreducible. To see this, if  $\mathfrak{a} \mid \mathfrak{b}\mathfrak{c}$ , then  $\mathfrak{a}$  contains one of  $\mathfrak{b}, \mathfrak{c}$ , and so either  $\mathfrak{a} = \mathfrak{b}$  or  $\mathfrak{a} = \mathfrak{c}$ .

*Remark 1.4.17.* Suppose  $\mathfrak{m} \subset A$  is a maximal ideal. Then any power  $\mathfrak{m}^n$  of  $\mathfrak{m}$  is primary.

*Proof.* We want to prove that the zero divisors of  $A/\mathfrak{m}^n$  are nilpotent. Because  $\mathfrak{m}$  is maximal, then  $A/\mathfrak{m}^n$  is a local ring with maximal ideal  $\mathfrak{m}/\mathfrak{m}^n$ . But then  $A/\mathfrak{m}^n \setminus \mathfrak{m}/\mathfrak{m}^n$  are all units, so everything in  $\mathfrak{m}$  is nilpotent.  $\square$

**Lemma 1.4.18.** *If  $A$  is Noetherian, then every irreducible ideal is primary.*

*Proof.* Let  $\mathfrak{a} \subset A$  be irreducible. Then we can pass to the quotient, so we may assume  $\mathfrak{a} = 0$ . Let  $x, y$  be nonzero with  $xy = 0$ . We want to show that  $x$  is nilpotent.

Because  $A$  is Noetherian, then there exists  $n$  such that  $\text{Ann } x^n = \text{Ann } x^{n+1}$ . We want to show that  $(x^n) \cap (y) = 0$ , so choose  $z = ax^n = by$ . Then  $zx = ax^{n+1} = byx = 0$ , so  $a \in \text{Ann } x^{n+1} = \text{Ann } x^n$ . However, this means  $z = 0$ . Because  $0$  is irreducible, then  $(x^n) = 0$ , so  $x^n = 0$ .  $\square$

**Corollary 1.4.19.** *If  $A$  is Noetherian, then every ideal of  $A$  has a primary decomposition. In other words, we can write  $I = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r$ , where each  $\mathfrak{a}_i$  is primary.*

*Proof.* Let  $S$  be the set of ideals with no primary decomposition. If  $S$  is nonempty, then  $S$  has a maximal element  $I$ . To see this, we can use the fact that  $A$  is Noetherian, so any chain of ideals in  $S$  eventually stabilizes. We know that  $I$  is not irreducible, we can write  $I = \mathfrak{a} \cap \mathfrak{b}$  such that  $I \neq \mathfrak{a}, \mathfrak{b}$ . In addition,  $\mathfrak{a}, \mathfrak{b} \notin S$ , so they have a primary decomposition. This implies that  $\mathfrak{a} \cap \mathfrak{b} = I$  has a primary decomposition.  $\square$

*Remark 1.4.20.* This decomposition is not unique. For example, consider  $I = \langle x^2, xy \rangle \subset k[x, y]$ . Then  $I = \langle x \rangle \cap \langle x^2, xy, y^n \rangle$  for all  $n > 0$ .

## 1.4.2 Artinian Rings

**Proposition 1.4.21.** *Assume that  $A$  is Artinian.*

1. *Every prime ideal of  $A$  is maximal.*
2.  *$A$  has finitely many maximal ideals.*
3. *The Jacobson radical of  $A$  is nilpotent.*

*Proof.* 1. Fix a prime ideal  $\mathfrak{p}$  and consider the domain  $B = A/\mathfrak{p}$ . Choose  $B \ni x \neq 0$  and consider the decreasing chain  $(x^n)$  of ideals. This stabilizes, so there exists  $(x^n) = (x^{n+1})$ , so we can write  $x^n = x^{n+1}y$  for some  $y \in B$ , and therefore  $1 = xy$  because  $B$  is a domain. Therefore  $x$  has an inverse, so  $B$  is a field. Thus  $\mathfrak{p}$  is maximal.

2. Suppose we have infinitely many maximal ideals  $\mathfrak{p}_1, \dots, \mathfrak{p}_n, \dots$  that are pairwise distinct. Then we form a chain

$$\mathfrak{p}_1 \supset \mathfrak{p}_1 \mathfrak{p}_2 \supset \cdots$$

which becomes stationary. Therefore  $\mathfrak{p}_1 \cdots \mathfrak{p}_n \subset \mathfrak{p}_{n+1}$ , so  $\mathfrak{p}_{n+1}$  contains some  $\mathfrak{p}_i$ . Because these ideals are maximal, this is a contradiction.

3. Consider  $I = \text{rad}(A) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ . Then the chain  $I \supset I^2 \supset \cdots$  stabilizes, so  $I^n = I^{n+1}$  for some  $n$ . Let  $J = ((0) : I^n)$ .<sup>1</sup> We will show that  $J = A$ . If not, let  $J' \supsetneq J$  such that  $J'$  is minimal for this property. Such a  $J'$  exists because  $A$  is Artinian.

Let  $x \in J' \setminus J$  and consider the ideal  $Ax + J$ . By minimality of  $J'$ , we see that  $I^n x + J \subsetneq J'$  (otherwise  $J = J'$  by Nakayama's lemma). Therefore  $I^n x + J = J$ , so  $I^n x \subset J$  and thus  $x \in (J : I^n)$ . Therefore,  $I^{n+1}x \subset I^n J = (0)$ . This implies  $I^n x = 0$ , so  $x \in J$  and thus  $J' = J$ .  $\square$

<sup>1</sup>Here,  $(\mathfrak{a} : \mathfrak{b}) = \{x \in A \mid x\mathfrak{b} \subset \mathfrak{a}\}$ .

**Definition 1.4.22.** An  $A$ -module  $M$  is called *irreducible* if  $0$  and  $M$  are the only submodules of  $M$ .

**Definition 1.4.23.** An  $A$ -module  $M$  is said to be of *finite length* if there exists a (finite) decreasing sequence of submodules

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n+1} = 0$$

such that  $M_i/M_{i+1}$  is irreducible for  $i = 0, \dots, n$ . In this case,  $n$  is actually unique and depends only on  $M$ . We will call  $n$  the *length* of  $M$ .

**Proposition 1.4.24.** Let  $A$  be a ring. Then  $A$  is Artinian if and only if  $A$  is of finite length as an  $A$ -module.

*Proof.* If  $A$  is of finite length, then we have a sequence  $A = M_0 \supsetneq \cdots \supsetneq M_{n+1} = 0$  where  $M_i/M_{i+1}$  is irreducible. If  $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$  is a decreasing chain of ideals, so  $\mathfrak{a}_i \cap M_n$  is a decreasing chain of ideals. However, each is either  $M_n$  or  $0$ , so this chain stabilizes. Similarly, the chain  $(M_j \cap \mathfrak{a}_i)/(M_{j+1} \cap \mathfrak{a}_i)$  also stabilizes for all  $j$ . Therefore, there exists  $N$  such that for all  $i > N$ ,  $M_j \cap \mathfrak{a}_i/(M_{j+1} \cap \mathfrak{a}_i)$  is constant for all  $j$ , so  $\mathfrak{a}_i$  is constant for all  $i > N$ .

Now suppose that  $A$  is Artinian. Choose  $I = \text{rad}(A) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$ , where the  $\mathfrak{m}_i$  are the maximal ideals of  $A$ . Then  $I$  is nilpotent, so there exists  $n > 0$  such that

$$0 = I^n = \mathfrak{m}_1^n \cdots \mathfrak{m}_m^n.$$

Then  $A = A/I^n = \prod A/\mathfrak{m}_j^n$  by the Chinese remainder theorem, so  $A/\mathfrak{m}_j^n$  is clearly a local ring and is of finite length as an  $A$ -module. Note that the  $A/\mathfrak{m}_j$ -vector space  $\mathfrak{m}_j^i/\mathfrak{m}_j^{i+1}$  is finite-dimensional because  $A$  is Artinian. Therefore  $\mathfrak{m}_j^i/\mathfrak{m}_j^{i+1}$  is of finite length.  $\square$

**Exercise 1.4.25.** If there is an exact sequence  $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$  of  $A$ -modules, then  $M$  is of finite length if and only if  $N$  and  $P$  are of finite length. Moreover,  $\ell(M) = \ell(P) + \ell(N)$ .

**Theorem 1.4.26.**  $A$  is Artinian if and only if  $A$  is Noetherian and  $\dim A = 0$ .

*Proof.* If  $A$  is Artinian, we have already proved that  $\dim A = 0$ . By the previous proposition, because  $A$  is of finite length,  $A$  is Noetherian. To see this, for a chain  $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ , note that  $\mathfrak{a}_m \cap M_i/\mathfrak{a}_m \cap M_{i+1}$  stabilizes. We can do this for each  $i$ , so any increasing chain stabilizes.

Now assume  $A$  is Noetherian and has dimension 0. We know that  $(0)$  has a primary decomposition, so we can write  $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ , where each  $\mathfrak{q}_i$  is primary. Then  $\mathfrak{m}_i = \sqrt{\mathfrak{q}_i}$  is a prime ideal, so it is maximal because  $\dim A = 0$ . Because  $A$  is Noetherian and for all  $x \in \mathfrak{m}_i$ ,  $x^n \in \mathfrak{q}_i$  for  $n \gg 0$ , so there exists  $N$  such that  $\mathfrak{m}_i^N \subset \mathfrak{q}_i$  for each  $i$ . Therefore

$$\mathfrak{m}_1^N \cdots \mathfrak{m}_r^N \subset \mathfrak{q}_1 \cdots \mathfrak{q}_r \subset \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r = 0,$$

so  $\mathfrak{m}_1^N \cdots \mathfrak{m}_r^N = 0$ . Therefore,  $A \cong A/\mathfrak{m}_1^N \times \cdots \times A/\mathfrak{m}_r^N$ . Each  $A/\mathfrak{m}_i^N$  is of finite length (because each  $\mathfrak{m}_i^j/\mathfrak{m}_i^{j+1}$  is a finite-dimensional vector space), so  $A$  is of finite length.  $\square$

**Proposition 1.4.27.** Let  $A$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ . Then one of the following holds:

- (a) Either  $\mathfrak{m}^n \supsetneq \mathfrak{m}^{n+1}$  for all  $n$ , or;
- (b)  $\mathfrak{m}^n = 0$  for  $n \gg 0$  and in this case,  $A$  is Artinian.

*Proof.* If  $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ , then  $\mathfrak{m}^n = 0$  by Nakayama's lemma. This implies that  $A = A/\mathfrak{m}^n$  is of finite length. Then if  $\mathfrak{p}$  is prime, then  $\mathfrak{m}^n = (0) \subset \mathfrak{p}$ , so  $\mathfrak{m} \subset \mathfrak{p}$ . Because  $\mathfrak{m}$  is maximal,  $\mathfrak{m} = \mathfrak{p}$ , so  $\dim A = 0$ .  $\square$

**Theorem 1.4.28** (Structure Theorem for Artinian Rings). *An Artinian ring is uniquely up to isomorphism a finite product of Artinian local rings.*

*Proof.* Previously, we proved that  $A = \prod A/\mathfrak{m}_i^N$ . Each of these is a local Artinian ring.  $\square$

## Linear Algebra of Modules

**Proposition 2.0.1.** *Assume  $M, N, P$  are  $A$ -modules.*

1. *The sequence  $N \rightarrow M \rightarrow P \rightarrow 0$  is exact if and only if for all  $A$ -modules  $Q$ , the sequence*

$$0 \rightarrow \text{Hom}(P, Q) \rightarrow \text{Hom}(M, Q) \rightarrow \text{Hom}(N, Q)$$

*is exact.*

2. *The sequence  $0 \rightarrow N \rightarrow M \rightarrow P$  is exact if and only if for all  $A$ -modules  $Q$ , the sequence*

$$0 \rightarrow \text{Hom}(Q, N) \rightarrow \text{Hom}(Q, M) \rightarrow \text{Hom}(Q, P)$$

*is exact.*

*Proof.* This is left as an exercise. □

**Remark 2.0.2.** In general, if  $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$  is exact, then

$$\begin{aligned} 0 \rightarrow \text{Hom}(P, Q) \rightarrow \text{Hom}(N, Q) \rightarrow \text{Hom}(M, Q) \\ 0 \rightarrow \text{Hom}(Q, M) \rightarrow \text{Hom}(Q, N) \rightarrow \text{Hom}(Q, P) \end{aligned}$$

are exact but the last morphism is not necessarily surjective.

**Definition 2.0.3.** A module  $Q$  is *projective* if the functor  $\text{Hom}(Q, -)$  is exact. Here, exact means that short exact sequences are preserved. Similarly, a module  $I$  is *injective* if the functor  $\text{Hom}(-, I)$  is exact.

**Proposition 2.0.4.** *A module  $Q$  is projective if and only if  $Q$  is a direct factor of a free module. In other words, there exists a free module  $F$  and  $A$ -module  $Q'$  such that  $F = Q \oplus Q'$ .*

*Proof.* Suppose  $Q$  is projective. Then there is a surjection  $\pi : A^{(S)} \rightarrow Q \rightarrow 0$ . Because  $Q$  is projective, there exists a map  $\theta$  such that  $\pi \circ \theta = \text{id}$ . Therefore  $A^{(S)} \cong Q \oplus Q'$ , where  $Q'$  is the kernel of  $\pi$ .

On the other hand, if  $A^{(S)} = Q \oplus Q'$ , then for any diagram of the form

$$(2.1) \quad \begin{array}{ccccc} M & \longrightarrow & P & \longrightarrow & 0 \\ & & \uparrow & & \\ & & Q & & \end{array}$$

we can embed  $Q$  in  $A^{(S)}$  and then use projectivity of free modules (because  $\text{Hom}(A, M) = M$ ). □

*Remark 2.0.5.* If  $M$  is projective and finitely generated, then it is a direct factor of a finite free module.

**Definition 2.0.6.** A *projective resolution* of an  $A$ -module  $M$  is a right bounded complex

$$\cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow 0$$

of projective modules such that there exists  $P_0 \rightarrow M$  such that

$$\cdots P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

is exact.

**Exercise 2.0.7.** Show that any module has a projective resolution (**Hint:** construct a free resolution). In addition, any two projective resolutions are homotopic.

**Definition 2.0.8.**  $k_q : M_q \rightarrow N_{q+1}$  such that  $\phi_q = d^N \circ k_q + k_{q-1} \circ d^M$ .

## 2.1 Tor and Ext Functors

Note that for a complex, we can compute the *homology*  $H_q(M_\bullet) := \ker d_q / \text{Im } d_{q+1}$ . This measures the defect of the complex from being exact. For functors that are not exact, we can construct *derived functors* that measure the defect of exactness. Let  $F : A\text{-Mod} \rightarrow A\text{-Mod}$  be right exact. Then for any  $M$ , we can consider a projective resolution  $P_\bullet \rightarrow M \rightarrow 0$ . Applying  $F$  to  $P_\bullet$ , then the *left derived functor*  $L_\bullet F(M)$  is defined by  $L_\bullet F(M) = H_\bullet(F(P_\bullet))$ .

**Proposition 2.1.1.** If  $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$  is exact, then we have a long exact sequence

$$\cdots \rightarrow L_1 F(M) \rightarrow L_1 F(N) \rightarrow L_1 F(P) \rightarrow L_0 F(M) \rightarrow L_0 F(N) \rightarrow L_0 F(P) \rightarrow 0.$$

Recall that the tensor product  $M \otimes N$  of two modules  $M, N$  is an  $A$ -module with a bilinear map  $M \times N \rightarrow M \otimes N$  such that all bilinear maps  $M \times N \rightarrow P$  factor through  $M \otimes N$ .

**Proposition 2.1.2.** The functors  $- \otimes N, \text{Hom}(N, -)$  are an adjoint pair.

**Corollary 2.1.3.** If  $N \rightarrow M \rightarrow P \rightarrow 0$  is exact, then

$$N \otimes Q \rightarrow M \otimes Q \rightarrow P \otimes Q \rightarrow 0$$

is exact.

**Definition 2.1.4.** A module  $Q$  is *flat* if  $- \otimes Q$  is exact.

We can define the left derived functors  $\text{Tor}_q(Q, M)$  of the tensor product.

**Proposition 2.1.5.** Any projective module is flat.

*Proof.* Clearly free modules are flat, so write  $Q \oplus Q' = A^{(S)}$  and then note that the tensor product distributes over the sum.  $\square$

**Proposition 2.1.6.** The following are equivalent:

1.  $M$  is flat over  $A$ .
2. If  $N' \hookrightarrow N$ , then  $M \otimes N' \hookrightarrow M \otimes N$ .

3. For all finitely generated ideals  $I \subset A$ ,  $I \otimes M \hookrightarrow M$ .
4. For any finitely generated ideals  $I \subset A$ ,  $\text{Tor}_1(M, A/I) = 0$ .
5. For any finitely generated module  $N$ , we have  $\text{Tor}(M, N) = 0$ .
6. For all  $a_i \in A$  and  $x_i \in M$  such that  $\sum a_i x_i = 0$  there exist  $y_1, \dots, y_s \in M$  and  $b_{ij}$  such that  $x_i = \sum b_{ij} y_j$ .

*Proof.* It is clear that 1 is equivalent to 2 implies 3 implies 4 implies 5. The directions 3 implies 2 and 4 implies 3 are left to the reader.

**1 implies 6** Choose  $a_i \in A, x_i \in M$  such that  $\sum_{i=1}^r a_i x_i = 0$ . Then define a map  $A^r \xrightarrow{f} A$  by

$$f(b_1, \dots, b_r) = \sum_{i=1}^r a_i b_i$$

and define  $K = \ker f$ . Because  $M$  is  $A$ -flat, we have an exact sequence

$$0 \rightarrow K \otimes M \rightarrow M^r \rightarrow M.$$

Then  $(x_1, \dots, x_r) \in \ker f \otimes \text{id}_M$ . Therefore there exists  $b_1, \dots, b_s \in K$  and  $y_1, \dots, y_s \in M$  such that

$$(x_1, \dots, x_r) = \sum_{j=1}^s b_j \otimes y_j.$$

Writing  $b_j = (b_{1j}, \dots, b_{rj})$ , we obtain the identity

$$\sum_{i=1}^r b_{ij} a_i = 0$$

and thus  $x_i = \sum b_{ij} y_j$ .

**6 implies 3** Choose an ideal  $I \subset A$ . Consider the map  $0 \rightarrow I \otimes M \rightarrow M$ . Then for any element in the kernel, we can write

$$\sum_i a_i \otimes x_i \mapsto \sum a_i x_i = 0.$$

Then we can write  $x_i = \sum b_{ij} y_j$  and so

$$\sum a_i \otimes x_i = \sum \sum a_i \otimes b_{ij} y_j = \sum (\sum a_i b_{ij}) \otimes y_j = 0$$

and thus  $I \otimes M \rightarrow M$  is injective.  $\square$

Let  $\phi : A \rightarrow B$  be a map of rings and let  $M$  be a  $B$ -module. Define  $\phi$  to be *flat* if  $B$  is flat as an  $A$ -module.

**Proposition 2.1.7.** *If  $\phi : A \rightarrow B$  is flat and  $M$  is a flat  $B$ -module, then  $M$  is also flat as an  $A$ -module.*

*Proof.* Let  $S$  be an  $A$ -module. Then  $S \otimes_A M = S \otimes_A (B \otimes_B M) = (S \otimes_A B) \otimes_B M$ . If  $0 \rightarrow N_1 \rightarrow N_2$  is an exact sequence of  $A$ -module, by flatness of  $B$  as an  $A$ -module, then

$$0 \rightarrow N_1 \otimes_A B \rightarrow N_2 \otimes_A B$$

is exact. Because  $M$  is flat over  $B$ , we see that

$$0 \rightarrow (N_1 \otimes_A B) \otimes_B M \rightarrow (N_2 \otimes_A B) \otimes_B M$$

is exact, as desired.  $\square$



Now let  $M$  be an  $A$ -module. Then for any map  $A \xrightarrow{\phi} B$ , we can consider the  $B$ -module  $M_{(B)} := M \otimes_A B$ .

**Proposition 2.1.8.** *If  $M$  is  $A$ -flat, then  $M_{(B)}$  is  $B$ -flat.*

*Proof.* For a  $B$ -module  $S$ , write

$$\begin{aligned} S \otimes_B M_{(B)} &= S \otimes_B (M \otimes_A B) \\ &\cong S \otimes_B (B \otimes_A M) \\ &\cong (S \otimes_B B) \otimes_A M \\ &\cong S \otimes_A M. \end{aligned}$$

Thus if  $0 \rightarrow S_1 \rightarrow S_2$  is exact, then  $0 \rightarrow S_1 \otimes_A M \rightarrow S_2 \otimes_A M$  is exact because  $M$  is  $A$ -flat, as desired.  $\square$

**Proposition 2.1.9.** *Let  $S \subset A$  be a multiplicative subset of  $A$ . Then the morphism of rings  $A \rightarrow S^{-1}A$  is flat.*

The proof is left to the reader. This can be reformulated as  $M \otimes_A S^{-1}A \cong S^{-1}M$ .

Now we will give some remarks about the Ext functors. For any left exact functor, we may define the right derived functors  $R^\bullet F$  by

$$R^i F(M) = H^i(F(I^\bullet))$$

where  $M \rightarrow I^\bullet$  is an injective resolution. Then we will define the right derived functors of  $\text{Hom}_A(N, -)$  by  $\text{Ext}_A^i(N, -)$ .

**Proposition 2.1.10.** *If  $M$  is injective, then  $\text{Ext}_A^i(N, M) = 0$  for all  $i > 0$ . Similarly, if  $N$  is projective, then  $\text{Ext}_A^i(N, M) = 0$  for all  $i > 0$ .*

*Remark 2.1.11.* We can compute  $\text{Ext}^i(N, M)$  using a projective resolution of  $N$ .

**Proposition 2.1.12.** *Let  $A \rightarrow B$  be a morphism of rings and let  $M, N$  be  $A$ -modules. Then let  $M_{(B)}, N_{(B)}$  be their base changes to  $B$ . Then we have*

$$\text{Ext}_B^i(M_{(B)}, N_{(B)}) = \text{Ext}_A^i(M, N)_{(B)}$$

and

$$\text{Tor}_i^B(M_{(A)}, N_{(B)}) = \text{Tor}_i^A(M, N)_{(B)}$$

if  $B$  is  $A$ -flat.

*Proof.* This follows from the definition of Ext, Tor using projective resolutions using the following facts.

1. If  $M$  is  $A$ -projective, then  $M_{(B)}$  is  $B$ -projective.
2. Since  $B$  is  $A$ -flat, for any complex  $X^\bullet$  of  $A$ -modules, then  $H^\bullet(X^\bullet_{(B)}) = H^\bullet(X^\bullet)_{(B)}$ .

$\square$

## 2.2 Flatness

**Proposition 2.2.1.** *Let  $A$  be a local ring. Then any finitely generated flat  $A$ -module is free. In particular, free, projective, and flat are equivalent for  $A$ -modules.*

*Proof.* We know that free implies projective implies flat. Therefore we will show that if  $M$  is flat, then it is free. Assume that  $M$  is finitely generated and  $A$ -flat. Let  $k = A/\mathfrak{m}$  be the residue field of  $A$ . Define  $\bar{M} = M \otimes_A k$ , which is a vector space of finite dimension over  $k$ . Then there exists  $x_1, \dots, x_r \in M$  that descend to a basis of  $\bar{M}$ .

Then the map  $A^r \rightarrow M, (a_i) \mapsto \sum a_i x_i$  is surjective by Nakayama's lemma. We will prove that this map is injective by induction. If  $r = 1$ , then suppose  $ax_1 = 0$ . Then there exist  $y_1, \dots, y_s, b_{11}, \dots, b_{1s}$  such that

$$x_1 = \sum_{j=1}^s b_{1j} y_j$$

where  $ab_{ij} = 0$  for all  $j = 1, \dots, s$ . Because  $\bar{x}_1 \neq 0$ , there exists  $j$  such that  $\bar{b}_{1j} \neq 0$  so  $b_{1j}$  is invertible in  $A$ . Thus  $a = 0$ .

Now suppose  $a_1 x_1 + \dots + a_r x_r = 0$ . Then there exist  $y_1, \dots, y_s$  and  $b_{ij}$  such that

$$x_i = \sum b_{ij} y_j$$

and

$$\sum a_i \begin{pmatrix} b_{i1} \\ \vdots \\ b_{ij} \end{pmatrix} = 0.$$

Because  $\bar{x}_r \neq 0$ , we see that  $\bar{b}_{rj} \neq 0$  for some  $j$  and thus  $b_{rj}$  is a unit. Then  $a_1 b_{1j} + \dots + a_r b_{rj} = 0$ , so we can write

$$\sum a_i (x_i - c_i x_r) = 0.$$

We know that  $\bar{x}_1 - c_1 \bar{x}_r, \dots, \bar{x}_{r-1} - c_{r-1} \bar{x}_r$  are linearly independent over  $k$ , so from the induction  $a_1 = \dots = a_r = 0$  and thus  $a_i = 0$ .  $\square$

*Remark 2.2.2.* If  $M$  is not finitely generated, the proposition is false. An example is given by taking the field of fractions of a local domain.

When proving the proposition, we in fact proved that

**Lemma 2.2.3.** *If  $x_1, \dots, x_r \in M$  with  $M$  a flat  $A$ -module for  $A$  a local ring and  $\bar{x}_1, \dots, \bar{x}_r$  are linearly independent in  $M \otimes_A k$ , then  $x_1, \dots, x_r$  are linearly independent in  $M$ .*

**Proposition 2.2.4.** *Suppose that  $A \rightarrow B$  is flat and  $I_1, I_2$  are ideals of  $A$ . Then*

1.  $(I_1 \cap I_2)B = I_1B \cap I_2B$ ;
2. If  $I_2$  is finitely generated, then  $(I_1 : I_2)B = (I_1B : I_2B)$ .

*Proof.* The proof is a formal consequence of flatness.

1. Consider the exact sequence  $0 \rightarrow I_1 \cap I_2 \rightarrow A \rightarrow A/I_1 \times A/I_2$ . Tensoring with  $B$ , we obtain an exact sequence

$$0 \rightarrow (I_1 \cap I_2) \otimes B \rightarrow B \rightarrow B/I_1B \times B/I_2B.$$

But then  $(I_1 \cap I_2) \otimes B = (I_1 \cap I_2)B$ , but the kernel of the last map is clearly  $I_1B \cap I_2B$ .

2. Set  $I_2 = (x_1, \dots, x_r)$ . Then because

$$(I_1 : I_2) = \bigcap_{i=1}^r (I_1 : x_i A),$$

it suffices to prove the result for  $I_2$  a principal ideal. We have an exact sequence

$$0 \rightarrow (I_1 : xA) \rightarrow A \xrightarrow{\times x} A/I_1.$$

Tensoring by  $B$ , we obtain

$$0 \rightarrow (I_1 : xA) \otimes B \rightarrow B \rightarrow B/I_1 B,$$

and by analysing the kernel, we see that  $(I_1 : xA)B = (I_1 B : xB)$ . By repeated application of the previous part, the desired result follows.  $\square$

**Example 2.2.5.** We will give an example where the previous proposition is not true in general. Let  $A = k[x, y]$  and  $B = A/xA = k[y]$ . Then choose  $I_1 = (x + y)$ ,  $I_2 = (y)$ , so  $I_1 \cap I_2 = I_1 I_2 = ((x + y)y)$ . But then we have  $(I_1 \cap I_2)B = y^2 B$ , but  $I_1 B \cap I_2 B = yB$ .

Another example is  $A = k[x, y]$ ,  $B = k[x, y, z]/(xz - y) \cong k[x, z]$ ,  $I_1 = xA$ ,  $I_2 = yA$ . Here we can check that  $(I_1 \cap I_2) = (xy)$ , that  $(I_1 \cap I_2)B = x^2 z B$ , but  $I_1 B \cap I_2 B = xz B$ . Viewing this geometrically as  $\text{Spec } B \rightarrow \text{Spec } A$ , we can check the fiber over  $(0, 0)$  and see that the map is not flat.

**Proposition 2.2.6.** Let  $A \xrightarrow{\varphi} B$  be a ring homomorphism. The following are equivalent:

1.  $B$  is flat over  $A$ ;
2.  $B_{\mathfrak{P}}$  is flat over  $A_{\mathfrak{p}}$  for all  $\mathfrak{P} \in \text{Spec } B$  and  $\mathfrak{p} = \varphi^{-1}(\mathfrak{P})$ .
3.  $B_{\mathfrak{P}}$  is flat over  $A_{\mathfrak{p}}$  for any  $\mathfrak{P}$  maximal.

*Proof.* **1 implies 2:** We know that  $B_{\mathfrak{p}}$  is flat over  $A_{\mathfrak{p}}$ . But then  $B_{\mathfrak{P}}$  is flat over  $B_{\mathfrak{p}}$  because it is a localization. By transitivity of flatness,  $B_{\mathfrak{P}}$  is flat over  $A_{\mathfrak{p}}$ .

**2 implies 3:** This is obvious.

**3 implies 1:** Note that for all  $\mathfrak{P}$  maximal,  $\text{Tor}_i^A(B, N)_{\mathfrak{P}} = 0$  for  $i > 0$ . This implies that  $\text{Tor}_i^A(B, N) = 0$ , and thus  $B$  is flat over  $A$ . To get that the first Tor is zero we need to use the lemma below.  $\square$

**Lemma 2.2.7.** Let  $\varphi : A \rightarrow B$  be a morphism of rings and choose  $\mathfrak{P} \in \text{Spec } B$ . Then let  $\mathfrak{p} = \varphi^{-1}(\mathfrak{P})$  and  $N$  an  $A$ -module. Then  $\text{Tor}_i^A(B, N)$  is a  $B$ -module and  $\text{Tor}_i^A(B, N)_{\mathfrak{P}} = \text{Tor}_i^{A_{\mathfrak{p}}}(B_{\mathfrak{P}}, N_{\mathfrak{p}})$ .

*Proof.* Let  $X_{\bullet} \rightarrow N$  be a projective resolution. Then Tor is computed by the homology of the complex  $B \otimes_A X_{\bullet}$ . When we localize, we localize the homology at the  $B$  term. However,  $B_{\mathfrak{p}} \otimes_A X_{\bullet} = B_{\mathfrak{P}} \otimes_{A_{\mathfrak{p}}}(X_{\bullet})_{\mathfrak{p}}$ , so because  $X_i$  is  $A$ -projective, then  $X_{\mathfrak{p}}$  is  $A_{\mathfrak{p}}$ -projective, and thus  $(X_{\bullet})_{\mathfrak{p}}$  is a projective resolution of  $N_{\mathfrak{p}}$ . Thus the complex  $B_{\mathfrak{P}} \otimes_{A_{\mathfrak{p}}}(X_{\bullet})_{\mathfrak{p}}$  computes the Tor as desired.  $\square$

**Definition 2.2.8.** An  $A$ -module  $N$  is said to be *faithfully flat* if

1.  $N$  is  $A$ -flat;
2. For any sequence  $P \rightarrow Q \rightarrow R$  of  $A$ -modules, if  $P \otimes N \rightarrow Q \otimes N \rightarrow R \otimes N$  is exact, then  $P \rightarrow Q \rightarrow R$  is exact.

**Theorem 2.2.9.** *Let  $M$  be an  $A$ -module. Then the following are equivalent:*

1.  $M$  is faithfully flat over  $A$ ;
2.  $M$  is flat and for any nonzero  $N$ ,  $M \otimes N \neq 0$ ;
3.  $M$  is flat and for all maximal ideals  $\mathfrak{m} \subset A$ ,  $\mathfrak{m}M \neq M$ .

*Proof.* **1 implies 2:** Choose the sequence  $0 \rightarrow N \rightarrow 0$ . Then tensor with  $M$ . If  $N \otimes M = 0$ , the sequence is now exact, then the original sequence is exact, and thus  $N = 0$ .

**2 implies 3:** Consider  $N = A/\mathfrak{m}$ . Then  $N \otimes M = M/\mathfrak{m}M \neq 0$ , so  $M \neq \mathfrak{m}M$ .

**3 implies 2:** Choose  $0 \neq x \in N$  and set  $I = \text{Ann}(x) \subset \mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$ . Then  $Ax = A/I$ , so  $Ax \otimes M \cong M/IM \neq 0$ . Because  $Ax$  injects into  $N$ ,  $Ax \otimes M$  injects into  $N \otimes M$ , which must be nonzero.

**2 implies 1:** Consider the sequence  $P \xrightarrow{f} Q \xrightarrow{g} R$ . Then because  $M$  is flat,  $\ker(g \otimes \text{id}_M) = \ker(g) \otimes M$  and  $\text{Im}(f \otimes \text{id}_M) = \text{Im}(f) \otimes M$ . If  $g \circ f = 0$ , then  $\text{Im}(g \circ f) = 0$ , which happens iff  $\text{Im}(g \otimes \text{id}_M) \otimes M = 0$ . Then  $\text{Im}((g \otimes \text{id}_M) \circ (f \otimes \text{id}_M)) = 0$ .

If  $P \otimes M \rightarrow Q \otimes M \rightarrow R \otimes M$  is exact, then  $P \rightarrow Q \rightarrow R$  is a complex. Finally, we need that  $\ker g = \text{Im } f$ . By flatness of  $M$ , we can tensor to find that  $\ker g / \text{Im } f \otimes M = 0$  and then we see that  $\ker g / \text{Im } f = 0$ .  $\square$

**Corollary 2.2.10.** *Let  $A \rightarrow B$  be a local homomorphism and let  $M$  be a finitely-generated  $B$ -module. Then  $M$  is flat over  $A$  if and only if  $M$  is faithfully flat over  $A$ .*

*Proof.* Clearly faithfully flat implies flat. Then we need to show that  $M \neq \mathfrak{m}_A M$ . By Nakayama's lemma, we know that  $M \otimes k_B \neq 0$ , so  $\mathfrak{m}_B M \neq M$ . In particular,  $\mathfrak{m}_A M \neq M$ . In particular, this implies that item 3 of the previous theorem holds. Thus  $M$  is faithfully flat over  $A$ .  $\square$

*Remark 2.2.11.* This also shows that flat and faithfully flat are equivalent over local rings. Alternatively, we can use the equivalence of flat and free.

*Remark 2.2.12.* Faithful flatness is transitive. In addition, if  $A \rightarrow B$  is a morphism of rings, and  $M$  is faithfully flat over  $A$ , then  $M \otimes_A B$  is faithfully flat over  $B$ .

**Proposition 2.2.13.** *Let  $M$  be a faithfully flat  $B$ -module which is faithfully flat over  $A$ . Then  $B$  is faithfully flat over  $A$ .*

*Proof.* Let  $N$  be an  $A$ -module. Then  $(B \otimes_A N) \otimes_B M = M \otimes_A N \neq 0$  if  $N \neq 0$ . This implies that  $B \otimes_A N$  is nonzero. Now it suffices to show that  $B$  is flat over  $A$ .

Let  $(S)$  be an exact sequence of  $A$ -modules. Then if we consider  $((S) \otimes_A M) = (S) \otimes_A M$ , this is exact by flatness of  $M$  over  $A$ . By faithful flatness of  $M$  over  $B$ , this implies that  $(S) \otimes_A B$  is exact.  $\square$

**Proposition 2.2.14.** *Let  $A \rightarrow B$  be faithfully flat. Then*

1. For any  $A$ -module  $N$ , the map  $N \rightarrow N \otimes_A B$  is injective;
2. If  $I \subset A$  is an ideal, then  $IB \cap A = I$ ;
3. The map  $\text{Spec } B \rightarrow \text{Spec } A$  is surjective.

*Proof.* 1. Let  $0 \neq x \in N$ . Then  $Ax \otimes B \hookrightarrow N \otimes B$ . Because  $B$  is faithfully flat,  $Ax \otimes B \neq 0$ .

2. Recall that  $B/IB = B \otimes A/I$ . Then the map  $A/I \rightarrow B/IB$  is injective. Therefore we have a map  $A \rightarrow B/IB$  which has kernel  $I = IB \cap A$ .
3. Choose  $\mathfrak{p} \in \text{Spec } A$ . Then  $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}$  and  $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$  is faithfully flat by base change. This means that  $\mathfrak{p}B_{\mathfrak{p}} \subsetneq B_{\mathfrak{p}}$ . Thus if we choose  $\mathfrak{m}$  to be a maximal ideal of  $B_{\mathfrak{p}}$  containing  $\mathfrak{p}B_{\mathfrak{p}}$ , we see that  $\mathfrak{m} \cap A_{\mathfrak{p}} \supset \mathfrak{p}A_{\mathfrak{p}}$  and thus they are equal by maximality of  $\mathfrak{p}A_{\mathfrak{p}}$ . Then if we choose  $\mathfrak{P} = \mathfrak{m} \cap B$ , we see that

$$\begin{aligned} \mathfrak{P} \cap A &= \mathfrak{m} \cap A \\ &= \mathfrak{m} \cap A_{\mathfrak{p}} \cap A \\ &= \mathfrak{p}A_{\mathfrak{p}} \cap A \\ &= \mathfrak{p}. \end{aligned}$$

Thus the image of  $\mathfrak{P}$  is  $\mathfrak{p}$ . □

**Theorem 2.2.15.** *Let  $\varphi : A \rightarrow B$  be a map of rings. The following are equivalent:*

1. *The map  $\varphi$  is faithfully flat.*
2. *The map  $\varphi$  is flat and  $\text{Spec } B \rightarrow \text{Spec } A$  is surjective.*
3. *The map  $\varphi$  is flat and for all maximal ideals  $\mathfrak{m}$  of  $A$ , there exists some maximal ideal  $\mathfrak{m}'$  of  $B$  such that  $\varphi^{-1}(\mathfrak{m}') = \mathfrak{m}$ .*

*Proof.* **1 implies 2:** This is the previous proposition.

**2 implies 3:** Choose a maximal ideal  $\mathfrak{m} \subset A$ . Then there exists  $\mathfrak{P} \in \text{Spec } B$  such that  $\varphi^{-1}(\mathfrak{P}) = \mathfrak{m}$ . But then if  $\mathfrak{m}'$  is any maximal ideal containing  $\mathfrak{P}$ , we see that  $\varphi^{-1}(\mathfrak{m}') = \varphi^{-1}(\mathfrak{P}) = \mathfrak{m}$  by maximality of  $\mathfrak{m}$ .

**3 implies 1:** We want to prove that  $B \neq \mathfrak{m}B$  for any maximal ideal  $\mathfrak{m}$  of  $A$ . Then there exists  $\mathfrak{m}'$  such that  $\varphi^{-1}(\mathfrak{m}') = \mathfrak{m}$ . But then  $B \supsetneq \mathfrak{m}' \supset \mathfrak{m}B$ . □

**Proposition 2.2.16 (Descent).** *Let  $A \rightarrow B$  be faithfully flat and  $M$  be an  $A$ -module. Then*

1.  *$M$  is flat (resp. faithfully flat) if and only if  $M \otimes_A B$  is  $B$ -flat (resp.  $B$ -faithfully flat).*
2. *Assume  $A$  is a local ring and  $M$  is finitely-generated. Then  $M$  is free if and only if  $M \otimes_A B$  is  $B$ -free.*

*Proof.* 1. Let  $(S)$  be an exact sequence. Then  $(S) \otimes_A B$  is exact, so  $S \otimes_A B \otimes M \otimes_A B = (S \otimes_A M) \otimes_A B$  is exact. By faithful flatness of  $B$ ,  $(S) \otimes_A M$  is exact. Now if  $N \neq 0$  is another  $A$ -module, we know that  $M_{(B)} \otimes N_{(B)} \neq 0$ , but this is the same as  $(M \otimes_A N)_{(B)}$ , so  $M \otimes_A N$  is nonzero.

2. Assume that  $A$  is local. Then suppose  $M \otimes_A B$  is free. Therefore  $M \otimes_A B$  is faithfully flat. But then,  $M$  is faithfully flat over  $A$ , which means that  $M$  is free because  $M$  is finitely generated. □

**Exercise 2.2.17.** Let  $A \subset B$  be integral domains. Assume that  $A$  and  $B$  have the same field of fractions. Prove that  $A \hookrightarrow B$  is faithfully flat if and only if  $A = B$ .

## 2.3 More on Integral Dependence

Recall Proposition 1.4.11.

**Corollary 2.3.1.** *Let  $x_1, \dots, x_n \in B$ . If each  $x_i$  is integral over  $A$ , then  $A[x_1, \dots, x_n]$  is a finitely-generated  $A$ -module.*

**Corollary 2.3.2.** *Let  $C \subset B$  be the set of integral elements over  $A$ . Then  $C$  is a subring of  $B$ .*

*Proof.* Note that  $x + y, xy \in A[x][y] \in A[x, y]$ , which is a finitely-generated  $A$  module. Therefore they are integral over  $A$ .  $\square$

**Remark 2.3.3.** The ring  $C$  is not necessarily finitely-generated over  $A$ . For an example, choose  $\mathbb{Z}\mathbb{Z} \subset \overline{\mathbb{Q}}$ .

**Definition 2.3.4.** Let  $A \subset B$ . Then we say that  $B$  is *integral over  $A$*  if all elements of  $B$  are integral over  $A$ .

**Corollary 2.3.5.** *Let  $A \subset B \subset C$  be extensions of rings. Then if  $B$  is integral over  $A$  and  $C$  is integral over  $B$ , then  $C$  is integral over  $A$ .*

Proof of this is left to the reader.

**Definition 2.3.6.** Let  $A$  be an integral domain. We say that  $A$  is *integrally closed* if for all  $x \in K = \text{Frac } A$ , then  $x$  is integral over  $A$  if and only if  $x \in A$ .

**Definition 2.3.7.** Assume that  $A \subset B$  is an inclusion of rings. Then the *integral closure of  $A$  inside  $B$*  is the set of all elements of  $B$  that are integral over  $A$ .

**Example 2.3.8.** A typical example of this situation is when  $A$  is a domain,  $K$  is its fraction field, and  $L/K$  is a field extension. Then we can consider the integral closure  $B$  of  $A$  inside  $L$ . In number theory, if  $K$  is a number field, we define its *ring of integers*  $\mathcal{O}_K$  to be the integral closure of  $\mathbb{Z}$  in  $K$ .

**Exercise 2.3.9.** If  $K = \mathbb{Q}(\zeta_p)$ , prove that  $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$ .

**Exercise 2.3.10.** Let  $B$  be an integral domain and  $A \subset B$ . Prove that the integral closure of  $A$  inside  $B$  is integrally closed.

**Lemma 2.3.11.** *Let  $B$  be a domain that is integral over  $A$ . Then  $A$  is a field if and only if  $B$  is a field.*

*Proof.* Assume that  $A$  is a field. Now choose  $0 \neq x \in B$ . But then we know that

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$$

for some  $a_i \in A$  and  $a_0 \neq 0$ . But then we have

$$x^{-1} = -a_0^{-1} \left( \sum_{i=1}^n a_i x^i \right)$$

and so  $x^{-1} \in B$ .

Now assume that  $B$  is a field. Choose  $0 \neq x \in A$ . Then  $x^{-1} \in B$ . This means that  $x^{-1}$  is integral over  $A$ , which means that

$$a^{-n} + a_{n-1}x^{-(n-1)} + \dots + a_0 = 0$$

for some  $a_i \in A, a_0 \neq 0$ . Then if we multiply by  $a^{n-1}$ , we obtain

$$a^{-1} + a_{n-1} + a_{n-2}x + \dots + a_0x^{n-1} = 0,$$

which means  $x^{-1} \in A$ .  $\square$

**Corollary 2.3.12.** *Let  $A \subset B$  and  $B$  be integral over  $A$ . Let  $\mathfrak{P} \in \text{Spec } B$  and  $\mathfrak{p} = A \cap \mathfrak{P}$ . Then  $\mathfrak{p}$  is maximal if and only if  $\mathfrak{P}$  is maximal.*

*Proof.* Note that  $B/\mathfrak{P}$  is integral over  $A/\mathfrak{p}$ . Then we apply the lemma to  $A/\mathfrak{p} \subset B/\mathfrak{P}$ .  $\square$

We can refine this into going-up and going-down. Let  $\phi : A \rightarrow B$  be a morphism of rings and let  $\phi^* : \text{Spec } B \rightarrow \text{Spec } A$  be the induced map of spaces.

**Definition 2.3.13** (Going-up). A ring homomorphism  $\phi$  satisfies the *Going-up property* if the following holds:

Let  $\mathfrak{p} \subset \mathfrak{p}'$  be prime ideals of  $A$  and suppose that  $\phi^*(\mathfrak{P}) = \mathfrak{p}$ . Then there exists  $\mathfrak{P}' \supset \mathfrak{P}$  such that  $\phi^*(\mathfrak{P}') = \mathfrak{p}'$ .

**Definition 2.3.14** (Going-down). A ring homomorphism  $\phi$  satisfies the *Going-down property* if the following holds:

Let  $\mathfrak{p} \subset \mathfrak{p}'$  be prime ideals of  $A$ . Then let  $\mathfrak{P}' \in \text{Spec } B$  satisfy  $\phi^*(\mathfrak{P}') = \mathfrak{p}'$ . Then there exists  $\mathfrak{P} \subset \mathfrak{P}'$  with  $\phi^*(\mathfrak{P}) = \mathfrak{p}$ .

**Lemma 2.3.15.** *The going-down property is equivalent to the following:*

*For all  $\mathfrak{p} \in \text{Spec } A$  and  $\mathfrak{P}$  a minimal prime ideal of  $B$  containing  $\mathfrak{p}B$ , we have  $\mathfrak{P} \cap A = \mathfrak{p}$ .*

*Proof.* First suppose that going-down holds. Then choose  $\mathfrak{P}$  be a minimal prime containing  $\mathfrak{p}B$ . Then  $\mathfrak{p}' = \phi^{-1}(\mathfrak{P}) \supset \mathfrak{p}$ . If  $\mathfrak{p}' \neq \mathfrak{p}$ , then there exists  $\mathfrak{P}_0 \subset \mathfrak{P}$  such that  $\phi^{-1}(\mathfrak{P}_0) = \mathfrak{P}$ , which contradicts minimality.

Now suppose the other condition holds. Suppose  $\mathfrak{P}'$  goes to  $\mathfrak{p}' \supset \mathfrak{p}$ . Then we know that  $\mathfrak{p}B \subset \mathfrak{p}'B \subset \mathfrak{P}'$ . If we fix  $\mathfrak{P}_0$  to be the minimal prime containing  $\mathfrak{p}B$ , then we see that  $\mathfrak{P}_0 \cap A = \mathfrak{p}$ .  $\square$

**Theorem 2.3.16.** *If  $\phi : A \rightarrow B$  is flat, then going-down holds.*

*Proof.* Fix  $\mathfrak{p} \subset \mathfrak{p}'$  and let  $\mathfrak{P}'$  lie over  $\mathfrak{p}'$ . Then we know that  $B_{\mathfrak{P}'}$  is flat over  $A_{\mathfrak{p}'}$ . Because  $A_{\mathfrak{p}'}$  is local and the map  $A_{\mathfrak{p}'} \rightarrow B_{\mathfrak{P}'}$  is local, it is faithfully flat. This implies that the map  $\text{Spec } B_{\mathfrak{P}'} \rightarrow \text{Spec } A_{\mathfrak{p}'}$  is surjective, so there exists  $\mathfrak{P}_1 \in \text{Spec } B_{\mathfrak{P}'}$  such that  $\phi^{-1}(\mathfrak{P}_1) = \mathfrak{p}A_{\mathfrak{p}'}$ .

Now set  $\mathfrak{P} := \mathfrak{P}_1 \cap B$ . Then we see that

$$\begin{aligned} \phi^{-1}(\mathfrak{P}) &= \phi^{-1}(\mathfrak{P}_1 \cap B) \\ &= \phi^{-1}(\mathfrak{P}_1) \cap A \\ &= \mathfrak{p}A_{\mathfrak{p}'} \cap A \\ &= \mathfrak{p}. \end{aligned} \quad \square$$

We will see consequences of this result in algebraic geometry.

We will now consider integral ring extensions  $A \subset B$ .

**Theorem 2.3.17** (Cohen-Seidenberg). *Suppose  $A \subset B$  is an integral extension. Then the following hold:*

1. *The map  $\text{Spec } B \rightarrow \text{Spec } A$  is surjective.*
2. *There are no inclusion relations between the prime ideals of  $B$  which are above a fixed prime ideal of  $A$ .*
3. *Going-up holds for  $A \subset B$ .*
4. *If  $A$  is local with maximal ideal  $\mathfrak{m}$ , then the prime ideals of  $B$  lying over  $\mathfrak{m}$  are precisely the maximal ideals of  $B$ .*

5. Assume further that  $A$  and  $B$  are integral domains and that  $A$  is integrally closed. Then going-down holds for  $A \subset B$ .
6. If  $B$  is the integral closure of  $A$  in a normal extension of field  $L$  of  $K := \text{Frac } A$ , then any two prime ideals of  $B$  lying over the same prime ideal of  $A$  are conjugate by an element of  $\text{Aut}(L/K)$ .

*Proof.* We prove **4**, then **1**, **2**, and **3**, then **6**, and finally **5**.

1. Let  $\mathfrak{p} \in \text{Spec } A$ . Then  $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}$  is integral over  $A_{\mathfrak{p}}$ . Applying **4**, we obtain the desired result.
2. Consider  $B_{\mathfrak{p}}$  again. By **4**, because ideals lying over  $\mathfrak{p}$  are maximal, there cannot be inclusion relations between them.
3. Let  $\mathfrak{p} \subset \mathfrak{p}'$  and  $\mathfrak{P} \in \text{Spec } B$  lying over  $\mathfrak{p}$ . Then  $B/\mathfrak{P}$  is integral over  $A/\mathfrak{p}$ . By **1**, we know that  $\text{Spec } B/\mathfrak{P} \rightarrow \text{Spec } A/\mathfrak{p}$  is surjective. Thus there exists  $\overline{\mathfrak{P}}' \in \text{Spec } B/\mathfrak{P}$  lying over  $\overline{\mathfrak{p}}' = \mathfrak{p}'/\mathfrak{p}$ . Then we know that  $\overline{\mathfrak{P}}' = \mathfrak{P}'/\mathfrak{P}$  for some prime ideal  $\mathfrak{P}'$  of  $B$ , and this is the ideal we are looking for.
4. This is a consequence of Lemma 2.3.15.
5. Write  $L = \text{Frac } B \supset K = \text{Frac } A$ . Then let  $L_1$  be the normal closure of  $L/K$ . Then let  $\mathfrak{p} \subset \mathfrak{p}'$  in  $A$  and  $\mathfrak{P}'$  in  $B$  lie over  $\mathfrak{p}'$ . Then let  $\mathfrak{P}_1 \subset \mathfrak{P}'_1$  in  $B_1$  the integral closure of  $A$  in  $L_1$ . These exist thanks to **1** and **3**.

Let  $\mathfrak{P}'_1$  in  $B_1$  such that  $\mathfrak{P}'_1 \cap B = \mathfrak{P}'$ . Then there exists  $\sigma$  such that  $\mathfrak{P}'_1 = \sigma(\mathfrak{P}_1)$  because both ideals are above  $\mathfrak{p}'$ . Then we can choose

$$\mathfrak{P} := \sigma(\mathfrak{P}_1) \cap B \subset \mathfrak{P}'_1 \cap B = \mathfrak{P}'.$$

We need to show that  $\mathfrak{P} \cap A = \mathfrak{p}$ . But this is simply

$$\begin{aligned} \mathfrak{P} \cap A &= \sigma(\mathfrak{P}_1) \cap A \\ &= \sigma(\mathfrak{P}_1 \cap A) \\ &= \sigma(\mathfrak{p}) \\ &= \mathfrak{p}. \end{aligned}$$

6. We know that  $A$  is integrally closed in  $K$ . Then let  $L/K$  be a finite Galois (we can always reduce to this case) extension and  $B$  the integral closure of  $A$  in  $L$ . Then let  $\mathfrak{P}, \mathfrak{P}' \in \text{Spec } B$  lie above  $\mathfrak{p} \in \text{Spec } A$ . We will show there exists  $\sigma \in \text{Gal}(L/K)$  such that  $\sigma(\mathfrak{P}) = \mathfrak{P}'$ .

Suppose that no such  $\sigma$  exists. Then for all  $\sigma \in \text{Gal}(L/K)$ ,  $\mathfrak{P}' \neq \sigma(\mathfrak{P})$ . In particular,  $\mathfrak{P}' \not\subset \sigma(\mathfrak{P})$ . Then there exists  $x \in \mathfrak{P}'$  which is not in any  $\sigma(\mathfrak{P})$  then we see that

$$y := \prod_{\sigma \in \text{Gal}(L/K)} \sigma(x) \in K$$

is integral over  $A$ , so  $y \in A$ . Also,  $y \notin \mathfrak{P}$  because  $x \notin \sigma(\mathfrak{P})$ , so  $x \in \mathfrak{P}'$  and thus  $y \in \mathfrak{P}'$ , so  $y \in \mathfrak{p} \subset \mathfrak{P}$ . This gives a contradiction.  $\square$

**Corollary 2.3.18.** Assume that  $B$  is integral over  $A$ .

1. If  $\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_r$  is a chain of prime ideals of  $B$ , then the  $\mathfrak{p}_i := \mathfrak{P}_i \cap A$  for a chain of prime ideals of  $A$ .



2. If  $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$  is a chain of prime ideals of  $A$ , then there exists a chain  $\mathfrak{P}_0 \subsetneq \cdots \subsetneq \mathfrak{P}_r$  of prime ideals of  $B$  above it.
3. If  $A$  is integrally closed and  $B$  is a domain, then for any chain  $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$  and  $\mathfrak{P}_r \in \text{Spec } B$  above  $\mathfrak{p}_r$ , there exists a chain  $\mathfrak{P}_0 \subsetneq \cdots \subsetneq \mathfrak{P}_r$  above the chain in  $A$ .

*Proof.* The proof is clear and left to the reader. □

**Definition 2.3.19.** Let  $\mathfrak{p} \in \text{Spec } A$ . Then define the *height* of  $\mathfrak{p}$  by

$$\text{ht}(\mathfrak{p}) = \max \{n \geq 0 \mid \exists \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

Then define the *dimension* of  $A$  by

$$\dim A = \max \{\text{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec } A\}.$$

**Corollary 2.3.20.** Let  $A \subset B$  be an integral extension. Then

1. Suppose  $\mathfrak{P} \in \text{Spec } B$  lies above  $\mathfrak{p} \in \text{Spec } A$ . Then  $\text{ht}(\mathfrak{P}) \leq \text{ht}(\mathfrak{p})$ .
2.  $\dim A = \dim B$ .
3. If  $A$  is integrally closed and  $B$  is a domain, then we have  $\text{ht}(\mathfrak{P}) = \text{ht}(\mathfrak{p})$ .

*Proof.* This is an immediate consequence of the previous corollary. □

## 2.4 Associated Primes

Let  $M$  be an  $A$ -module and  $\mathfrak{p} \in \text{Spec } A$ .

**Definition 2.4.1.** We say that  $\mathfrak{p}$  is an *associated prime* of  $M$  if one of the two following equivalent conditions hold.

1. There exists  $x \in M$  such that  $\text{Ann}_A(x) = \mathfrak{p}$ ;
2. There is an injection  $A/\mathfrak{p} \hookrightarrow M$ .

We will denote the set of associated primes using the unfortunate notation  $\text{Ass}_A(M)$ . Then the set of primes  $\mathfrak{p}$  such that  $M_{\mathfrak{p}} \neq 0$  will be denoted  $\text{Supp}_A(M)$ .

**Proposition 2.4.2.** Let  $\mathfrak{p}$  be a maximal element of  $\{\text{Ann}(x) \mid x \in M, x \neq 0\}$ . Then  $\mathfrak{p} \in \text{Ass}_A(M)$ .

*Proof.* We will show that such a maximal element is actually a prime ideal. Suppose  $ab \in \mathfrak{p}$ . Then  $\mathfrak{p} = \text{Ann}(x)$  for some nonzero  $x$ , so  $b \cdot x \neq 0$ . Then  $\text{Ann}(x) \subset \text{Ann}(bx) \neq A$ . By maximality,  $\text{Ann}(x) = \text{Ann}(bx)$ . Because  $abx = 0$ , then  $a \in \text{Ann}(bx) = \mathfrak{p}$ . □

**Corollary 2.4.3.** Let  $A$  be Noetherian.

1.  $M$  is nonzero if and only if  $\text{Ass}_A(M)$  is nonempty.
2. The set of zero divisors for  $M$  is the union of the associated primes of  $M$ .

*Proof.* 1. If there is some associated prime, then clearly  $M \neq 0$ . In the other direction, the set of annihilators has a maximal element because  $A$  is Noetherian, so there must be an associated prime.

2. Let  $a \in \text{Ann}(x)$  for some nonzero  $x \in M$ . Then  $\text{Ann}(x) \subset \mathfrak{p}$  is contained in some associated prime (because it is contained in some maximal element), and thus every zero divisor is contained in an associated prime. The other direction is obvious.  $\square$

**Lemma 2.4.4.** *Let  $S \subset A$  be a multiplicative set and  $M$  an  $A$ -module. Then*

$$\text{Ass}_A(S^{-1}M) = \varphi^*(\text{Ass}_{S^{-1}A}(S^{-1}M)).$$

*Proof.* Let  $\mathfrak{p} \in \text{Ass}_A(S^{-1}M)$ . Then  $\mathfrak{p} = \text{Ann}_A \frac{x}{1}$  for some  $x \in M$ , so  $\mathfrak{p} \cap S$  must be empty. Next, we see that the set  $\{\text{Ann}_A(sx) \mid s \in S\}$  contains some maximal element  $\mathfrak{m}$  because  $A$  is Noetherian. But then  $\mathfrak{m} = \text{Ann}_A(s_0 \cdot x) = \mathfrak{p}$ .

On the other hand, if  $a \in \mathfrak{p}$ , then  $\frac{ax}{1} = 0$ , which means  $asx = 0$  for some  $s \in S$ . Then  $a \in \text{Ann}_A(sx) \subset \text{Ann}(s_0sx) = \text{Ann}(s_0x)$ . Thus  $\mathfrak{p} \subset \text{Ann}(s_0x)$ . Thus we have shown that

$$\text{Ass}_A(S^{-1}M) \subset \varphi^* \text{Ass}_{S^{-1}A}(S^{-1}M).$$

The other inclusion is clear.  $\square$

**Theorem 2.4.5.** *Let  $A$  be Noetherian and  $M$  an  $A$ -module. Then  $\text{Ass}_A(M) \subset \text{Supp}_A(M)$  and any minimal element of  $\text{Supp}_A(M)$  is inside  $\text{Ass}_A(M)$ .*

*Proof.* Let  $\mathfrak{p} \in \text{Ass}_A(M)$ . Then  $A/\mathfrak{p}$  injects in  $M$ , so we have an injection  $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$ . Thus  $\mathfrak{p} \in \text{Supp}_A(M)$ .

Now choose a minimal  $\mathfrak{p} \in \text{Supp}_A(M)$ . Thus  $M_{\mathfrak{p}}$  is nontrivial, so there exists a prime ideal  $\mathfrak{q} \subset \mathfrak{p}$  such that  $\mathfrak{q}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ . Thus  $M_{\mathfrak{q}} = (M_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}}$  is nonzero, so  $\mathfrak{q} \in \text{Supp}(M)$ . By minimality,  $\mathfrak{q} = \mathfrak{p}$  and thus  $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ . Therefore  $\mathfrak{p} \in \text{Ass}_A(M)$ .  $\square$

**Definition 2.4.6.** If  $\mathfrak{p} \in \text{Ass}_A(M)$ , then  $\mathfrak{p}$  is not necessarily minimal in the support of  $M$ . Then such a prime is called an *embedded prime*.

**Proposition 2.4.7.** *Let  $A$  be Noetherian and  $M$  a finitely-generated  $A$ -module. Then*

1. *There exists a chain*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

*such that  $M_i/M_{i-1} \cong A/\mathfrak{p}_i$  for  $i = 1, \dots, n$  and  $\mathfrak{p}_i \in \text{Supp}_A(M)$ .*

2. *Given such a sequence, we have  $\text{Ass}_A(M) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ . In particular this set is finite.*

*Proof.* 1. Suppose  $M \neq 0$ . Then choose  $\mathfrak{p}_1 \in \text{Ass}_A(M)$  and let  $M_1$  be the image of  $A/\mathfrak{p}_1$  in  $M$ . Then if  $M/M_1$  is nonzero, choose  $\mathfrak{p}_2 \in \text{Ass}_A(M/M_1)$  and  $M_2$  defined analogously to  $M_1$ . This gives a sequence of submodules of  $M$  such that  $A/\mathfrak{p}_i \cong M_i/M_{i-1}$ . Because  $M$  is Noetherian, this sequence becomes stationary. Thus there exists  $n$  such that  $M_n = M$ .

2. This is a consequence of the next lemma.  $\square$

**Remark 2.4.8.** In general the support of a module is **not** finite.

**Lemma 2.4.9.** *Assume we have an exact sequence of modules  $0 \rightarrow M' \rightarrow M \rightarrow M''$ . Then  $\text{Ass}(M) \subset \text{Ass}(M') \cup \text{Ass}(M'')$ .*

*Proof.* If  $\mathfrak{p} \in \text{Ass}(M)$ , there exists  $N \subset M$  such that  $N \cong A/\mathfrak{p}$ . Then if  $N \cap M' = 0$ ,  $N \hookrightarrow M''$  and  $\mathfrak{p} \in \text{Ass}(M'')$ . If the intersection is nonzero, then there exists some nonzero  $x \in N \cap M'$  such that  $\text{Ann}_A(x) = \mathfrak{p}$  because  $A/\mathfrak{p}$  is a domain. Thus  $\mathfrak{p} \in \text{Ass}(M')$ .  $\square$

**Definition 2.4.10.** We say that  $M$  is *coprimary* if  $\text{Ass}_A(M) = \{\mathfrak{p}\}$ .

**Definition 2.4.11.** Let  $N \subset M$ . Then we say that  $N$  is  $\mathfrak{p}$ -*primary* if  $\text{Ass}_A(M/N) = \{\mathfrak{p}\}$ . Alternatively, we say that  $N$  *belongs to*  $\mathfrak{p}$ .

**Lemma 2.4.12.** A module  $M$  is coprimary if and only if  $M$  is nonzero and any zero divisor for  $M$  is locally nilpotent (for all  $x \in M$ , there exists  $n > 0$  such that  $a^n \cdot x = 0$ ).

*Proof.* Suppose that  $M$  is coprimary. Now suppose that  $a \in \mathfrak{p}$  and  $x \in M$ . Then  $\text{Ass}(Ax) = \{\mathfrak{p}\}$ , so  $\mathfrak{p}$  is minimal in the support of  $Ax$ , which is  $V(\text{Ann}(x))$ . Therefore,  $\mathfrak{p} = \sqrt{\text{Ann}(x)}$ . Thus for  $a \in \mathfrak{p}$ ,  $a^n \in \text{Ann}(x)$ .

In the other direction, let  $\mathfrak{p}$  be the set of locally nilpotent elements with respect to  $M$ . This is clearly an ideal of  $A$ . Then let  $\mathfrak{q} \in \text{Ass}_A(M)$ . Then  $x \in M$ , so  $\mathfrak{q} = \text{Ann}(x)$ . Therefore  $\mathfrak{p} \subset \mathfrak{q}$  because  $\mathfrak{q}$  is a prime ideal. However,  $\mathfrak{q}$  is contained in the set of zero divisors, which is precisely  $\mathfrak{p}$ .  $\square$

**Remark 2.4.13.** Let  $I \subset A$  be an ideal. Then  $\text{Ass}_A(A/I) = \{\mathfrak{p}\}$  if and only if the zero divisors of  $A/I$  are locally nilpotent. This is equivalent to  $I$  being primary.

**Lemma 2.4.14.** 1. Let  $Q_1, Q_2 \subset M$  be  $\mathfrak{p}$ -primary submodules. Then  $Q_1 \cap Q_2$  is  $\mathfrak{p}$ -primary.

2. Let  $N = Q_1 \cap \cdots \cap Q_r$  be an irredundant decomposition (i.e.  $Q_i$  is  $\mathfrak{p}_i$ -primary) for distinct  $\mathfrak{p}_i$ . Then  $\text{Ass}_A(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ .

*Proof.* 1. Note that  $M/Q_1 \cap Q_2$  injects in  $M/Q_1 \oplus M/Q_2$ . The desired result follows from the previous lemma.

2. First, note that  $M/N \hookrightarrow \bigoplus M/Q_i$ . Then suppose  $\text{Ass}(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ . Then we have an injection

$$\frac{Q_2 \cap \cdots \cap Q_r}{N} \hookrightarrow M/N$$

and thus  $\text{Ass}((Q_2 \cap \cdots \cap Q_r)/N)$  is contained in  $\text{Ass}(M/N)$ . By the exact sequence

$$0 \rightarrow N \rightarrow Q_2 \cap \cdots \cap Q_r \rightarrow M/Q_1,$$

we see that  $\text{Ass}(Q_2 \cap \cdots \cap Q_r/N) = \{\mathfrak{p}_1\}$ .  $\square$

**Theorem 2.4.15.** Let  $M$  be a module over a Noetherian ring  $A$ . Then for all  $\mathfrak{p} \in \text{Ass}(M)$ , there exists a  $\mathfrak{p}$ -primary submodule  $Q(\mathfrak{p}) \subset M$  such that

$$\bigcap_{\mathfrak{p} \in \text{Ass}(M)} Q(\mathfrak{p}) = \{0\}.$$

*Proof.* Fix  $\mathfrak{p} \in \text{Ass}(M)$ . Consider the set

$$\mathcal{S}_{\mathfrak{p}} = \{N \subseteq M \mid 0 \notin \text{Ass}(N)\}.$$

This set is nonempty because  $0 \in \mathcal{S}_{\mathfrak{p}}$ . Next, if  $N_{\lambda} \in \mathcal{S}_{\mathfrak{p}}$  is a chain, then the module  $N = \bigcup N_{\lambda}$  is a submodule of  $M$ . In addition,  $\text{Ass}(N) \subset \bigcup \text{Ass}(N_{\lambda})$ . This implies that  $\mathcal{S}_{\mathfrak{p}}$  contains a maximal element by Zorn's lemma. Choose such a maximal element  $Q(\mathfrak{p})$ .

We will show that  $M/Q(\mathfrak{p})$  is coprimary. By the exact sequence

$$0 \rightarrow Q(\mathfrak{p}) \rightarrow M \rightarrow M/Q(\mathfrak{p}),$$

if  $\mathfrak{p}' \in \text{Ass}(M/Q(\mathfrak{p}))$ , then  $\mathfrak{p}' = \mathfrak{p}$  because otherwise  $A/\mathfrak{p}'$  would inject in  $M/Q(\mathfrak{p})$  as  $Q'/Q(\mathfrak{p})$ . Then  $\text{Ass}(Q') \subset \text{Ass}(Q(\mathfrak{p})) \cup \text{Ass}(Q'/Q(\mathfrak{p}))$ , so  $Q' \supsetneq Q(\mathfrak{p})$ , contradicting minimality. Thus  $\text{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$ .

The second part of the claim follows immediately from the fact that  $\text{Ass}(\bigcap Q(\mathfrak{p})) = \bigcap \text{Ass}(Q(\mathfrak{p})) = \emptyset$ .  $\square$

**Corollary 2.4.16.** *Let  $M$  be an  $A$ -module of finite type. Then any  $N \subset M$  has a primary decomposition*

$$N = Q_1 \cap \cdots \cap Q_r$$

such that

1. The  $Q_i$  are  $\mathfrak{p}_i$ -primary;
2. No  $Q_i$  can be omitted;
3. This decomposition is irredundant:  $\text{Ass}(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ .

*Proof.* Apply the previous theorem to  $M/N$ . Because  $M/N$  is of finite type,  $\text{Ass}(M/N)$  is finite. Then use the previous lemma.  $\square$

**Exercise 2.4.17.** Let  $A \xrightarrow{\varphi} B$  be a morphism of rings and let  $M$  be a  $B$ -module. Then prove that

$$\varphi^*(\text{Ass}_B(M)) = \text{Ass}_A(M)$$

where  $\varphi^* : \text{Spec } B \rightarrow \text{Spec } A$  is the induced map of spaces.

## Dimension Theory

### 3.1 Graded Rings and Modules

Let  $A = \bigoplus_{n \geq 0} A_n$  be a graded ring. This means that  $A_n \cdot A_m \subset A_{n+m}$ . Then an  $A$ -module  $M$  is a graded module if

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

such that  $A_n \cdot M_m \subset M_{n+m}$ . We will call  $M_m$  the homogeneous elements of degree  $m$  on  $M$ .

Now let  $N \subset M$  be a submodule. We say that  $N$  is a graded submodule if  $N = \bigoplus N \cap M_m$ .  $N$  is also called homogeneous. A homogeneous element of  $M$  is an element of some  $M_m$ . Being a graded submodule is the same as every element being a sum of homogeneous elements.

**Lemma 3.1.1.** *The following are equivalent:*

1.  $N$  is a homogeneous submodule.
2.  $N$  is generated by homogeneous elements.
3. If  $x = x_r + \cdots + x_n \in N$  for  $x_i \in M_i$ , then for all  $i$ ,  $x_i \in N$ .

Moreover, if  $N \subset M$  is homogeneous, then so is  $M/N$ , and

$$M/N = \bigoplus_m M_m/N_m.$$

*Proof.* The proof is left as an exercise to the reader. □

**Example 3.1.2.** Let  $k$  be any ring. Then the ring  $A = k[x_1, \dots, x_r]$  is a graded ring where the grading is by the degree of each monomial. In particular,  $A_0 = k$ . Then an ideal  $I \subset A$  is graded if  $I = \bigoplus_n I_n$  where  $I_n = I \cap A_n$ . In addition,  $A/I$  is a graded ring.

**Proposition 3.1.3.** *Let  $A$  be a Noetherian graded ring and  $M$  a graded  $A$ -module. Then*

1. If  $\mathfrak{p} \in \text{Ass}(M)$ , then  $\mathfrak{p}$  is a graded ideal of  $A$  and there exists a homogeneous  $x \in M$  such that  $\mathfrak{p} = \text{Ann}(x)$ .
2. One can choose a  $\mathfrak{p}$ -primary graded submodule  $Q(\mathfrak{p})$  such that  $0 = \bigcap_{\mathfrak{p} \in \text{Ass}(M)} Q(\mathfrak{p})$ .

*Proof.* Let  $x \in M$  and set  $\mathfrak{p} = \text{Ann}(M)$ . Then write  $x = x_e + x_{e-1} + \cdots + x_0$ . Then for  $f \in \mathfrak{p}$ , write  $f = f_r + \cdots + f_s$ . If  $fx = 0$ , then we can write

$$0 = fx + f_r x_e + (f_{r-1} x_e + f_r x_{e-1}) + \cdots$$

and deduce that  $0 = f_r x_e = f_r^2 x_{e-1} = \cdots$ . Then  $f_r^e \in \mathfrak{p}$ , so  $f_r \in \mathfrak{p}$ . By induction, all  $f_i \in \mathfrak{p}$ , so  $\mathfrak{p}$  is graded.

The proof of the second part is simply the following lemma.  $\square$

**Lemma 3.1.4.** *Let  $\mathfrak{p}$  be a graded prime ideal and  $Q \subset M$  such that  $Q$  is  $\mathfrak{p}$ -primary. Let  $Q' \subset Q$  be the submodule of  $Q$  generated by the homogeneous elements of  $Q$ . Then  $Q'$  is  $\mathfrak{p}$ -primary.*

*Proof.* This will be proved later.  $\square$

We will now discuss filtrations of rings. A *filtration* is a sequence of subgroups

$$A = J_0 \supset J_1 \supset J_2 \cdots$$

such that  $J_n \cdot J_m = J_{n+m}$ . If we set

$$A' = \bigoplus_{n=0}^{\infty} J_n / J_{n+1},$$

then  $A'$  is a graded ring.

The basic example is  $J_m = I^m$  for some fixed ideal  $I \subset A$ . In this case, the filtration is called the *I*-adic filtration.

**Lemma 3.1.5.** *Let  $A$  be a Noetherian ring and set  $I \subset A$ . Then*

$$\text{gr}^I A = \bigoplus_{n=0}^{\infty} I^n / I^{n+1}$$

*is a Noetherian graded ring.*

*Proof.* Because  $I$  is finitely-generated, then  $I/I^2$  is a finitely-generated  $A/I$ -module. Thus  $\text{gr}^I(A)$  is a finitely-generated  $A/I$ -algebra. If  $x_1, \dots, x_r$  is a set of generators of  $I$ , then

$$A/I[x_1, \dots, x_r] \rightarrow \text{gr}^I A$$

is surjective, so because  $A/I$  is Noetherian, so is  $A/I[x_1, \dots, x_r]$  and thus so is  $\text{gr}^I A$ .  $\square$

Let  $A$  be an Artinian ring and  $B = A[x_1, \dots, x_r]$  be a graded ring. Then let  $M$  be a finitely-generated graded  $B$ -module. Each graded piece  $M_n$  is an  $A$ -module, so write  $F_M(n) = \ell_A(M_n)$ . Because  $M$  is finitely generated, we have a map

$$\bigoplus_{i=1}^r B(d_i) \twoheadrightarrow M.$$

Here,  $B(d_i) = B$  as a  $B$ -module with the gradation  $B(d)_n = B_{n-d}$ . Thus  $M$  is generated by homogeneous elements  $x_{d_i}$  of degree  $d_i$ . This gives us the map

$$\begin{aligned} \bigoplus_{i=1}^r B_{m-d_i} &\hookrightarrow M_m \\ (f_i) &\mapsto \sum_{i=1}^r f_i x_{d_i} \end{aligned}$$

and thus  $\ell_A(M_m) \leq \sum_{i=1}^r \ell_A(B_{m-d_i})$ . But then  $B_m$  is a free  $A$ -module, and thus

$$\ell_A(B_m) \leq \binom{r+m-1}{m-1} \ell(A).$$

**Theorem 3.1.6.** *Let  $A, B, M$  be as above. Then there is a polynomial  $f_M(x) \in \mathbb{Q}[x]$  such that*

$$\ell_A(M_n) = f_M(n)$$

for  $n \gg 0$ . This is called the Hilbert-Samuel polynomial for  $M$ . The degree of this polynomial will give the first definition for the dimension of  $M$ .

*Proof.* Say that  $M$  satisfies the property  $P(M)$  if there exists  $f \in \mathbb{Q}(x)$  such that  $\ell(M_n) = f(n)$  for  $n \gg 0$ .

1. First, we will show that if  $N_1, N_2 \subset M$  and  $P(M/N_1), P(M/N_2)$  hold, then  $P(M/N_1 \cap N_2)$  holds.
2. Second, if  $N$  is irreducible, then  $P(M/N)$  holds.

If we prove these two things, then we simply use a primary decomposition of  $0 \subset M$ . Then the second statement implies that  $P(M/N_i)$  holds and then we simply repeatedly apply the first to see that  $P(M)$  holds.

1. Suppose  $N = N_1 \cap N_2$  with  $N, N_1, N_2$  graded. We then have an exact sequence

$$0 \rightarrow N_1/(N_1 \cap N_2) \rightarrow M/(N_1 \cap N_2) \rightarrow M/N_1 \rightarrow 0$$

and we know that  $N_1/(N_1 \cap N_2) = (N_1 + N_2)/N_1$  is graded. Then we see that  $F_{M/(N_1 \cap N_2)} = F_{M/N} + F_{(N_1+N_2)/N_1}$ , so we only need to prove that  $F_{N_1+N_2/N_1}$  exists. But then  $F_{N_1+N_2/N_1} = F_{M/N_2} - F_{M/N_1+N_2}$  and because  $P(M/N_1 + N_2)$  holds, so does  $P(M/N_1 \cap N_2)$ .

2. Let  $N$  be irreducible. We know that  $M' = M/N$  is coprimary, so  $N$  is  $\mathfrak{p}$ -primary for some prime ideal  $\mathfrak{p} \subset B$ . Write  $I = (x_1, \dots, x_m)$ . If  $I \subset \mathfrak{p}$ , then  $M'_n = 0$  for  $n \gg 0$ . Indeed, if  $d$  is the maximal degree of a system of generators of  $M'$ , then  $M'_{n+d} = I^n \cdot M'_d$ . On the other hand, because  $M'$  is  $\mathfrak{p}$ -primary, then elements in  $\mathfrak{p}$  are locally nilpotent. Thus there exists  $k \gg 0$  such that  $p^k \cdot M'_d = 0$  and thus  $M'_{n+d} = 0$  for  $n \geq k$ . Thus  $F_{M/N}$  exists and is identically zero.

In the second case,  $I \not\subset \mathfrak{p}$ . Then suppose that  $x_1 \notin \mathfrak{p}$ . Thus  $x_1$  is not a zero divisor for  $M'$ . Thus, we have an exact sequence

$$0 \rightarrow M' \rightarrow x_1 M' \rightarrow M'/x_1 M' \rightarrow 0$$

which then gives

$$0 \rightarrow (M/N)_{n-1} \rightarrow (M/N)_n \rightarrow (M/N + x_1 M)_n \rightarrow 0$$

when restricting to a single graded piece. Thus  $N \subsetneq N + x_1 M$ . This implies that  $f_{M/N+x_1 M}$  exists because above, we proved that if  $P(M/N')$  holds for any  $N' \supsetneq N$  implies that  $P(M/N)$  holds. Then for  $n \geq n_0$ , we have  $\ell((M/N)_m) - \ell((M/N)_{n-1}) = f_{M/N+x_1 M}(n)$ . This implies that

$$\ell((M/N)_m) = f(n) + f(n-1) + \dots + \ell(M/N)_{n_0}.$$

Then  $f(n) + \dots + f(n_0) = g(n)$  for some polynomial  $g$  of degree  $\deg f + 1$  and then  $f_{M/N} = g + \ell((M/N)_{n_0})$ .  $\square$

Now let  $A \supset I$  and  $M$  be an  $A$ -module with filtration

$$M_0 = M \supset M_1 \supset \cdots \supset M_n \supset \cdots$$

We say that the filtration is

1. *I*-admissible if  $IM_n \subset M_{n+1}$  for all  $n \gg 0$ ;
2. *I*-adic if  $IM_n = M_{n+1}$  for all  $n \geq 0$ ;
3. essentially *I*-adic if  $IM_n = M_{n+1}$  for  $n \gg 0$ .

*Remark 3.1.7.* A filtration on  $M$  defines a topology on  $M$  so that  $M$  is a topological group. Here, a system of neighborhoods of 0 is  $(M_n)_{n \gg 0}$ . If  $\bigcap M_n = 0$ , then the topology is Hausdorff. If the filtration is essentially *I*-adic, then the topology is called the *I*-adic topology.

**Lemma 3.1.8.** *Let  $A$  and  $I$  be as before. Let  $M$  be an  $A$ -module with an admissible filtration. Let  $A' = \bigoplus_{n=1}^{\infty} I^n x^n \subset A[x]$  and*

$$M' = \bigoplus M_n \otimes_A A x^n = \bigoplus M_n x^n.$$

1.  $M'$  is a  $A'$ -module.
2. The filtration is essentially *I*-adic if and only if  $M'$  is a finitely-generated  $A'$ -module.

*Proof.* 1. This is trivial.

2. Note that  $M'$  is a graded  $A'$ -module. If  $M'$  is finitely generated, then write  $M' = A' m_1 + \cdots + A' m_r$ . Then we see that  $M'_n = I x M'_{n-1}$  for  $n > \max \{\deg m_i\}$ . Thus  $M_n$  is essentially *I*-adic.

Conversely, if  $M_n = I^{n-n_0} M_{n_0}$  for  $n \geq n_0$ , then, then it is clear that  $M'$  is generated by  $M_{n_0} x^{n_0} + \cdots + M_1 x + M_0$  and is thus finitely generated.  $\square$

**Theorem 3.1.9 (Artin-Rees).** *Let  $A$  be a Noetherian ring and  $I \subset A$ . Then let  $M$  be a finitely-generated  $A$ -module and  $N \subset M$  be a submodule. Then there exists  $r > 0$  such that*

$$I^n M \cap N = I^{n-r} (I^r M \cap N)$$

for all  $n \geq r$ .

*Proof.* Let  $M_n = I^n M$  be the *I*-adic filtration. Then  $N_n = I^n M \cap N$  is *I*-admissible. Then both  $N' \subset M'$  are both  $A'$ -modules. We know that  $A'$  is Noetherian, so because  $M_n$ ,  $M'$  is finitely generated. Thus  $N'$  is also Noetherian, so it is also finitely-generated. This implies that  $N_m$  is essentially *I*-adic, as desired.  $\square$

*Remark 3.1.10.* This theorem is saying that the filtration  $I^n M \cap N$  is essentially *I*-adic.

**Corollary 3.1.11 (Krull Intersection Theorem).** *Let  $A, I, M$  as above.*

1. If  $N = \bigcap_{n=0}^{\infty} I^n M$ , then  $IN = N$ .
2. If  $I \subset \text{rad}(A)$ , then  $\bigcap_{n=0}^{\infty} I^n M = 0$ .

*Proof.* 1. Note that  $N \subset M$ . Then apply the Artin-Rees theorem to  $N = I^n M \cap N$ .

2. Apply Nakayama's lemma.  $\square$



**Corollary 3.1.12.** *Let  $A$  be a Noetherian domain and let  $I \subset A$  be a proper ideal. Then  $\bigcap_n I^n = 0$ .*

*Proof.* Let  $N = \bigcap_{n=0}^{\infty} I^n$ . By the previous corollary,  $IN = N$ . Then  $N$  is finitely generated because  $A$  is Noetherian. Thus there exists  $x \in I$  such that  $(1+x)N = 0$ , which implies that  $N = 0$  because  $A$  is a domain.  $\square$

**Exercise 3.1.13.** Let  $A$  be Noetherian and  $M$  a finitely-generated  $A$ -module. Then let  $I, J$  be generated by  $M$ -regular elements. Then there exists  $r > 0$  such that  $(I^n M : J) = I^{n-r}(I^r M : J)$ . Here,  $(N : J) = \{m \in M \mid Jm \subset N\}$ .

## 3.2 Other Notions of Dimension

Let  $A$  be a ring. Then we define the *Krull dimension*

$$\dim A = \sup \{\text{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec } A\}.$$

Then for any ideal  $I \subset A$ , define the height of  $I$  to be

$$\text{ht}(I) = \inf \{\text{ht}(\mathfrak{p}) \mid \mathfrak{p} \supset I\}.$$

**Proposition 3.2.1.** *For any ideal  $I$ , we have  $\dim(A/I) + \text{ht}(I) \leq \dim A$ .*

*Proof.* Consider a chain

$$\mathfrak{p}_{d'} \supsetneq \cdots \supsetneq \mathfrak{p}_0 \supset I$$

where  $d' = \dim(A/I)$ . Then we see that  $\dim A \geq d' + \text{ht}(\mathfrak{p}_0) \geq d' + \text{ht}(I)$ , as desired.  $\square$

Then if  $M$  is an  $A$ -module, define  $\dim M = \dim(A / \text{Ann}_A M)$ .

**Proposition 3.2.2.** *Assume that  $A$  is Noetherian and  $M$  is a finite  $A$ -module. Then the following are equivalent:*

1.  $M$  is of finite length.
2.  $A / \text{Ann}_A M$  is Artinian.
3.  $\dim M = 0$ .

*Proof.* Clearly conditions 2 and 3 are equivalent. Then  $M$  is a quotient of  $(A / \text{Ann}_A M)^r$ , so 2 implies 1. Thus we need to prove that 1 implies 3.

Assume that  $\ell(M) < \infty$ . If we write  $A' = A / \text{Ann}_A(M)$ , then  $M$  is a finite  $A'$ -module. If  $\dim A' > 0$ , then there exists  $\mathfrak{p} \subset A'$  that is minimal but not maximal. Then because  $\text{Ann}_{A'}(M) = 0$ , we have  $\mathfrak{p} \in V((0)) = \text{Supp}(M)$ . But then  $\mathfrak{p} \in \text{Ass}_{A'}(M)$  and thus we have an embedding  $A'/\mathfrak{p} \hookrightarrow M$ . But then  $\dim M'/\mathfrak{p} > 0$ , so  $\ell(A'/\mathfrak{p}) = \infty$  and thus  $\ell(M) = \infty$ .  $\square$

Now let  $A$  be a semilocal ring. Let  $\mathfrak{m} = \text{Rad } A$ . Then an ideal  $I \subset A$  is called an *ideal of definition* of  $A$  if there exists  $s > 0$  such that  $\mathfrak{m}^s \subset I \subset \mathfrak{m}$ .

*Remark 3.2.3.*  $I$  is an ideal of definition if and only if  $A/I$  is Artinian.

Let  $A^* = \text{gr}^I(A) = \bigoplus_{n=0}^{\infty} I^n / I^{n+1}$  be the graded ring with respect to the  $I$ -adic filtration and let  $M^* = \text{gr}^I(M)$  be the corresponding graded  $A^*$ -module. Then if  $I = (x_1, \dots, x_r) \subset A$ , define

$$B = A/I[x_1, \dots, x_r].$$

Then we have a map  $B \twoheadrightarrow A^*$ , so  $M^*$  is a  $B$ -module. Now define  $\chi(M, I, n) := \ell(M/I^n M)$ . If  $M$  is a finite  $A$ -module, then  $M/I^n M$  is of finite length (because  $A/I^n$  is Artinian) and thus

$$\ell(M/I^n M) = \ell(M/IM) + \ell(IM/I^2 M) + \dots + \ell(I^{n-1}M/I^n M).$$

Then if  $\ell(I^s M/I^{s+1} M)$  is a polynomial in  $s$  of degree at most  $r-1$  for  $s \gg 0$ , then  $\ell(M/I^n M)$  is a polynomial of degree at most  $r$  for  $n \gg 0$ .

Now if  $J$  is another ideal of definition, then there exists  $s$  such that  $J^s \subset I$  and thus  $\chi(M, J, ns) \geq \chi(M, I, n)$ . Therefore

$$d^\bullet \chi(M, J, n) \geq d^\bullet \chi(M, I, n)$$

and so the degree of  $\chi(M, I, n)$  is independent of  $I$ . Denote this degree by  $d(M)$ . We know that  $d(M) \geq r$ , which is the number of generators of  $I$ .

**Lemma 3.2.4.** *Assume we have an exact sequence of finite  $A$ -modules*

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$

Then  $d(M) = \max\{d(M'), d(M'')\}$  and  $\chi(M, I, n) - \chi(M', I, n) - \chi(M'', I, n)$  is a polynomial of degree strictly less than  $d(M'')$ .

*Proof.* For each  $n$  we have an exact sequence

$$0 \rightarrow \frac{M'}{I^n M \cap M'} \rightarrow \frac{M}{I^n M} \rightarrow \frac{M''}{I^n M''} \rightarrow 0.$$

Then  $\Delta := \chi(M, I, n) - \chi(M'', I, n) = \ell(M/I^n M) - \ell(M''/I^n M) = \ell(M'/I^n M \cap M')$ . By Artin-Rees, there exists  $r$  such that  $M' \cap I^n M = I^{n-r} \cdot (M' \cap I^r M)$ . But then

$$\chi(M', I, n-r) \leq \Delta \leq \chi(M', I, n)$$

because  $\chi(M', I, n) - \chi(M', I, n-r)$  has degree strictly less than  $d(M')$ , and the desired result follows.  $\square$

**Lemma 3.2.5.** *Let  $A$  be a Noetherian semilocal ring. Then  $d(A) \geq \dim A$ . In particular,  $\dim A < \infty$ .*

*Proof.* We will induct on  $d(A)$ . If  $d(A) = 0$ , then  $\mathfrak{m}^n = \mathfrak{m}^{n+1}$  for  $n \gg 0$ . By Nakayama, we see that  $\mathfrak{m}^n = 0$ , so  $\ell(A) < \infty$  and thus  $A$  is Artinian.

Assume that  $d(A) > 0$  and  $\dim A > 0$ . Let

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \dots \supset \mathfrak{p}_\ell = \mathfrak{p}$$

be a chain of prime ideals of length  $\ell > 0$ . Choose  $x \in \mathfrak{p}_{\ell-1} \setminus \mathfrak{p}_\ell$ . Then  $\dim(A/\mathfrak{p} + xA) \geq \ell - 1$ . Because we have the exact sequence

$$0 \rightarrow A/\mathfrak{p} \xrightarrow{\times x} A/\mathfrak{p} \rightarrow A/\mathfrak{p} + xA \rightarrow 0,$$

we have  $d(A/\mathfrak{p}) = \max(d(A/\mathfrak{p}), d(A/\mathfrak{p} + xA))$  and that  $\chi(A/\mathfrak{p} + xA, I, n)$  has degree less than  $d(A/\mathfrak{p})$ . Therefore

$$d(A/\mathfrak{p} + xA) < d(A/\mathfrak{p}) \leq d(A).$$

By induction,  $\dim(A/\mathfrak{p} + xA) \leq d(A/\mathfrak{p} + xA)$  and thus  $\ell - 1 \leq d(A/\mathfrak{p} + xA) \leq d(A) - 1$ . This holds for any chain of ideals, so  $\dim A \leq d(A)$ .  $\square$

**Corollary 3.2.6.** *If  $A$  is Noetherian and  $\mathfrak{p} \in \text{Spec } A$ , then  $\text{ht}(\mathfrak{p}) < \infty$ .*

*Proof.*  $\text{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}} < \infty$  by the previous lemma.  $\square$

**Lemma 3.2.7.** *Let  $A$  be a Noetherian semilocal ring and  $M$  a finite  $A$ -module. Choose  $x \in \text{Rad}(A)$ . Then  $d(M) \geq d(M/xM) \geq d(M) - 1$ .*

*Proof.* Let  $I \subset A$  be an ideal of definition such that  $x \in I$ . Then

$$\chi(M/xM, I, n) = \ell(M/xM + I^n M) = \ell(M/I^n M) - \ell\left(\frac{xM + I^n M}{I^n M}\right).$$

Then because  $x \in I$ , we see that  $I^{n-1}M \subset (I^n M : x)$ , so

$$\ell\left(\frac{xM + I^n M}{I^n M}\right) \leq \ell(M/I^{n-1}M).$$

This implies that  $\chi(M/xM, I, n) \geq \chi(M, I, n) - \chi(M, I, n-1)$ , so  $d(M/xM) \geq d(M) - 1$ .  $\square$

**Lemma 3.2.8.** *Let  $A$  and  $M$  be as before. Let  $r = \dim M > 0$ . Then there exists  $x_1, \dots, x_r \in \text{Rad}(A)$  such that  $\ell(M/x_1M + \dots + x_rM) < \infty$ .*

*Proof.* Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_t$  be the minimal prime ideals containing  $\text{Ann}_A(M)$  such that  $\dim(A/\mathfrak{p}_i) = r$ . Because  $r > 0$ , then the  $\mathfrak{p}_i$  are not maximal and therefore  $\text{Rad}(A) \not\subset \mathfrak{p}_i$ . In particular, it is not contained in  $\bigcup \mathfrak{p}_i$ . Choose  $x_1 \in \text{Rad}(A) \setminus \bigcup \mathfrak{p}_i$ .

If  $\mathfrak{q} \supset \text{Ann}(M/x_1M) \supset \text{Ann}(M) + x_1A$  is prime and minimal, then  $\mathfrak{q} \not\subset \mathfrak{p}_i$  because  $x_1 \notin \mathfrak{p}_i$ . This implies  $\dim A/\mathfrak{q} \leq r - 1$ . By induction, we can then find  $x_2, \dots, x_r$  such that

$$\ell(\overline{M}/x_2\overline{M} + \dots + x_r\overline{M}) < \infty,$$

where  $\overline{M} = M/x_1M$ .  $\square$

**Theorem 3.2.9.** *Let  $A$  be semilocal and  $M$  a finite  $A$ -module. Then  $d(M) = \dim M$  is the smallest integer  $r$  such that there exists  $x_1, \dots, x_r \in \text{Rad}(A)$  such that  $\ell(M/x_1M + \dots + x_rM) < \infty$ .*

*Proof.* Choose  $x_1, \dots, x_r \in \mathfrak{m} = \text{Rad}(A)$ . If  $\ell(M/x_1M + \dots + x_rM) < \infty$ , then we know that  $d(M/x_1M + \dots + x_rM) \geq d(M) - r$ . Then because  $M/x_1M + \dots + x_rM$  has finite length, its dimension is zero and thus  $r \geq d(M)$ . Then let  $r_0$  be the smallest such integer. By the previous lemma, we deduce that  $\dim M \geq r_0 \geq d(M)$ .

We will show that  $d(M) \geq \dim M$ . Consider a sequence

$$M = M_0 \supset M_1 \supset \dots \supset M_{n+1} = 0$$

such that  $M_i/M_{i+1} \cong A/\mathfrak{p}_i$  for some prime ideals  $\mathfrak{p}_i$ . Then  $\text{Ass}(M) \subset \{\mathfrak{p}_0, \dots, \mathfrak{p}_n\} \subset \text{Supp } M$  are the minimal primes containing  $\text{Ann}_A(M)$ , so we see that

$$d(M) = \max \{d(A/\mathfrak{p}_i)\} \geq \max \{\dim(A/\mathfrak{p}_i)\} = \dim(A/\text{Ann}_A M). \quad \square$$

*Remark 3.2.10.* If  $M = A$ , then  $d(M)$  is the smallest integer  $r$  such that there exists  $x_1, \dots, x_r \in \text{Rad}(A)$  such that  $(x_1, \dots, x_r)$  is an ideal of definition.

**Corollary 3.2.11.** *If  $A$  is Noetherian and  $I = (x_1, \dots, x_r) \subset A$ , then any minimal prime ideal  $\mathfrak{p}$  containing  $I$  has height at most  $r$ . In particular,  $\text{ht}(I) \leq r$ .*

*Proof.* First, note that  $A_{\mathfrak{p}}/IA_{\mathfrak{p}}$  is Artinian because the image of  $\mathfrak{p}A_{\mathfrak{p}}$  is both maximal and minimal. Then  $\ell(A_{\mathfrak{p}}/x_1A_{\mathfrak{p}} + \cdots + x_rA_{\mathfrak{p}}) < \infty$  and thus  $\text{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}) \leq r$ .  $\square$

Now let  $M$  be a finitely generated  $A$ -module. Define  $\widehat{M} = \lim_{\leftarrow} M/I^n M$  for an ideal of definition  $I$ . We call  $\widehat{M}$  the  $I$ -adic completion of  $M$ .

**Corollary 3.2.12.**  $\dim \widehat{M} = \dim M$ .

*Proof.* We know that  $\widehat{M}/I^n \widehat{M} = M/I^n M$ . Thus the two modules have the same Hilbert-Samuel polynomial.  $\square$

**Corollary 3.2.13.** Let  $A$  be Noetherian with  $\mathfrak{p} \in \text{Spec } A$ . Let  $n$  be an integer. The following are equivalent:

1.  $\text{ht}(\mathfrak{p}) \leq n$ .
2. There exists  $I$  generated by  $n$  elements such that  $\mathfrak{p}$  is minimal in  $V(I)$ .

*Proof.* **1 implies 2** Suppose that  $\text{ht}(\mathfrak{p}) \leq n$ . Then there exists an ideal of definition  $J$  of  $A_{\mathfrak{p}}$  generated by  $n$  elements. If  $J = (\frac{x_1}{s}, \dots, \frac{x_n}{s})$ , then  $I = (x_1, \dots, x_n) \subset \mathfrak{p}$  and  $\mathfrak{p}$  is minimal containing  $I$ .

**2 implies 1** Let  $I = (x_1, \dots, x_n)$  such that  $\mathfrak{p} \supset I$  is minimal. Therefore  $A_{\mathfrak{p}}/IA_{\mathfrak{p}}$  is Artinian, so it has finite length. Thus  $\dim A_{\mathfrak{p}} \leq n$ , as desired.  $\square$

**Definition 3.2.14.** A system of parameters for  $M$  is a set of elements  $x_1, \dots, x_s \in \text{Rad}(A)$  such that

- $\ell(M/x_1M + \cdots + x_sM) < \infty$ ;
- $s = \dim M$ .

**Proposition 3.2.15.** Let  $x_1, \dots, x_r \in \text{Rad}(A)$ . Then  $\dim(M/(x_1, \dots, x_r)M) \geq \dim M - r$  and we have equality if and only if  $x_1, \dots, x_r$  belong to a system of parameters for  $M$ .

*Proof.* By induction  $d(M/xM) \geq d(M) - 1$  for any  $x \in \text{Rad}(A)$ . Then we know that

$$d(M/(x_1, \dots, x_r)M) \geq d(M) - r = \dim M - r.$$

Then assume that we have equality. Let  $y_1, \dots, y_p$  be a system of parameters for  $M/(x_1, \dots, x_r)M$ . Then  $\dim(M/(x_1, \dots, x_r)M) = p = \dim M - r$ . However, if  $\overline{M} = M/(x_1, \dots, x_r)$ , then

$$\ell(\overline{M}/(y_1, \dots, y_p)\overline{M}) = \ell(M/(y_1, \dots, y_p, x_1, \dots, x_r)M) < \infty$$

and thus  $y_1, \dots, y_p, x_1, \dots, x_r$  is a system of parameters for  $M$ .

Conversely suppose that  $x_1, \dots, x_r, y_1, \dots, y_p$  is a system of parameters for  $M$ . Then

$$\dim(M/(x_1, \dots, x_r)M) \geq \dim M - r = p,$$

but we have equality because

$$0 = d(M/x_1, \dots, x_r, y_1, \dots, y_p) \geq d(M/(x_1, \dots, x_r)M) - p,$$

and so  $p \geq d(M/(x_1, \dots, x_r)M)$ .  $\square$

Now we turn to the case of local Noetherian rings  $A$  with maximal ideal  $\mathfrak{m}$  and residue field  $k$ . Then if  $d = \dim A$ , any ideal of definition has at least  $d$  generators. Then let  $x_1, \dots, x_d \in \mathfrak{m}$  such that  $\ell(A/(x_1, \dots, x_d)) < \infty$ . Thus  $I = (x_1, \dots, x_d)$  is an ideal of definition and  $(x_1, \dots, x_d)$  is a system of parameters of  $A$ .

**Definition 3.2.16.** A local ring  $A$  is a *regular local ring* if there is a system of parameters generating the maximal ideal of  $A$ . Such a system is called a *regular system of parameters*.

Note that  $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$  and that  $A$  is regular if and only if  $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$ .

**Proposition 3.2.17.** Let  $(A, \mathfrak{m}, k)$  be a local Noetherian ring. Let  $(x_1, \dots, x_d)$  be a system of parameters of  $A$ . Then  $\dim(A/(x_1, \dots, x_i)A) = d - i$  and the image of  $(x_{i+1}, \dots, x_d)$  in  $A/(x_1, \dots, x_i)A$  is a system of parameters of this quotient.

### 3.3 Dimension in the Relative Setting

Consider a morphism  $A \xrightarrow{\varphi} B$ . We have the pullback  $\varphi^*: \text{Spec } B \rightarrow \text{Spec } A$ . For  $\mathfrak{p} \in \text{Spec } A$ , we have a bijection between  $(\varphi^*)^{-1}(\mathfrak{p})$  and  $\text{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ . This latter ring is isomorphic to  $B \otimes k(\mathfrak{p})$ , where  $k(\mathfrak{p})$  is the residue field of  $A_{\mathfrak{p}}$ .

**Theorem 3.3.1.** Let  $\mathfrak{P} \in \text{Spec } B$  lie over  $\mathfrak{p}$ . Then

1.  $\text{ht}(\mathfrak{P}) \leq \text{ht}(\mathfrak{p}) + \text{ht}(\mathfrak{P}/\mathfrak{p}\mathfrak{P})$ . Equivalently,  $\dim B_{\mathfrak{P}} \leq \dim A_{\mathfrak{p}} + \dim(B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}})$ .
2. Equality holds is equivalent to the going-down property for  $\varphi$  and in particular if  $\varphi$  is flat.
3. If  $\varphi^*$  is surjective and the going-down property holds, then  $\dim B \geq \dim A$  and  $\text{ht}(I) = \text{ht}(IB)$  for  $I \subset A$ .

*Proof.* 1. Set  $A = A_{\mathfrak{p}}, B = B_{\mathfrak{P}}$ . We need to prove that  $\dim B \leq \dim A + \dim B/\mathfrak{p}B$ , where  $\mathfrak{p}$  is the maximal ideal of  $A$ . Write  $r = \dim A$  and let  $x_1, \dots, x_r$  be a system of parameters for  $A$ . Then  $I = (x_1, \dots, x_r)$  is an ideal of definition, so  $\mathfrak{p}^n \subset I \subset \mathfrak{p}$  for some  $n$ . Thus  $\mathfrak{p}^n B \subset IB \subset \mathfrak{p}B$  and all of these ideals have the same nilradical. Therefore

$$\dim B/OB = \dim B/\mathfrak{p}^n B = \dim B/\mathfrak{p}B = s$$

for some integer  $s$ . If  $y_1, \dots, y_s$  is a system of parameters for  $B/IB$ , then  $x_1, \dots, x_r, y_1, \dots, y_s$  generate an ideal of definition for  $B$ , so  $r + s \geq \dim B$ .

2. Let  $\mathfrak{P} = \mathfrak{P}_0 \supseteq \mathfrak{P}_1 \supseteq \dots \supseteq \mathfrak{P}_s$  be a chain of ideals of  $B/\mathfrak{p}B$  of length  $s = \dim B/\mathfrak{p}B$ . Then for  $i = 0, \dots, s$  we know  $\mathfrak{p} \subset \varphi^*(\mathfrak{P}_i)$  and thus  $\varphi^*(\mathfrak{P}_i) = \mathfrak{p}$  for all  $i$ . Now by the Going Down property we can find

$$\mathfrak{P}_s \supseteq \dots \supseteq \mathfrak{P}_{r+s}$$

such that  $\mathfrak{p}_i = \varphi^{-1}(\mathfrak{P}_{s+i})$ . Thus we have

$$\mathfrak{p} = \mathfrak{p}_0 \supseteq \mathfrak{p}_1 \supseteq \dots \supseteq \mathfrak{p}_r$$

where  $r = \dim A$ . This gives us the chain

$$\mathfrak{P}_0 \supseteq \dots \supseteq \mathfrak{P}_{r+s}$$

and thus  $\dim B \geq r + s$ .

3. The first inequality follows from 2. Note that  $\dim B = \dim A + \dim(B/\mathfrak{p}B) \geq \dim A$ .

To prove the equality, let  $\mathfrak{P} \in V(IB)$  be minimal such that  $\text{ht}(\mathfrak{P}) = \text{ht}(IB)$ . Let  $\mathfrak{p} = \varphi^*(\mathfrak{P})$ . Then  $\mathfrak{p} \supseteq I$  and  $\mathfrak{P}/\mathfrak{p}B$  is minimal, so  $\text{ht}(\mathfrak{P}/\mathfrak{p}B) = 0$ . This tells us that  $\dim B_{\mathfrak{P}} = \dim A_{\mathfrak{p}}$  and thus  $\text{ht}(\mathfrak{P}) = \text{ht}(\mathfrak{p})$ . Thus  $\text{ht}(\mathfrak{P}) \geq \text{ht}(I)$ .

Conversely, let  $\mathfrak{p} \supset I$  be minimal with  $\text{ht}(\mathfrak{p}) = \text{ht}(I)$ . Let  $\mathfrak{P} \in \text{Spec } B$  such that  $\varphi^*(\mathfrak{P}) = \mathfrak{p}$ . Then  $\mathfrak{P} \supset \mathfrak{p}B \supset IB$  and so we may suppose it is minimal for this property. Then we see that

$$\text{ht}(IB) \leq \text{ht}(\mathfrak{P}) = \text{ht}(\mathfrak{p}) = \text{ht}(I)$$

as desired.  $\square$

**Corollary 3.3.2.** *Let  $B \supset A$  be Noetherian rings such that  $B$  is integral over  $A$ .*

1.  $\dim A = \dim B$ ;
2. For all  $\mathfrak{P} \in \text{Spec } B$ ,  $\text{ht}(\mathfrak{P}) \leq \text{ht}(\mathfrak{P} \cap A)$ .
3. If the going-down property holds, then for any ideal  $J \subset B$ , have  $\text{ht}(J) = \text{ht}(J \cap A)$ .

*Proof.* The proof of this is left as an exercise to the reader.  $\square$

**Exercise 3.3.3.** Let  $A \xrightarrow{\phi} B$  be a morphism of rings and assume that going-down holds for  $\phi$ . Let  $\mathfrak{p} \supset \mathfrak{q}$  be prime ideals of  $A$ . Prove that  $\dim(B \otimes k(\mathfrak{p})) \geq \dim(B \otimes k(\mathfrak{q}))$ .

Now we will consider finitely generated extensions of rings. Here  $B$  will be a finitely-generated  $A$ -algebra.

**Theorem 3.3.4.** *Let  $A$  be Noetherian. Then  $\dim A[X] = \dim A + 1$ .*

*Proof.* Let  $\mathfrak{p} \in \text{Spec } A$  and let  $\mathfrak{P} \in \text{Spec } B$  such that  $\mathfrak{P} \cap A = \mathfrak{p}$ . Choose  $\mathfrak{P}$  to be maximal for this property. We will show that  $\text{ht}(\mathfrak{P}/\mathfrak{p}B) = 1$ . After localization at  $\mathfrak{p}$ , we may assume that  $\mathfrak{p}$  is maximal and  $A$  is local. Then  $B/\mathfrak{p}B = A/\mathfrak{p}[X]$ , and  $A/\mathfrak{p}$  is a field. Thus  $B/\mathfrak{p}B$  is a PID, so  $\mathfrak{P}/\mathfrak{p}B$  is a nonzero principal ideal, so it must have height exactly equal to 1.

Previous we have seen that because  $B$  is flat,  $\dim B_{\mathfrak{P}} = \dim A_{\mathfrak{p}} + 1$ , and thus  $\text{ht} \mathfrak{P} = \text{ht}(\mathfrak{p}) + 1$ , and we obtain the desired result.  $\square$

**Corollary 3.3.5.** 1.  $\dim A[x_1, \dots, x_m] = \dim A + m$ .

2. If  $k$  is a field, then  $\dim k[x_1, \dots, x_m] = m$ . Moreover,  $\text{ht}((x_1, \dots, x_i)) = i$ .

*Proof.* We only need to prove the part about the height of  $(x_1, \dots, x_i)$ . Then we have

$$0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_i) \subsetneq \cdots \subsetneq (x_1, \dots, x_n).$$

Then clearly  $\text{ht}((x_1, \dots, x_i)) \geq i$  and the inequality cannot be strict because otherwise  $(x_1, \dots, x_n)$  has height strictly larger than  $n$ .  $\square$

**Exercise 3.3.6.** Let  $A$  be Noetherian,  $I \subset A$ , and  $I' \subset A[X]$ . Suppose  $I' = I[X]$ . Show that  $\text{ht}(I') = \text{ht}(I)$ .

**Theorem 3.3.7 (Noether Normalization).** *Let  $A$  be a finitely generated  $k$ -algebra over a field  $k$ . Let  $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots \subsetneq \mathfrak{a}_p$  be a chain of prime ideals of  $A$ . Then there exist elements  $x_1, \dots, x_m \in A$  algebraically independent such that:*

1.  $A$  is integral over  $B = k[x_1, \dots, x_m]$ ;
2. For all  $i = 1, \dots, p$ , there exists an integer  $h(i) \geq 0$  such that  $\mathfrak{a}_i \cap B = (x_1, x_2, \dots, x_{h(i)})$ . In particular,  $\text{ht}(\mathfrak{a}_i) = h(i)$ .

*Remark 3.3.8.* Recall that  $x_1, \dots, x_m$  are algebraically independent means that the map  $k[X_1, \dots, X_m] \rightarrow A$  sending  $X_i \mapsto x_i$  is injective.

*Proof of Theorem.* We will treat the case where  $A = k[y_1, \dots, y_m]$  because any finitely generated algebra over  $k$  which is an integral domain is integral over such a ring. We will prove the result by induction on  $m$  and  $p$ . The case when  $m = 1$  is clear because  $k[y_1]$  is a PID.

We will assume the result is true for  $m - 1$ . Now we will form an induction on  $p$ . In the case  $p = 1$ , first assume that  $\mathfrak{a}_1 = (x_1)$ , where  $x_1 \notin k$ . Then  $x_1 = f(y_1, \dots, y_m) \in k[y_1, \dots, y_m]$ . For  $i = 2, \dots, m$  we will introduce  $x_i = y_i - y_1^{r_i}$  for some integer  $r_i$ . We want to choose the  $r_i$  such that  $y_1$  is integral over  $k[x_1, \dots, x_m]$ . Writing

$$\begin{aligned} x_1 &= f(y_1, \dots, y_m) \\ &= \sum_{\underline{p}} a_{\underline{p}} y^{\underline{p}} \\ &= \sum_{\underline{p}} a_{\underline{p}} y_1^{p_1} (x_2 + y_1^{r_2})^{p_2} \cdots (x_m + y_1^{r_m})^{p_m}, \end{aligned}$$

we see that  $f(\underline{p}) = p_1 + r_2 p_2 + \cdots + r_m p_m$  is the maximal degree of  $y_1$  in this expression. Then it is possible to choose  $r_2, \dots, r_m$  such that  $f(\underline{p})$  are all distinct, for example  $r_i = k^i$  for  $k > \max \{p_i\}$ , where the max is taken over all  $p_i$  that occur in the polynomial.

Then choosing the  $\underline{p}$  for which  $f(\underline{p})$  is maximal, we can write

$$x_1 = a_{\underline{p}} y_1^{f(\underline{p})} + \sum_{j \leq f(\underline{p})} Q_j(\underline{x}) y_1^j.$$

Thus  $y_1$  is integral over  $k[x_1, \dots, x_m]$ , so  $y_i = x_i + y_1^{r_i}$  is integral over  $k[x_1, \dots, x_m]$ . Therefore  $A$  is integral over  $k[x_1, \dots, x_m]$ . Finally,  $x_1, \dots, x_m$  are algebraically independent because otherwise the transcendence degree of  $\text{Frac}(A)$  is smaller than  $m$ . We now show that  $\mathfrak{a}_1 \cap B = (x_1)$ . If  $y \in \mathfrak{a}_1 \cap B$ , then write  $y = b' x_1$  for some  $b' \in A$ . But then  $b' \in A \cap \text{Frac}(B)$ , so because  $B$  is integrally closed,  $b' \in B$  and thus  $y \in Bx_1$ .

For the general case, suppose  $\mathfrak{a}_1$  is generated by more than one element. Choose  $x_1 \in \mathfrak{a}_1 \setminus k$  and choose  $t_2, \dots, t_m$  such that  $A$  is integral over  $C = k[x_1, t_2, \dots, t_m]$  and  $x_1 A \cap C = Cx_1$ . By the induction hypothesis on  $m$ , there exist  $x_2, \dots, x_m$  such that  $k[t_2, \dots, t_m]$  is integral over  $k[x_1, \dots, x_m]$  and  $\mathfrak{a}_1 \cap k[t_1, \dots, t_m] \cap k[x_2, \dots, x_m] = (x_2, \dots, x_h)$ . To see this, choose  $z \in \mathfrak{a}_1 \cap k[x_1, \dots, x_h]$ . Then there exist  $h_j \in k[x_2, \dots, x_m]$  such that

$$z = \sum_{j=1}^d h_j x_1^j$$

because  $x_1 \in \mathfrak{a}_1 \cap k[x_1, \dots, x_m]$ . Thus  $h_0 \in \mathfrak{a}_1 \cap k[x_1, \dots, x_m] = (x_2, \dots, x_h)$  and thus  $z \in (x_1, \dots, x_h)$ . This finishes the case  $p = 1$ .

Now we complete the induction on  $p$ . Suppose we have a chain of prime ideals  $\mathfrak{a}_1 \subsetneq \cdots \subsetneq \mathfrak{a}_p$  in  $A$ . Then we choose  $t_1, \dots, t_m$  such that

- $A$  is integral over  $k[t_1, \dots, t_m]$
- $\mathfrak{a}_i \cap k[t_1, \dots, t_m] = (t_1, \dots, t_{h(i)})$  for  $i \leq p - 1$ .

Now we apply the case  $p = 1$  to the ideal  $\mathfrak{a}_p \cap k[t_{r+1}, \dots, t_m]$  where  $r = h(p - 1)$ . Thus there exist  $x_{r+1}, \dots, x_m$  such that  $k[t_{r+1}, \dots, t_m]$  is integral over  $k[x_1, \dots, x_m]$  and  $\mathfrak{a}_p \cap k[x_{r+1}, \dots, x_m] = (x_{r+1}, \dots, x_h)$ .

First, it is clear that  $A$  is integral over  $k[t_1, \dots, t_r, x_{r+1}, \dots, x_m]$ . If we set  $x_i = t_i$  for  $i \leq r$ , then we will show that

$$\mathfrak{a}_p \cap k[x_1, \dots, x_m] = (x_1, \dots, x_r).$$

One direction is obvious from the inductive hypothesis. In the other direction, if we write  $y = \sum a_h x^h$ , then because  $x_1, \dots, x_r \in \mathfrak{a}_{p-1} \in \mathfrak{a}_{p-1} \cap k[x_1, \dots, x_m]$ , we see that

$$a_0 \in \mathfrak{a}_p \cap k[x_{r+1}, \dots, x_m] = (x_{r+1}, \dots, x_m).$$

Thus  $y \in (x_1, \dots, x_r)$ , as desired.  $\square$

**Corollary 3.3.9.** *Let  $A$  be an integral domain of finite type over a field  $k$ . Then  $\dim A$  equals the transcendence degree of the fraction field of  $A$ .*

*Proof.* There exist  $x_1, \dots, x_m$  such that  $A$  is integral over  $k[x_1, \dots, x_m]$ . Then  $\text{Frac}(A)$  is algebraic over  $k(x_1, \dots, x_m)$ . On the other hand, we know that  $\dim A = m$ , which is the transcendence degree of  $k(x_1, \dots, x_m)$ .  $\square$

**Corollary 3.3.10 (Nullstellensatz).** *Let  $A$  be an algebra of finite type over a field  $k$ . Then for any maximal ideal  $\mathfrak{m} \subset A$ ,  $A/\mathfrak{m}$  is algebraic over  $k$ .*

*Proof.* Note that  $\dim A/\mathfrak{m} = 0$ , but this is also the transcendence degree over  $k$  by the previous corollary.  $\square$

**Proposition 3.3.11.** *Let  $A$  be an integral domain of finite type over a field  $k$ . Then for any prime ideal  $\mathfrak{p} \in \text{Spec } A$ , we have  $\text{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} = \dim A$ .*

*Proof.* Let  $\mathfrak{p} \in \text{Spec } A$  and  $h$  be its height. Then by Noether normalization, there exist  $x_1, \dots, x_m$ , where  $m = \dim A$ , such that  $A$  is integral over  $A' = k[x_1, \dots, x_m]$  and  $\mathfrak{p}' = \mathfrak{p} \cap A' = (x_1, \dots, x_h)$ . Then we know that  $A'/\mathfrak{p}' \cong k[x_{h+1}, \dots, x_m]$ . Because  $A/\mathfrak{p}$  is integral over  $A'/\mathfrak{p}'$ , we see that  $\dim A/\mathfrak{p} = \dim A'/\mathfrak{p}' = m - h$ . But then  $\text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{p}') = h$  because  $A'$  is integrally closed and  $A$  is integral over  $A'$ . Therefore,

$$\text{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} = h + m - h = m = \dim A. \quad \square$$

*Remark 3.3.12.* Let  $A$  be a finitely generated  $k$ -algebra. Then for any maximal ideal  $\mathfrak{m} \subset A$ , we know that  $A/\mathfrak{m}$  is an algebraic extension of  $k$ . Therefore we have a correspondence

$$\left\{ \begin{array}{c} \text{Maximal ideals} \\ \text{of } \text{Spec } A \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Galois orbits of points in } \bar{k}^n \\ \text{satisfying certain algebraic equations} \end{array} \right\}.$$

In addition, any prime ideal  $\mathfrak{p}$  defines a subvariety  $\text{Spec } A/\mathfrak{p} = V(\mathfrak{p})$  of  $\text{Spec } A$ .

**Proposition 3.3.13.** *Let  $A, A'$  be two finitely-generated  $k$ -algebras that are domains. Then for any minimal prime ideal  $\mathfrak{p} \subset A \otimes_k A'$ , we have  $\dim A \otimes_k A'/\mathfrak{p} = \dim A + \dim A'$ .*

*Proof.* Choose  $B, B'$  polynomials over  $k$  such that  $A$  (resp  $A'$ ) is integral over  $B$  (resp  $B'$ ). Then write  $d, d' = \dim A, \dim A'$ . Then  $A \otimes A'$  is torsion free over  $B \otimes B'$ . Then because  $\mathfrak{p} \subset \text{Spec}(A \otimes A')$  is minimal, we see that  $B \otimes B' \cap \mathfrak{p} = 0$ . Therefore  $A \otimes A'/\mathfrak{p}$  is integral over  $B \otimes B'$  and thus the desired result follows using integrality.  $\square$

*Remark 3.3.14.* We can think of  $\text{Spec } A \otimes A'$  as the product  $\text{Spec } A \times \text{Spec } A'$ . The proposition says that irreducible components of the product variety have the expected dimension.



**Proposition 3.3.15** (Hilbert's Nullstellensatz). *Let  $k$  be a field,  $A$  be a finitely-generated  $k$ -algebra and  $I \subsetneq A$  be a proper ideal. Then  $\sqrt{I} = \bigcap_{\substack{\mathfrak{m} \supset I \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}$ .*

*Proof.* One direction is obvious because  $\sqrt{I}$  is the intersection of all prime ideals containing  $I$ . Let  $a \in J = \bigcap \mathfrak{m} \setminus \sqrt{I}$ . Then  $S = \{1, a, a^2, \dots\} \cap I = \emptyset$  and thus  $S^{-1}I \subset S^{-1}A$  is a proper ideal. Thus there exists a maximal ideal of  $S^{-1}A$  such that  $S^{-1}I \subset \mathfrak{m}_0$ . Because  $S^{-1}A$  is a finitely generated  $k$ -algebra, we have

$$\dim(S^{-1}A/\mathfrak{m}_0) = \text{trdeg}_k S^{-1}A/\mathfrak{m}_0 = 0.$$

Then writing  $\mathfrak{m} = \mathfrak{m}_0 \cap A$ , we see that  $k \subset A/\mathfrak{m} \subset S^{-1}A/\mathfrak{m}_0$  and thus  $\dim A/\mathfrak{m} = 0$  and therefore  $\mathfrak{m} \supset I$  is maximal. However,  $a \notin \mathfrak{m}$  by hypothesis, which gives us a contradiction.  $\square$

### 3.4 Rings of Dimension 1

**Definition 3.4.1.** A local ring  $A$  is called a *discrete valuation ring* if it is a principal ideal domain and has a nonzero prime ideal.

This prime ideal is naturally maximal because if  $\mathfrak{p} \subset \mathfrak{m} \subset A$  given by  $(a) \subset (b)$ , then we know  $a = bs$  for some  $s \in A$ , but then  $s \in \mathfrak{p}$ , so  $s = as'$  and thus  $a = bas'$ , so  $bs' = 1$ . In particular,  $\dim A = 1$ .

**Definition 3.4.2.** A *discrete valuation* on  $A$  is a surjective function  $v: A^* \rightarrow \mathbb{Z}$  such that

- $v(xy) = v(x) + v(y)$ ;
- $v(x + y) \geq \min\{v(x), v(y)\}$ .

We define  $v(0) = \infty$ .

If  $A$  is a DVR, then choose  $x \neq 0$ . We define  $v(x) = \sup\{n \geq 0 \mid x \in (\pi^n)\}$ , where  $\pi$  generates the maximal ideal of  $A$ . Then  $v(x)$  is well-defined because  $\bigcap_0(\pi^n) = 0$  by the Krull intersection theorem. We can extend this valuation to  $K = \text{Frac } A$  by  $v(x/y) = v(x) - v(y)$ .

**Proposition 3.4.3.** *If  $K$  is a field and  $v: K^* \rightarrow \mathbb{Z}$  is a valuation, then  $A = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$  is a discrete valuation ring.*

Proof of this is left as an exercise.

**Example 3.4.4.** Consider the ring  $\mathbb{Z}_{(p)}$  where  $p$  is a prime number. Then for any  $\frac{x}{y}$  with  $y$  coprime to  $p$ , define  $v_p\left(\frac{x}{y}\right)$  to be the maximal power of  $p$  dividing  $x$ .

Now let  $k$  be a field and let  $k[[T]]$  be the ring of formal power series in  $T$ . Then any series  $a_0 + a_1T + \dots$  is invertible iff  $a_0 \neq 0$ , and thus any element is a product of  $T^n$  and a unit for some  $n$ . Thus  $v(F)$  is the degree of the first monomial with nonzero coefficient.

**Proposition 3.4.5.** *Let  $A$  be a ring. Then the following are equivalent:*

1.  $A$  is a DVR.
2.  $A$  is a local noetherian ring and  $\mathfrak{m}_A$  is generated by an element  $\pi$  which is not nilpotent.

*Proof.* One direction is clear. If  $A$  is a DVR, then it is a domain, and thus the generator of the prime ideal is not nilpotent.

In the other direction, suppose  $\mathfrak{m} = (\pi)$ . Then by the Krull intersection theorem,  $\bigcap_n (\pi^n) = 0$ . Then for  $0 \neq x \in A$ , there exists a maximal  $n$  such that  $x \in (\pi^n)$ , so  $x = \pi^n u$  for some  $u \in A^\times$ . But then for  $y \in A, y \neq 0$ , write  $y = \pi^m v$  for  $v \in A^\times$ . Thus  $xy = \pi^{m+n} uv \neq 0$  because  $\pi$  is not nilpotent. Therefore  $A$  is a domain and thus is a DVR.  $\square$

**Proposition 3.4.6.** *Let  $A$  be a local Noetherian ring. Then  $A$  is a DVR if and only if*

1.  $A$  is integrally closed;
2.  $A$  has a unique nonzero prime ideal.

*Proof.* One direction is clear. In the other direction, assume  $A$  is integrally closed and has a unique nonzero prime. First, we note that  $A$  must be local and  $\mathfrak{m} \neq 0$ . Then if we write

$$\mathfrak{m}' = \{x \in K \mid x\mathfrak{m} \subset A\} \supset A,$$

this is an  $A$ -module. If we write  $y \in \mathfrak{m}$ , then  $y\mathfrak{m}' \subset A$  and thus  $\mathfrak{m}' \subset Ay^{-1}$ . This implies that  $\mathfrak{m}'$  is finitely generated. Then we have  $\mathfrak{m} \subset \mathfrak{m}\mathfrak{m}' \subset A$ . We will show that this cannot equal  $\mathfrak{m}$  by contradiction.

Let  $x \in \mathfrak{m}'$ . Then  $x\mathfrak{m} \subset \mathfrak{m}$  and thus  $x$  is integral over  $A$ . Because  $A$  is integrally closed,  $x \in A$ . Thus  $\mathfrak{m}' = A$ . Now we set  $S = \{1, x, x^2, \dots\}$ . Then  $S^{-1}A = K$  because it has no nonzero prime ideals. If we choose  $z \in A \setminus 0$ , then we can write  $\frac{1}{z} = \frac{y}{x^n}$  for some  $n \geq 0$ . This tells us that  $x^n \in (z)$ . Because  $\mathfrak{m}$  is finitely generated, we see that  $\mathfrak{m}^N \subset (z)$ . Then let  $N_0$  be the smallest integer such that  $\mathfrak{m}^{N_0} \subset (z) \subset \mathfrak{m}$  and let  $y \in \mathfrak{m}^{N_0-1} \setminus (z)$ . Then we have  $\mathfrak{m}y \subset \mathfrak{m}^{N_0} \subset (z)$  and thus  $\frac{y}{z} \in \mathfrak{m}'$ . This implies that  $\mathfrak{m}' \supsetneq A$  and thus  $\mathfrak{m}\mathfrak{m}' = A$ . Therefore we can write

$$1 = \sum x_i y_i^{-1}$$

where  $x_i \in \mathfrak{m}, y_i \in \mathfrak{m}'$ . Therefore there exists  $i$  such that  $x_i y_i^{-1} \notin \mathfrak{m}$  and thus  $y_i^{-1} \mathfrak{m} = A$ . Therefore  $\mathfrak{m} = (y_i)$  is a principal ideal, so  $A$  is a DVR.  $\square$

**Proposition 3.4.7.** *Let  $A$  be a Noetherian domain. The following are equivalent:*

1. For all  $0 \neq \mathfrak{p} \in \text{Spec } A$ , the localization  $A_{\mathfrak{p}}$  is a discrete valuation ring.
2.  $A$  is integrally closed and of dimension 1.

*Proof.* **1 implies 2:** Let  $x \in K$  be integral over  $A$ . But then  $x \in A_{\mathfrak{p}}$  for all  $\mathfrak{p} \neq 0$  and thus  $x \in \bigcap_{\mathfrak{p} \neq 0} A_{\mathfrak{p}} = A$ . Being of dimension 1 is easy. If  $0 \neq \mathfrak{p} \subset \mathfrak{m} \subset A$ , we localize at  $\mathfrak{m}$  and see that  $\mathfrak{p}A_{\mathfrak{m}} = \mathfrak{m}A_{\mathfrak{m}}$ , so  $\mathfrak{p} = \mathfrak{m}$ .

**2 implies 1:** For all  $\mathfrak{p}$ , we know that  $A_{\mathfrak{p}}$  is integrally closed. Because it has dimension 1, it must be a DVR.  $\square$

**Definition 3.4.8.** A ring  $A$  is called a *Dedekind domain* if it is a domain satisfying the properties of the previous proposition.

**Example 3.4.9.**  $\mathbb{Z}$  is a Dedekind domain. It is clearly a domain, having dimension 1 follows from being a PID, and being integrally closed is obvious. More generally, any principal ideal domain is a Dedekind domain.

**Example 3.4.10.** Let  $A$  be a Dedekind domain and  $K = \text{Frac}(A)$ . Then let  $L/K$  be a finite extension and  $B$  be the integral closure of  $A$  in  $L$ . Then  $B$  is a Dedekind domain. In particular, if  $K$  is a number field, then its ring of integers  $\mathcal{O}_K$  is a Dedekind domain.

Here is another example of this. Let  $k$  be a field and  $A = k[x], k = k(X)$ . Then if  $L/K$  is a field extension and  $B$  is the integral closure of  $A$  in  $L$ , then  $\text{Spec } B \rightarrow \text{Spec } A$  is a smooth affine curve with a map to  $\mathbb{A}^1$ .

*Remark 3.4.11.* All smooth curves can be obtained in this way (of taking the integral closure of some ring). Also, normalization resolves all singularities of curves.

**Definition 3.4.12.** A fractional ideal  $\mathfrak{a} \subset K = \text{Frac}(A)$  is an  $A$ -submodule of finite type.

Note that for  $\mathfrak{a}, \mathfrak{b}$  nonzero fractional ideals, then  $\mathfrak{a} \cdot \mathfrak{b}$  is a fractional ideal.

**Proposition 3.4.13.** If  $A$  is a Dedekind domain, then the set of nonzero fractional ideals form an abelian group.

*Proof.* It is easy to see that the multiplication is associative and commutative. Now we need to show that inverses exist. For  $\mathfrak{a} \subset K$  a fractional ideal, we need to find another fractional ideal  $\mathfrak{b}$  such that  $\mathfrak{a}\mathfrak{b} = A$ . Set

$$\mathfrak{b} = \mathfrak{a}^{-1} := \{x \in K \mid x \cdot \mathfrak{a} \subset A\}.$$

Clearly we know that for  $0 \neq x \in \mathfrak{a}$ , we have  $\mathfrak{a}^{-1} \subset x^{-1}A$  and thus  $\mathfrak{a}^{-1}$  is of finite type. Then for any prime ideal  $\mathfrak{p}$  we know that  $\mathfrak{a}_{\mathfrak{p}} \cdot \mathfrak{b}_{\mathfrak{p}} = (\mathfrak{a} \cdot \mathfrak{b})_{\mathfrak{p}}$ , and thus  $\mathfrak{a}_{\mathfrak{p}}^{-1} = (\mathfrak{a}^{-1})_{\mathfrak{p}}$ . Therefore we have

$$(\mathfrak{a} \cdot \mathfrak{a}^{-1})_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}} \cdot \mathfrak{a}_{\mathfrak{p}}^{-1} = A_{\mathfrak{p}},$$

and thus  $\mathfrak{a} \cdot \mathfrak{a}^{-1} = A$ . □

Now if  $A$  is a Dedekind domain and  $0 \neq \mathfrak{p} \neq \mathfrak{p}' \subset A$ , then  $\mathfrak{p} + \mathfrak{p}' = A$ , so  $\mathfrak{p}\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{p}'$ . This implies that for any  $0 \neq \mathfrak{a} \subset A$  that  $\mathfrak{p} \cdot \mathfrak{a} \subsetneq \mathfrak{a}$  because otherwise  $\mathfrak{p} = A$ .

**Lemma 3.4.14.** If  $x \in A$  is nonzero, then there are only finitely many maximal ideals  $\mathfrak{p}$  such that  $x \in \mathfrak{p}$ .

*Proof.* Let  $x \in \mathfrak{p}$ . Then  $\mathfrak{a}^{-1} \subset x^{-1}A$ , so if  $x \in \mathfrak{p}_1, \mathfrak{p}_2, \dots$  for infinitely many maximal ideals, then

$$x \in \mathfrak{p}_1 \supsetneq \mathfrak{p}_1\mathfrak{p}_2 \supsetneq \dots \subset \prod_{i=1}^n \mathfrak{p}_i \supsetneq \dots$$

is a strictly decreasing infinite chain of ideals containing  $x$ . Thus we have

$$\mathfrak{p}_1^{-1} \subset \mathfrak{p}_1^{-1}\mathfrak{p}_2^{-1} \subset \dots \subset \prod_{i=1}^n \mathfrak{p}_i^{-1} \subset \dots \subset x^{-1}A,$$

but this is impossible because  $x^{-1}A$  is Noetherian. □

*Remark 3.4.15.* For any nonzero ideal  $I \subset A$ , there are only finitely many prime ideals  $\mathfrak{p}$  such that  $I \subset \mathfrak{p}$ .

**Definition 3.4.16.** Let  $\mathfrak{p}$  be a maximal ideal of  $A$ . Then let  $v_{\mathfrak{p}}(I)$  be the  $n_{\mathfrak{p}}$  such that  $IA_{\mathfrak{p}} = \omega_{\mathfrak{p}}^{n_{\mathfrak{p}}} A_{\mathfrak{p}}$ . Thus  $I \subset \mathfrak{p}^{n_{\mathfrak{p}}}$  but  $I \not\subset \mathfrak{p}^{n_{\mathfrak{p}}+1}$ .

**Corollary 3.4.17.** Let  $I$  be a fractional ideal of  $A$ , where  $A$  is a Dedekind domain. Then

$$I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(I)}.$$

*Proof.* If we denote the product by  $J$ , then  $I_{\mathfrak{p}} = J_{\mathfrak{p}}$  for all  $\mathfrak{p}$  and thus  $I = J$ .  $\square$

*Remark 3.4.18.* This corollary gives the primary decomposition of an ideal in a Dedekind domain. In number theory, this replaces the prime factorization of an integer.

### 3.5 Depth

Let  $A$  be a ring and  $M$  be an  $A$ -module. Then for  $a_1, \dots, a_r \in A$ , write  $\underline{a} = (a_1, \dots, a_r) \subset A$ .

**Definition 3.5.1.** The sequence  $a_1, \dots, a_r$  is an  $M$ -regular sequence if it satisfies the following:

1.  $a_i$  is not a zero divisor of  $M/(a_1, \dots, a_{i-1})M$  for  $i = 1, \dots, r$ .
2.  $\underline{a} \cdot M \subsetneq M$ . Therefore we have

$$a_1M \subsetneq (a_1, a_2)M \subsetneq \dots \subsetneq (a_1, \dots, a_r)M.$$

**Lemma 3.5.2.** Assume that  $\underline{a} = (a_1, \dots, a_r)$  is  $M$ -regular and let  $m_1, \dots, m_r \in M$  such that  $\sum_{i=1}^r a_i m_i = 0$ . Then  $m_i \in \underline{a} \cdot M$  for all  $i$ .

*Proof.* We will induct on  $r$ . If  $r = 1$ , then  $a_1 m_1 = 0$  implies  $m_1 = 0$ . Now assume  $a_1 m_1 + \dots + a_r m_r = 0$  implies that  $a_r \bar{m}_r = 0$  in  $M/(a_1, \dots, a_{r-1})M$ . Then there exists  $n_1, \dots, n_{r-1}$  such that  $m_r = a_1 n_1 + \dots + a_{r-1} n_{r-1}$ , and thus

$$\sum_{i=1}^{r-1} a_i (m_i + a_r n_i) = 0.$$

Thus  $m_i + a_r n_i \in (a_1, \dots, a_{r-1})M$  and thus  $m_i \in (a_1, \dots, a_r)M$ .  $\square$

**Theorem 3.5.3.** Assume that  $(a_1, \dots, a_r)$  is an  $M$ -regular sequence. Then for any integers  $n_1, \dots, n_r$ , the sequence  $a_1^{n_1}, \dots, a_r^{n_r}$  is  $M$ -regular.

*Proof.* It is sufficient to prove that  $a_1^n, a_2, \dots, a_r$  is an  $M$ -regular sequence. We will induct on  $n$ . Assume that  $a_1^{n-1}, \dots, a_2, a_r$  is  $M$ -regular. First, multiplication by  $a_1^n$  is clearly injective.

Now if  $a_1^n, a_2, \dots, a_{j-1}$  is  $M$ -regular, then let  $m \in M$  such that

$$a_j m = a_1^n m_1 + \dots + a_{j-1} m_{j-1}.$$

By induction on  $n$ , we can write

$$m = a_1^{n-1} m'_1 + \dots + a_{j-1} m'_{j-1}.$$

Multiplying this by  $a_j$  and combining the two equations, we obtain

$$0 = a_1^{n-1} (a_1 m_1 - a_j m'_1) + a_2 (m_2 - a_j m'_2) + \dots + a_{j-1} (m_{j-1} - a_j m'_{j-1}).$$

By the previous lemma, we see that  $a_1 m_1 - a_j m'_1 \in (a_1^{n-1}, \dots, a_{j-1})M$ . Therefore,  $a_j m'_1 \in (a_1, \dots, a_{j-1})M$ , so  $m'_1 \in (a_1, \dots, a_{j-1})M$ . This implies that  $m \in (a_1^n, \dots, a_{j-1})M$ .  $\square$

**Definition 3.5.4.** The sequence  $(a_1, \dots, a_r)$  is said to be  $M$ -quasi-regular if one of the following equivalent conditions holds:

- For all  $F(x_1, \dots, x_r) \in M[x_1, \dots, x_r] = A[x_1, \dots, x_r] \otimes_A M$  homogeneous of degree  $n$  such that  $F(a_1, \dots, a_r) \in I^{n+1}M$ , this implies that  $F(x_1, \dots, x_r) \in IM[x_1, \dots, x_r]$ , where  $I = (a_1, \dots, a_r)$ .
- If  $F(x_1, \dots, x_r) \in M[x_1, \dots, x_r]$  is homogeneous and such that  $F(a_1, \dots, a_r) = 0$ , then  $F \in IM[x_1, \dots, x_r]$
- The map

$$M/IM[x_1, \dots, x_r] \rightarrow \text{gr}^I M = \bigoplus_{n=0}^{\infty} I^n M / I^{n+1} M, F \mapsto F(a_1, \dots, a_r)$$

is an isomorphism.

**Lemma 3.5.5.** Assume that  $(a_1, \dots, a_r)$  is  $M$ -quasi-regular and  $x \in A$ . Then if  $(IM : x) = IM$ , then  $(I^n M : x) = I^n M$  for all  $n \geq 1$ .

*Proof.* We will induct on  $n$ . Suppose that  $m \in (I^n M : x)$ . Then  $xm \in I^n M \subset I^{n-1}M$  and thus  $m \in I^{n-1}M$ . Therefore there exists  $g(X_1, \dots, X_r)$  homogeneous of degree  $n-1$  such that  $m = g(a_1, \dots, a_r)$ . This implies that  $xg(X_1, \dots, X_r) \in M[\underline{X}]$ , so

$$xg(a_1, \dots, a_r) = xm \in I^n M$$

and then quasi-regularity gives us that  $xg(X_1, \dots, X_r) \in IM[\underline{X}]$ . This implies that

$$g(X_1, \dots, X_r) \in (IM : X)[\underline{X}] = IM[\underline{X}],$$

as desired.  $\square$

**Proposition 3.5.6.** Using the same notation, if  $(a_1, \dots, a_r)$  is  $M$ -regular, then it is  $M$ -quasi-regular. Conversely, if  $(a_1, \dots, a_r)$  is  $M$ -quasi-regular and  $M, M/a_1 M, \dots, M/(a_1, \dots, a_r)M$  are Hausdorff in the  $I$ -adic topology, then  $(a_1, \dots, a_r)$  is  $M$ -regular.

*Proof.* First we prove that regular implies quasi-regular by induction on  $r$ . Clearly  $r = 1$  is obvious. Suppose  $g(x_1) \in M[x_1]$  is homogeneous of degree  $n$ . Then if  $g(a_1) \in a_1^{n+1}M$ , we have  $a_1^{n+1}m' = a_1^n m$  and thus  $a_1^n(m - a_1 m') = 0$ . By regularity, we have  $m = a_1 m'$  and thus  $g(x_1) \in a_1 M[x_1]$ .

For the inductive step, suppose that  $(a_1, \dots, a_r)$  is regular. Then we know that  $(a_1, \dots, a_{r-1})$  is  $M$ -quasi-regular, so now choose  $F(x_1, \dots, x_r) \in M[x_1, \dots, x_r]$  homogeneous of degree  $q$  and such that  $F(a_1, \dots, a_r) = 0$ . Then we can write

$$F(x_1, \dots, x_r) = G(x_1, \dots, x_{r-1}) + x_r H(x_1, \dots, x_r)$$

where  $H$  is homogeneous of degree  $q-1$ . Then  $G(a_1, \dots, a_{r-1}) \in I_0^q M$ , where  $I_0 = (a_1, \dots, a_{r-1})$ . This implies that  $a_r H(a_1, \dots, a_{r-1}) \in I_0^q M$ , which implies that  $H(a_1, \dots, a_{r-1}) \in (I_0^q M : a_r)$ . Because  $a_1, \dots, a_{r-1}$  is quasi-regular and  $a_1, \dots, a_r$  is regular, we have  $(I_0 M : a_r) = I_0 M$ . This implies that  $(I_0^q M : a_r) = I_0^q M$  and thus  $H(a_1, \dots, a_r) \in I_0^q M$ . Then  $H(a_1, \dots, a_r) = h(a_1, \dots, a_{r-1})$  where  $h$  is homogeneous of degree  $q$ . Now let

$$g(x_1, \dots, x_{r-1}) = G(x_1, \dots, x_{r-1}) + a_r h(x_1, \dots, x_{r-1}).$$

Because  $g(a_1, \dots, a_{r-1}) = F(a_1, \dots, a_r) = 0$ , we see that  $g(x_1, \dots, x_{r-1}) \in I_0M[x_1, \dots, x_{r-1}]$  by induction. We conclude that  $G \in IM[x_1, \dots, x_{r-1}]$ . Because  $H \in I_0M[x_1, \dots, x_r]$ , we have  $F \in IM[x_1, \dots, x_r]$ .

Now in the other direction, we will induct on  $r$ . If  $r = 1$ , assume that  $a_1$  is  $M$ -quasi-regular. We need to show that  $m \mapsto a_1m$  is injective. Suppose that  $a_1m = 0$ . Then if we consider the polynomial  $g_0(x) = m$ , we see that  $a_1g_0(a_1) = 0$  and thus  $x_1g_0(x_1) \in a_1M[x_1]$ . This means that  $g_0(x_1) \in a_1M[x_1]$  and thus  $m \in a_1M$ . Then there exists  $g_1$  homogeneous of degree 1 such that  $m = g_1(a_1)$ . Then we see that  $x_1g_1(x_1) \in IM[x_1]$  and thus  $g_1(x_1) \in IM[x_1]$ , so  $m = g_1(a_1) \in I^2M$ . Then there exists  $g_2 \in M[x_1]$  homogeneous of degree 2 such that  $m = g_2(a_1)$ . Then  $a_1m = a_1g_2(a_1) = 0$ , so we deduce that  $x_1g_2(x_1) \in IM$  and thus  $g_2(x_1) \in IM$ . Evaluating at  $a_1$ , we see that  $m \in I^3M$ . In particular, we see that  $m \in \bigcap_{n \geq 1} I^nM = 0$ , where the last equality uses the Hausdorff condition, and thus  $m = 0$ .

Now for the induction, we know that  $a_1$  is  $M$ -regular. We need to show that  $(a_2, \dots, a_r)$  is  $M/a_1M$ -regular. This follows from the inductive hypothesis if we check that  $(a_2, \dots, a_r)$  is  $M/a_1M =: \bar{M}$ -quasi-regular. Choose  $F(x_2, \dots, x_r) \in M[x_2, \dots, x_r]$  homogeneous of degree  $n$  such that  $F(a_2, \dots, a_r) \in a_1M$ . Then we can write  $F(a_1, \dots, a_r) = a_1m$ , so let  $i$  be such that  $m \in I^iM$ . Then let  $G \in M[x_1, \dots, x_r]$  be homogeneous of degree  $i$  and satisfy  $m = G(a_1, \dots, a_r)$ . Then the polynomial

$$F(x_2, \dots, x_r) - x_1G(x_1, \dots, x_r)$$

vanishes at  $(a_1, \dots, a_r)$ . If  $i < n - 1$ , then  $a_1G(a_1, \dots, a_r) = F(a_2, \dots, a_r) \in I^nM \subset I^{i+2}M$ . But then  $x_1G(x_1, \dots, x_r)$  is homogeneous of degree  $i + 1$  and thus  $x_1G(x_1, \dots, x_r) \in IM[x_1, \dots, x_r]$  by quasi-regularity. This implies that

$$m = G(a_1, \dots, a_n) \in I^{i+1}M.$$

We can repeat this until  $m \in I^{n-1}M$  and  $G$  is of degree  $n - 1$ . Then

$$g(x) = F(x_2, \dots, x_r) - x_1G(x_1, \dots, x_r)$$

is homogeneous of degree  $n$  and  $g(a_1, \dots, a_r) = 0$ . This implies that  $g(x_1, \dots, x_r) \in IM[x_1, \dots, x_r]$  and thus  $F(x_2, \dots, x_r) \in IM[x_2, \dots, x_r]$ . Thus

$$\bar{F}(x_2, \dots, x_r) \in \bar{IM}[x_2, \dots, x_r].$$

Then  $(a_2, \dots, a_r)$  is  $\bar{M}$ -quasi-regular, so they are  $\bar{M}$ -regular by induction.  $\square$

*Remark 3.5.7.* For  $A$  Noetherian and  $M$  of finite type, regular and quasi-regular are equivalent.

**Definition 3.5.8.** Let  $I \subset A$  be an ideal and  $M$  and  $A$ -module. Then the  $I$ -depth of  $M$  is the (possibly infinite) length of the longest  $M$ -regular sequence in  $I$ .

Before we continue, recall that  $\text{Ext}^\bullet$  are the right derived functors of  $\text{Hom}$  and are computed by taking projective resolutions of the first argument or injective resolution of the second argument. In particular, if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence, we have an exact sequence

$$\dots \rightarrow \text{Ext}^i(M, A) \rightarrow \text{Ext}^i(M, B) \rightarrow \text{Ext}^i(M, C) \rightarrow \text{Ext}^{i+1}(M, A) \rightarrow \dots$$

and similarly,

$$\dots \rightarrow \text{Ext}^i(C, N) \rightarrow \text{Ext}^i(B, N) \rightarrow \text{Ext}^i(A, N) \rightarrow \text{Ext}^{i+1}(C, N) \rightarrow \dots$$

**Theorem 3.5.9.** Assume that  $A$  is Noetherian,  $M$  a finite  $A$ -module, and  $I \subset A$  an ideal such that  $IM \neq M$ . Let  $m \in \mathbb{Z}_{>0}$ . Then the following are equivalent:

1.  $\text{Ext}^n(N, M) = 0$  for all  $i < n$  and any finitely  $A$ -module  $N$  such that  $\text{supp}(N) \subset V(I)$ .
2.  $\text{Ext}^i(A/I, M) = 0$  for all  $i < n$ .
3.  $\text{Ext}^i(N, M) = 0$  for all  $i < n$  for some finite  $A$ -module  $N$  such that  $\text{supp}(N) = V(I)$ .
4. There exists a  $M$ -regular sequence  $(a_1, \dots, a_n)$  of length  $n$  inside  $I$ .

*Proof.* Clearly **1** implies **2** implies **3**. Now we prove **3** implies **4** implies **1**.

**3 implies 4:** Assume that  $\text{Hom}(N, M) = 0$ . If  $I$  does not contain any  $M$ -regular element, then  $I \subset \bigcup_{\mathfrak{p} \in \text{Ass } M} \mathfrak{p}$  and thus  $I \subset \mathfrak{p}$  for some  $\mathfrak{p} \in \text{Ass}(M)$ . Then  $A/\mathfrak{p} \hookrightarrow M$ , which is equivalent to  $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$ . On the other hand, we know  $\mathfrak{p} \in V(I) = \text{supp}(N)$ , so  $N_{\mathfrak{p}} \neq 0$ . By Nakayama, we see that  $N_{\mathfrak{p}} \otimes k(\mathfrak{p}) \neq 0$ , and thus  $\text{Hom}_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$ , but this is simply  $\text{Hom}_A(N, M)_{\mathfrak{p}}$ . Thus  $\text{Hom}(N, M) \neq 0$  and thus there exists  $a_1 \in I$  that is  $M$ -regular. Then we have an exact sequence

$$0 \rightarrow M \rightarrow M \rightarrow M/a_1M \rightarrow 0.$$

Writing  $M_1 = M/a_1M$ , we have proved the case  $n = 1$  and now proceed by induction on  $n$ . Applying  $\text{Ext}^n(N, -)$  to the above exact sequence, we have the exact sequence

$$\dots \rightarrow \text{Ext}^i(N, M) \rightarrow \text{Ext}^i(N, M_1) \rightarrow \text{Ext}^{i+1}(N, M)$$

and deduce that  $\text{Ext}^i(N, M_1) = 0$  for  $i < n - 1$ . Applying the case  $n - 1$  to  $M_1$ , we obtain an  $M_1$ -regular sequence  $a_2, \dots, a_n \in I$  and thus  $(a_1, \dots, a_n)$  is  $M$ -regular.

**4 implies 1:** We will induct on  $n$ . We have a sequence  $a_1, \dots, a_n \in I$  that is  $M$ -regular. Then set  $M_1 = M/a_1M$ , so we have an exact sequence

$$0 \rightarrow M \rightarrow M \rightarrow M_1 \rightarrow 0.$$

Then choose  $N$  such that  $\text{supp}(N) \subset V(I)$ . Then we have an exact sequence

$$\text{Ext}^{i-1}(N, M_1) \rightarrow \text{Ext}^i(N, M) \rightarrow \text{Ext}^i(N, M).$$

If  $i < n$ , then  $i - 1 < n - 1$ . By the inductive hypothesis, this implies that  $\text{Ext}^{i-1}(N, M_1) = 0$ . Thus the map  $\text{Ext}^i(N, M) \xrightarrow{\times a_1} \text{Ext}^i(N, M)$  is injective, so because  $\text{supp}(N) \subset V(I)$ , we have  $I \subset \sqrt{\text{Ann}(N)}$ . Thus there exists  $m$  such that  $a_1^m \in \text{Ann}(N)$ . From the exact sequence, we know  $\text{Ext}^i(N, M) \xrightarrow{\times a_1^m} \text{Ext}^i(N, M)$  is injective, which means that  $\text{Ext}^i(N, M) = 0$ .  $\square$

**Definition 3.5.10.** If  $A$  is a local Noetherian ring and  $M$  is an  $A$ -module, then we define the depth of  $M$  to be

$$\text{depth}(M) := \mathfrak{m}\text{-depth of } M.$$

This is the same as the maximal length of an  $M$ -regular sequence in  $\mathfrak{m}$ .

**Corollary 3.5.11.** Let  $A$  be local Noetherian and  $M$  be a finitely-generated  $A$ -module. Then  $\text{depth}(M) = n$  if and only if there exists a  $M$ -regular sequence  $a_1, \dots, a_n$  such that  $\text{Ext}^i(k, M) = 0$  for all  $i < n$  and  $\text{Ext}^n(k, M) = \text{Hom}(k, \overline{M})$ , where  $\overline{M} = M/(a_1, \dots, a_n)M$  and  $k = A/\mathfrak{m}$ .

*Proof.* We know that the equivalence of **2** and **4** from the theorem implies the corollary except for  $\text{Ext}^n(k, M) = \text{Hom}(k, \overline{M})$ , but this fact be proved by induction on  $n$  using the fact that

$$0 \rightarrow M \xrightarrow{\times a_1} M \rightarrow M/a_1M \rightarrow 0$$

is exact because this implies that  $\text{Ext}^{n-1}(k, M/a_1M) = \text{Ext}^n(k, M)$ . This gives the desired result.  $\square$

**Lemma 3.5.12.** *Let  $A$  be a local Noetherian and  $M, N$  finite  $A$ -modules. Then if  $k = \text{depth}(M)$  and  $r = \dim(N)$ , then  $\text{Ext}^i(N, M) = 0$  for all  $i < k - r$ .*

*Proof.* We use induction on  $r$ . For the case  $r = 0$ , we know that  $\text{supp}(N) = \{\mathfrak{m}\}$  and this follows by the previous theorem. Now for the inductive hypothesis, we may assume that  $N = A/\mathfrak{p}$  and  $\dim A/\mathfrak{p} = r$ . This is possible because we can consider a filtration on  $N$  with successive quotients of the form  $A/\mathfrak{p}_i$  with  $\dim A/\mathfrak{p}_i \leq r$ . We know that  $\mathfrak{m} \neq \mathfrak{p}$ , so there exists  $x \in \mathfrak{m} \setminus \mathfrak{p}$ , and this  $x$  is  $N$ -regular. Then we have the exact sequence

$$0 \rightarrow N \xrightarrow{\cdot x} N \rightarrow N' \rightarrow 0,$$

where  $N' = N/xN$ . Because  $\bar{x}$  is not in any minimal prime ideal of  $A/\mathfrak{p}$ , we know that  $r' := \dim N' < \dim A/\mathfrak{p}$ . By induction, we know that  $\text{Ext}^i(N', M) = 0$  for  $k < k - r'$ . Now if  $i < k - r$ , we know that  $i + 1 < k - r'$  and thus  $\text{Ext}^{i+1}(N', M) = 0$ . Now considering the exact sequence of Ext groups, we have

$$\text{Ext}^i(N', M) \rightarrow \text{Ext}^i(N, M) \rightarrow \text{Ext}^i(N, M) \rightarrow \text{Ext}^{i+1}(N', M)$$

and thus  $\text{Ext}^i(N, M)/x \text{Ext}^i(N, M) = 0$  and thus  $\text{Ext}^i(N, M) = 0$  by Nakayama's lemma.  $\square$

*Remark 3.5.13.* If  $N, M$  are finitely-generated, then  $\text{Ext}^i(N, M)$  is also finitely-generated.

**Theorem 3.5.14.** *Let  $A$  be local Noetherian and  $M$  be a finitely-generated  $A$ -module. Then for all  $\mathfrak{p} \in \text{Ass}(M)$ , we have  $\text{depth } M \leq \dim A/\mathfrak{p}$ .*

*Proof.* We know  $A/\mathfrak{p} \hookrightarrow M$ , so  $\text{Hom}(A/\mathfrak{p}, M) \neq 0$ . Thus  $0 \geq \text{depth } M - \dim A/\mathfrak{p}$  by the lemma.  $\square$

**Corollary 3.5.15.** *Let  $A$  be local Noetherian. Then  $\text{depth } A \leq \dim A$ .*

In general, this inequality is strict, so we will later study the rings for which this is an equality.

**Lemma 3.5.16.** *Let  $A$  be a local Noetherian ring. Then let  $M$  be a finitely-generated  $A$ -module and  $(a_1, \dots, a_r)$  be an  $M$ -regular sequence. Then  $\dim M/(a_1, \dots, a_r) \leq \dim M - r$ .*

*Proof.* We prove this by induction on  $M$ . It suffices to do this for  $r = 1$ , so choose  $x \in A$  an  $M$ -regular element. We know that  $\dim M/xM \geq \dim M - 1$  for any  $x \in A$ , so we need to prove this is an equality. Then we know  $\text{supp}(M/xM) = \text{supp}(M) \cap V(x)$ , and thus  $x$  is not contained in any minimal prime ideal in  $\text{supp}(M)$  by regularity. Therefore  $\text{supp}(M/xM)$  does not contain any minimal ideal of  $V(\text{Ann}(M)) = \text{supp}(M)$ . In particular, this means that  $\dim M/xM < \dim M$ , as desired.  $\square$

**Lemma 3.5.17.** *Let  $M, N$  be finitely-generated  $A$ -modules. Then  $\text{supp}(M \otimes N) = \text{supp}(M) \cap \text{Supp}(N)$ .*



*Proof.* Let  $\mathfrak{p} \in \text{Supp}(M) \cap \text{Supp}(N)$ . Then  $M_{\mathfrak{p}}, N_{\mathfrak{p}} \neq 0$ . By Nakayama, we have  $M_{\mathfrak{p}} \otimes k(\mathfrak{p}) \neq 0$  and  $N_{\mathfrak{p}} \otimes k(\mathfrak{p}) \neq 0$  where  $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ . This implies that

$$M_{\mathfrak{p}} \otimes k(\mathfrak{p}) \otimes_{k(\mathfrak{p})} \otimes N_{\mathfrak{p}} \otimes k(\mathfrak{p}) \neq 0$$

and therefore  $M_{\mathfrak{p}} \otimes k_{\mathfrak{p}} N_{\mathfrak{p}} = (M \otimes_A N)_{\mathfrak{p}} \neq 0$ . Thus  $\text{supp}(M) \cap \text{supp}(N) \subset \text{supp}(M \otimes N)$ . Note that for  $N = A/xA$ , we have  $\text{supp}(M/xM) = \text{supp}(M) \cap V(x)$  because  $M \otimes N = M/xM$ .  $\square$

## Geometric Aspects of Commutative Algebra

### 4.1 Cohen-Macaulay Rings and Modules

Let  $A$  be a local Noetherian ring and  $M$  be a finitely-generated  $A$ -module.

**Definition 4.1.1.** Recall that  $\dim M \geq \text{depth } M$ . Then  $M$  is *Cohen-Macaulay* if  $\dim M = \text{depth } M$ .

**Theorem 4.1.2.** *Let  $A$  be local Noetherian and  $M$  be finitely generated.*

1. *If  $M$  is Cohen-Macaulay, then for any  $\mathfrak{p} \in \text{Ass}(M)$ ,  $\text{depth } M = \dim A/\mathfrak{p}$ .*
2. *If  $f \in A$  is  $M$ -regular and  $M' = M/fM$ , then  $M$  is Cohen-Macaulay if and only if  $M'$  is.*
3. *If  $M$  is Cohen-Macaulay, then for all  $\mathfrak{p} \in \text{Spec } A$ ,  $M_{\mathfrak{p}}$  is Cohen-Macaulay and  $\text{depth}_{\mathfrak{p}} M = \text{depth}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$ .*

*Proof.* 1. Let  $M \neq 0$  and  $\dim M = \text{depth } M$ . Then let  $\mathfrak{p} \in \text{Ass}(M) \subset \text{supp}(M)$ . This implies that  $\dim A/\mathfrak{p} \leq \dim M$ , but also  $\dim A/\mathfrak{p} \geq \text{depth}(M) = \dim M$ .

2. Let  $f$  be  $M$ -regular. Then we know  $\text{depth } M/fM = \text{depth } M - 1$  (this follows from the Theorem 3.5.9, applying  $\text{Ext}(k, -)$  to the exact sequence  $0 \rightarrow M \rightarrow M \rightarrow M/fM \rightarrow 0$ ). But then we proved earlier that  $\dim M/fM = \dim M - 1$ , and thus  $M$  is C-M if and only if  $M/fM$  is C-M.
3. Let  $\mathfrak{p} \in \text{supp}(M)$ . Then  $\mathfrak{p} \supset \text{Ann}(M)$  and  $M_{\mathfrak{p}} \neq 0$ . But then if  $x_1, \dots, x_r \in \mathfrak{p}$  is  $M$ -regular where  $r = \text{depth}_{\mathfrak{p}} M$ , then  $x_1, \dots, x_r \in \mathfrak{p}A_{\mathfrak{p}}$  is  $M_{\mathfrak{p}}$ -regular by exactness of localization. This implies that  $\text{depth } M_{\mathfrak{p}} \geq r = \text{depth}_{\mathfrak{p}} M$ . We know that  $\dim M_{\mathfrak{p}} \geq \text{depth } M_{\mathfrak{p}}$ , so we need to prove that  $r = \dim M_{\mathfrak{p}}$ . We do this by induction on  $\text{depth}_{\mathfrak{p}} M$ . If  $r = 0$ , then we know  $\text{Hom}(A/\mathfrak{p}, M) \neq 0$ . Thus there exists  $\mathfrak{p}' \supset \mathfrak{p}$  such that  $A/\mathfrak{p}' \hookrightarrow M$ , so  $\mathfrak{p}' \in \text{Ass } M$ . By minimality of associated primes, we have  $\mathfrak{p}' = \mathfrak{p}$ . Now  $\dim M_{\mathfrak{p}'} = 0$  because  $\mathfrak{p}$  is maximal in  $A_{\mathfrak{p}}$  and minimal in  $\text{supp}(M)$ .

Now in general, assume  $\text{depth}_{\mathfrak{p}}(M) > 0$ . Let  $a \in \mathfrak{p}$  be  $M$ -regular. Thus  $a$  is  $M_{\mathfrak{p}}$ -regular, so set  $M_1 = M/aM$ . We know that  $\dim(M_1)_{\mathfrak{p}} = \text{depth}_{\mathfrak{p}} M_1$  by the inductive hypothesis. This implies that  $\dim M_{\mathfrak{p}} = \text{depth}_{\mathfrak{p}} M$  because  $\dim(M_1)_{\mathfrak{p}} = \dim M_{\mathfrak{p}} - 1$  and  $\text{depth}_{\mathfrak{p}} M_1 = \text{depth}_{\mathfrak{p}} M - 1$ .  $\square$

**Theorem 4.1.3.** *Let  $A$  be local Noetherian and Cohen-Macaulay. Then*

1. For all proper ideals  $I \subsetneq A$ , we have  $\text{ht}(I) + \dim A/I = \dim A$ .
2. For all  $P \supset Q$  prime ideals, then  $\text{ht}(P) - \text{ht}(Q) = \text{ht}(P/Q)$ , where the last height is taken in  $A/Q$ .
3. Let  $x_1, \dots, x_r \in \mathfrak{m}$ . Then the following are equivalent:
  - a)  $x_1, \dots, x_r$  are  $A$ -regular.
  - b)  $\text{ht}(x_1, \dots, x_i) = i$  for all  $i \leq r$ .
  - c)  $\text{ht}(x_1, \dots, x_r) = r$ .
  - d) There exists  $x_{r+1}, \dots, x_n \in \mathfrak{m}$  with  $n = \dim A$  such that  $x_1, \dots, x_n$  is a system of parameters for  $A$ .

*Proof.* We will first prove **3** and then use this to prove **1** and **2**.

**a implies b:** We already know that  $\text{ht}(x_1, \dots, x_i) \leq i$ . First, we know that  $x_1$  does not belong to any minimal prime ideal of  $A$  because  $x_1$  is  $A$ -regular. Thus  $\text{ht}(x_1) = 1$ . Then in  $A/x_1$ , the sequence  $\bar{x}_1, \dots, \bar{x}_r$  is  $A/x_1$ -regular, and thus  $\text{ht}(\bar{x}_2, \dots, \bar{x}_i) = i - 1$ . Thus we can find

$$\mathfrak{p}_i \supsetneq \mathfrak{p}_{i-1} \subsetneq \dots \supsetneq \mathfrak{p}_1 \supset (x_1)$$

such that  $\bar{\mathfrak{p}}_i \supsetneq \dots \supsetneq \bar{\mathfrak{p}}_1 \supsetneq 0$ . This implies that  $\text{ht}(\mathfrak{p}_i) \geq 1$  and thus  $\text{ht}(x_1, \dots, x_r) \geq i$ .

**b implies c:** This is trivial.

**c implies d:** Assume  $r < \dim A$ . Then there exists  $x_{r+1} \in \mathfrak{m}$  that does not belong to any minimal prime ideal containing  $(x_1, \dots, x_r)$ . Therefore  $\text{ht}(x_1, \dots, x_{r+1}) = r + 1$  and then we can find  $x_{r+2}, \dots, x_n$  such that  $\text{ht}(x_1, \dots, x_n) = n$ . But this implies that  $I = (x_1, \dots, x_n)$  is an ideal of definition for  $A$  and so  $(x_1, \dots, x_n)$  is a system of parameters.

**d implies a:** Note that so far we have not used the fact that  $A$  is Cohen-Macaulay. We show that a system of parameters is  $A$ -regular. Let  $\mathfrak{p} \in \text{Ass}(A)$ . Then  $\dim A/\mathfrak{p} = n$ . This implies that  $\mathfrak{p}$  is minimal and therefore  $\bar{x}_1 \neq 0$  in  $A/\mathfrak{p}$  because otherwise  $\bar{x}_1, \dots, \bar{x}_n$  would be a system of parameters in  $A/\mathfrak{p}$ . But then this means  $x_1 \notin \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(A)$ . This implies that  $x_1$  is  $A$ -regular, and thus  $A/x_1A$  is Cohen-Macaulay of dimension  $n - 1$ . On the other hand,  $\bar{x}_2, \dots, \bar{x}_n$  is a system of parameters for  $A/x_1A$  and the desired result follows by induction.

Now we will prove **1** and **2**.

1. Let  $I \subsetneq A$  be an ideal with height  $\text{ht}(I) = r$ . Then we can choose  $a_1, \dots, a_r \in I$  such that  $\text{ht}(a_1, \dots, a_r) = r$ . By **3a**, this means that  $a_1, \dots, a_r$  is  $A$ -regular and thus  $r \leq \text{depth}_I(A)$ . On the other hand, if we have  $b_1, \dots, b_s \in I$  an  $A$ -regular sequence, we see that  $s \leq \text{ht}(I)$ . This tells us that  $\text{ht}(I) = \text{depth}_I A$ . But then recall that

$$\text{ht}(I) = \inf \{ \text{ht}(\mathfrak{p}) \mid \mathfrak{p} \supset I \}$$

and that

$$\dim A/I = \sup \{ \dim A/\mathfrak{p}, \mathfrak{p} \supset I \}.$$

This means we can assume that  $I = \mathfrak{p}$  is a prime ideal with  $\mathfrak{p} = r$ . In this case, we know that  $A_{\mathfrak{p}}$  is Cohen-Macaulay and

$$\text{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}} = \text{depth}_{\mathfrak{p}}(A)$$

so there exists  $x_1, \dots, x_r \in I$  an  $A$ -regular sequence such that  $A/(x_1, \dots, x_r)$  is Cohen-Macaulay of dimension  $n - r$ . Then  $\bar{\mathfrak{p}}$  is minimal in  $A/(x_1, \dots, x_r)$  because otherwise we

would be able to find  $x_{r+1} \in \mathfrak{p}$  which is not a zero divisor in  $A/(x_1, \dots, x_r)$  and then  $\text{depth}(\mathfrak{p}) \geq r + 1$ . This implies

$$\dim A/\mathfrak{p} = \dim A/(x_1, \dots, x_r) = n - r$$

and thus  $\dim A/\mathfrak{p} + \text{ht}(\mathfrak{p}) = n - r + r = n = \dim A$ .

2. Suppose  $P \supset Q$  are prime ideals. We know  $A_P$  is Cohen-Macaulay. Then we know that  $\dim A_P = \text{ht}(QA_P) + \dim(A_P/QA_P)$  by **1** for  $A_P$ . But then we have the desired result because  $\dim A_P = \text{ht}(P)$ ,  $\text{ht}(QA_P) = \text{ht}(Q)$ , and  $\dim(A_P/QA_P) = \text{ht}(P/Q)$ .  $\square$

## 4.2 Normal and Regular Rings

Let  $A$  be a domain and  $K = \text{Frac}(A)$ . If  $x \in K$ , we say that it is *almost integral* if there exists  $a \in A$  such that  $ax^m \in A$  for all  $n \geq 1$ .

*Remark 4.2.1.* Clearly if  $x$  is integral, then  $x$  is almost integral. Also, if  $x$  is almost integral and  $A$  is Noetherian,  $x$  is integral.

To prove the first part, write

$$x^N = a_0 + a_1x + \dots + a_{N-1}x^{N-1}.$$

Then choose  $a$  such that  $ax^i \in A$  for  $i \leq N - 1$  and then by induction we have  $ax^i \in A$  for  $i \geq n$ .

To prove the second part, note that if  $ax^i \in A$  for all  $a \geq 1$ , then  $A[x] \subset a^{-1}A$ . Because  $A$  is noetherian, then  $A[x]$  is a finitely-generated  $A$ -module and then use the characterization of integral elements over a ring.

**Definition 4.2.2.** We say that  $A$  is a *normal domain* (resp. *completely normal domain*) if every integral element (resp. almost integral element) of  $K$  over  $A$  belongs to  $A$ .

*Remark 4.2.3.* Thering  $A$  is normal if and only if  $A_{\mathfrak{p}}$  is normal for all prime ideals  $\mathfrak{p}$ . Additionally, for any normal ring  $A$ , all localizations  $S^{-1}A$  are normal.

We also note that if  $A$  is normal (resp. completely normal) then so is  $A[x]$  when  $A$  is Noetherian. The proof is easy and uses the fact that  $K[x]$  is a UFD and is thus completely normal. Then for  $f \in K(x)$  is almost integral, it belongs to  $K[x]$  and then we write

$$f = a_r x^r + \dots + a_0$$

where  $a_i \in K$ . Then there exists  $Q \in A[x]$  such that  $Qf^n \in A[x]$  for all  $n \geq 1$  and thus  $\alpha_d a_r^n \in A$  for all  $n \geq 1$ , where  $Q = \alpha_d x^d + \dots + \alpha_0$ . This means that  $a_r$  is almost integral and therefore  $a_r \in A$ . Then  $f - a_r x^r$  is almost integral and finally we proceed by induction to see that  $f \in A[x]$ .

Now we consider the normal rings of dimension 1.

**Proposition 4.2.4.** *Let  $A$  be a local Noetherian ring. Then the following are equivalent:*

1.  $A$  is a discrete valuation ring.
2.  $A$  is normal and  $\dim A = 1$ .
3.  $A$  is normal and there exists  $0 \neq a \in \mathfrak{m}$  such that  $(a)$  is  $\mathfrak{m}$ -primary.
4. The maximal ideal  $\mathfrak{m}$  of  $A$  is principal.

*Proof.* We will proceed quickly.

**1 implies 2:** This was done previously.

**2 implies 3:** Note that  $\mathfrak{m}$  is the only prime ideal containing  $(a)$  because  $A$  has dimension 1 and is a local domain. This means that  $(a)$  is  $\mathfrak{m}$ -primary.

**3 implies 4:** We know that  $\mathfrak{m} \in \text{Ass}(A/(a))$  and thus  $\mathfrak{m} = \text{Ann}_A(\bar{x})$  for some  $0 \neq \bar{x} \in A/aA$ . If  $x \in A$  is a lift of  $\bar{x}$ , then  $xa^{-1} \notin A$ . However, we know that  $\mathfrak{m} \cdot x \subseteq (a)$ , so  $mxa^{-1} \subset A$ . This tells us that  $\mathfrak{m} \subsetneq mxa^{-1}$  is a strict inclusion because otherwise  $xa^{-1}$  is integral over  $A$  and thus  $xa^{-1} \in A$ . Therefore  $mxa^{-1} = A$ , so there exists  $t \in \mathfrak{m}$  such that  $txa^{-1} = 1$  and thus  $tA \subset \mathfrak{m} \ni y$  and thus  $y = txa^{-1}y = t(xa^{-1}y)$ , where  $xa^{-1}y \in A$ , and thus  $y \in tA$ , so  $\mathfrak{m} = tA$  is principal.

**4 implies 1:** If  $\mathfrak{m} = (t)$ , then we know that

$$\bigcap_{n \geq 0} \mathfrak{m}^n = 0$$

and so for all  $x \neq 0$  there exists  $m$  such that  $x = ut^m$  where  $u \in A^\times$ , and thus  $A$  is a DVR.  $\square$

Now let  $A$  be a ring and  $I \subset A$  such that  $\bigcap_n I^n = (0)$  and  $A$  is  $I$ -separated. Then for all  $0 \neq a \in A$ , there exists  $n$  such that  $a \in I^n \setminus I^{n+1}$ . Then define  $a^*$  to be the class of  $a$  in  $I^n/I^{n+1}$  and

$$A^* = \text{gr}^I(A) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}.$$

**Theorem 4.2.5.** *Let  $A, I$  be as before. Then*

1. *If  $A^*$  is a domain, then so is  $A$ .*
2. *If  $A$  is Noetherian and  $I \subset \text{rad}(A)$ , then if  $A^*$  is a normal domain, so is  $A$ .*

*Proof.* 1. Choose  $a, b \in A$  nonzero. Then  $a^*, b^* \neq 0$  and so  $a^*b^* \neq 0$  because  $A^*$  is a domain. This means that if  $a \in I^n \setminus I^{n+1}, b \in I^q \setminus I^{q+1}$ , we know that  $ab \in I^{n+q} \setminus I^{n+q+1}$  and thus  $ab \neq 0$ .

2. Let  $x \in K$  be integral over  $A$ . Write  $x = \frac{a}{b}$  where  $a, b \in A$ . Then  $A/bA$  is Noetherian and therefore by the Krull intersection theorem this is the same thing as

$$\bigcap_{n \geq 0} I^n + bA = bA.$$

To show that  $x \in A$ , we will prove that  $a \in bA$  or that  $a \in I^n + bA$  for all  $n \geq 0$ . If  $a \in I^{n-1} + bA$ , then we know that  $a = a' + bs$ , where  $a' \in I^{n-1}$  and  $s \in A$ . Then we know that because  $\frac{a}{b}$  is integral, so is  $\frac{a'}{b}$ . Then we may assume that  $a \in I^{n-1}$  and  $\frac{a}{b}$  is almost integral. Then there exists  $c \in A$  such that  $ca^m \in b^m A$  for all  $m \geq 1$  and thus because  $a^* \neq 0$ , we know that  $(a^*)^m = (a^m)^* \neq 0$  and thus  $c^*(a^*)^m \in (b^*)^m A$ . Because  $A^*$  is Noetherian and normal, it is completely normal, and thus  $\frac{a^*}{b^*} \in A^*$  and thus  $a^* = b^*d^*$  for some  $0 \neq d \in A$ . This means that  $a - bd \in I^{N+1}$  for  $N$  such that  $a \in I^N \setminus I^{N+1}$ . But we know this for  $N = n - 1$ , and thus  $a - bd \in I^n$ , so  $a \in I^n + bA$ .  $\square$

Recall that if  $A$  is local Noetherian, we say that  $A$  is *regular* if there exists a system of parameters that generates the maximal ideal  $\mathfrak{m}$  of  $A$ .

**Theorem 4.2.6.** *Let  $A$  be local Noetherian,  $\mathfrak{m}$  be its maximal ideal, and  $k$  be the residue field. Then  $A$  is regular if and only if*

$$\mathrm{gr}^{\mathfrak{m}}(A) \cong k[x_1, \dots, x_d],$$

where  $d = \dim A$ .

*Proof.* Assume that  $A$  is regular. Then  $\mathfrak{m} = (x_1, \dots, x_d)$  is a system of parameters because  $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A = d$ . Therefore we have a map

$$k[x_1, \dots, x_d] \longrightarrow \mathrm{gr}^{\mathfrak{m}}(A) \quad Q(X_1, \dots, X_n) \mapsto Q(x_1, \dots, x_d).$$

This map is clearly surjective, so we need to show that it is injective. If  $Q \neq 0$  is in the kernel, then let  $q = \deg Q$ . Then  $(Q) \subset \ker$ . We know that

$$\ell(A/\mathfrak{m}^{n+1}) \leq \binom{n+d}{d} - \binom{n-q+d}{d}$$

and thus  $\dim A = d(A) \leq d - 1$ , a contradiction.

Now assume that  $\mathrm{gr}^{\mathfrak{m}}(A) \cong k[x_1, \dots, x_d]$ . By definition, this means that  $d(A) = d$ . Also  $\dim \mathfrak{m}/\mathfrak{m}^2$  is the dimension of the space of homogeneous polynomials of degree 1, so  $\dim \mathfrak{m}/\mathfrak{m}^2 = d$ . Thus  $A$  is regular.  $\square$

**Corollary 4.2.7.** *Let  $A$  be a local Noetherian ring. Then the following are equivalent:*

1.  $A$  is a discrete valuation ring.
2.  $A$  is a normal ring of dimension 1.
3.  $A$  is regular of dimension 1.

*Proof.* We have seen before that **1** is equivalent to **2** which implies **3**. To prove that **3** implies **1**, we need to show that  $A$  is a domain. But then  $\mathrm{gr}^{\mathfrak{m}}(A) \cong k[X]$  is a domain and thus  $A$  is a domain.  $\square$

**Corollary 4.2.8.** *If  $A$  is regular, then  $A$  is a normal domain.*

**Theorem 4.2.9.** *Let  $A$  be a regular local Noetherian ring and  $(x_1, \dots, x_d)$  be a regular system of parameters. Then*

1.  $A$  is a normal domain.
2.  $x_1, \dots, x_d$  is an  $A$ -regular sequence and  $A$  is Cohen-Macaulay.
3. The ideal  $\mathfrak{p}_i = (x_1, \dots, x_i)$  is a prime ideal of height  $i$  and  $A/\mathfrak{p}_i$  is regular of dimension  $d - i$ .
4. If  $\mathfrak{p} \in \mathrm{Spec}(A)$  and  $A/\mathfrak{p}$  is regular, then there exists a system of parameters  $y_1, \dots, y_d$  such that  $\mathfrak{p} = (y_1, \dots, y_i)$  and  $\dim A/\mathfrak{p} = d - i$ .

*Proof.* We prove each part.

1. We already proved that the graded ring  $\mathrm{gr}^{\mathfrak{m}}(A)$  is a normal domain.
2. Use the previous theorem. Recall that  $k[x_1, \dots, x_d] \cong \mathrm{gr}^{\mathfrak{m}}(A)$ . This means that  $x_1, \dots, x_d$  is an  $A$ -quasi-regular sequence. Because  $A$  is Noetherian, then  $x_1, \dots, x_d$  is an  $A$ -regular sequence. This implies that

$$d \leq \mathrm{depth}(A) \leq d$$

and thus  $A$  is Cohen-Macaulay.

3. The previous implies that  $\dim A/(x_1, \dots, x_i) = d - i$ . Then the ideal  $\bar{m} = m/p_i$  is generated by the  $d - i$  elements  $\bar{x}_{i+1}, \dots, \bar{x}_d$  and these form a system of parameters. By definition,  $A/p_i$  is regular.
4. Write  $\bar{m} = m/p$ . Then let  $i \leq d$  such that  $\dim \bar{m}/\bar{m}^2 = d - i$ . This means that  $i = \dim(\mathfrak{p} + m^2)/m^2$ , so we can choose  $y_1, \dots, y_i \in \mathfrak{p}$  such that  $\bar{y}_1, \dots, \bar{y}_i$  is a basis of  $(\mathfrak{p} + m^2)/m^2$ . We then complete this to a basis of  $m/m^2$ . Thus  $y_1, \dots, y_d$  is a regular system of parameters for  $A$ , and is thus a regular sequence. Then  $\mathfrak{p}' = (y_1, \dots, y_i) \subset \mathfrak{p}$  is a prime ideal and  $\dim A/\mathfrak{p}' = d - i$ . But then  $\dim A/\mathfrak{p} = p - i$  because  $A/\mathfrak{p}$  is regular by assumption, so we must have  $\mathfrak{p} = \mathfrak{p}'$ .  $\square$

We now return to some definitions.

**Definition 4.2.10.** Let  $M$  be an  $A$ -module. Then  $\mathfrak{p} \in \text{Ass}(M)$  is called an *embedded prime* if it is not minimal in the support.

Recall that if  $M$  is Cohen-Macaulay, then for any  $\mathfrak{p} \in \text{Ass}(M)$ , then  $\dim A/\mathfrak{p} = \dim M$ . Therefore  $M$  has no embedded primes.

**Definition 4.2.11.** Let  $I$  be an ideal of  $A$ . We say that  $I$  is *unmixed* if for all  $\mathfrak{p} \in \text{Ass}(A/I)$  we have  $\text{ht}(\mathfrak{p}) = \text{ht}(I)$ . Note this implies that  $A/I$  has no embedded primes.

**Definition 4.2.12.** We say that the *unmixed theorem holds for  $A$*  if for all  $I = (a_1, \dots, a_r)$  with  $r = \text{ht}(I)$ , then  $I$  is unmixed.

*Remark 4.2.13.* This is equivalent to saying that for all  $I = (x_1, \dots, x_r)$  of height  $\text{ht}(I) = r$ ,  $A/I$  has no embedded primes.

To see this, note that if  $A/I$  has no embedded primes, choose a minimal  $Q \in \text{Ass}(A/I)$ . We know that  $\text{ht}(Q) \leq r$  because  $Q$  is minimal containing  $I$ . However, this is at least

$$\inf \{\text{ht}(\mathfrak{p}) \mid \mathfrak{p} \supset I\} = \text{ht}(I) = r.$$

**Theorem 4.2.14.** Let  $A$  be Noetherian. Then  $A$  is Cohen-Macaulay if and only if the unmixed theorem holds for  $A$ .

*Proof.* Suppose that the unmixed theorem holds for  $A$ . Let  $\mathfrak{p}$  be a prime ideal. Then we know  $r = \text{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}}$ , so we can choose  $a_1, \dots, a_r \in \mathfrak{p}$  such that  $\text{ht}((a_1, \dots, a_i)) = i$ . However, if  $A$  satisfies the unmixed theorem, then  $a_{i+1}$  is not in any prime ideal of  $\text{Ass}(A/(a_1, \dots, a_i))$ . Here, we use the fact that  $A/(a_1, \dots, a_i)$  has no embedded primes because  $(a_1, \dots, a_i)$  is unmixed. Thus  $a_1, \dots, a_r$  is an  $A$ -regular sequence and is also  $A_{\mathfrak{p}}$ -regular. This implies that  $r \leq \text{depth}(A_{\mathfrak{p}}) \leq \dim A_{\mathfrak{p}} = \text{ht}(\mathfrak{p}) = r$  and thus  $A_{\mathfrak{p}}$  is Cohen-Macaulay.

Now assume that  $A$  is Cohen-Macaulay. Then if  $I = (a_1, \dots, a_r)$  has height  $r$ , then  $a_1, \dots, a_r$  is an  $A$ -regular sequence. Thus  $A/(a_1, \dots, a_r)$  is Cohen-Macaulay and thus has no embedded primes, so  $(a_1, \dots, a_r)$  is unmixed.  $\square$

Returning to normal domains, we have the following result:

**Theorem 4.2.15.** Let  $A$  be a Noetherian normal domain. Then any nonzero principal ideal is unmixed and

$$A = \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

In particular,  $A$  is Cohen-Macaulay if  $\dim A \leq 2$ .

*Proof.* Let  $0 \neq a \in A$ . To show that  $(a)$  is unmixed, we need to show that  $\text{ht}((a)) = 1$ . Let  $\mathfrak{p} \in \text{Ass}(A/(a))$ . Then  $\mathfrak{p} = \text{Ann}_A(\bar{b})$  for some  $b \neq 0$ . This means that

$$\begin{aligned} \mathfrak{p} &= \{x \in K \mid xb \subset (a)\} \\ &= \left\{x \in K \mid \frac{xb}{a} \in A\right\}. \end{aligned}$$

This means that

$$\mathfrak{p}^{-1} = \{y \in K \mid y\mathfrak{p} \subset A\}$$

and thus we can replace  $A$  by  $A_{\mathfrak{p}}$ . This implies that  $\mathfrak{p}^{-1}A_{\mathfrak{p}} \neq A_{\mathfrak{p}}$ , so

$$\mathfrak{p}A_{\mathfrak{p}} \subset \mathfrak{p}^{-1}A_{\mathfrak{p}} \cdot \mathfrak{p}A_{\mathfrak{p}} \subset A_{\mathfrak{p}}.$$

Then one of these two inclusions is an equality, so if the first inclusion is an equality, we know that elements of  $\mathfrak{p}^{-1}A_{\mathfrak{p}}$  are integral over  $A_{\mathfrak{p}}$  and thus  $\mathfrak{p}^{-1}A_{\mathfrak{p}} \subset A_{\mathfrak{p}}$  because  $A$  is a normal domain. This is impossible, so  $\mathfrak{p}^{-1}A_{\mathfrak{p}} \cdot \mathfrak{p}A_{\mathfrak{p}} = A_{\mathfrak{p}}$  and thus  $\mathfrak{p}A_{\mathfrak{p}}$  is a principal ideal. Therefore  $A_{\mathfrak{p}}$  is a DVR, so  $\text{ht}(\mathfrak{p}) = 1$ . Thus  $(a)$  is unmixed.

Now we will prove that  $A$  is the intersection of the localizations at the height 1 prime ideals. One direction is obvious, so now let  $\frac{b}{a} \in \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$ . Then if  $\frac{b}{a} \notin A$ , we know that  $A/(a) \neq 0$  and thus there exists  $\mathfrak{p}$  such that  $\text{Ass}(A/(a)) \neq \emptyset$ . We know that  $\text{ht}(\mathfrak{p}) = 1$ , so  $A_{\mathfrak{p}} \neq \mathfrak{p} = \text{Ann}(b)$ , a contradiction with the fact that  $\frac{b}{a} \in A_{\mathfrak{p}}$ .  $\square$

### 4.3 Homological Theory

**Lemma 4.3.1.** *Let  $A$  be a ring and  $M$  be an  $A$ -module. Then*

1.  $M$  is projective if and only if  $\text{Ext}_A^1(M, N) = 0$  for all  $A$ -modules  $N$ .
2.  $M$  is injective if and only if  $\text{Ext}_A^1(A/I, M) = 0$  for all ideals  $I \subset A$ .

*Proof.* We only need to prove the second part. One direction follows from the direction, so now suppose we have  $0 \rightarrow N \rightarrow N'$  injective. We need to show that  $\text{Hom}(N', M) \rightarrow \text{Hom}(N, M)$  is surjective. Consider the set

$$\mathcal{S} = \{N \subset Q \subset N' \mid \text{Hom}(Q, M) \twoheadrightarrow \text{Hom}(N, M)\}.$$

We will show that  $N' \in \mathcal{S}$ . First,  $\mathcal{S}$  contains a maximal element  $N_1$  by an application of Zorn's lemma. If  $N_{\alpha}$  is a chain, then  $\bigcup_{\alpha} N_{\alpha} \in \mathcal{S}$ . Now we show that  $N_1 = N'$ . Otherwise, there exists  $x \in N' \setminus N_1$  and set  $N_2 = N_1 + Ax \subset N'$ . This gives an exact sequence

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow A/I \rightarrow 0.$$

Now we apply the functor  $\text{Hom}(-, M)$  and we see that  $\text{Hom}(N_2, M) \twoheadrightarrow \text{Hom}(N_1, M)$  because  $\text{Ext}_A^1(A/I, M) = 0$ . But this implies that  $N_2 \in \mathcal{S}$  and this contradicts maximality of  $N_1$ .  $\square$

**Definition 4.3.2.** Let  $A$  be a ring and  $M$  an  $A$ -module. Then we define the *projective dimension*  $\text{proj. dim } M$  to be the length of the shortest projective resolution of  $M$ . Equivalently, we define the *injective dimension*  $\text{inj. dim } M$  to be the length of the shortest injective resolution of  $M$ .

**Lemma 4.3.3.** *Let  $n$  be a positive integer. Then the following are equivalent:*

1. All  $A$ -modules  $M$  have projective dimension at most  $n$ ;



2. All  $A$ -modules  $M$  of finite type have projective dimension at most  $n$ ;
3. All  $A$ -modules  $M$  have injective dimension at most  $n$ ;
4.  $\text{Ext}_A^{n+1}(M, N) = 0$  for all  $A$ -modules  $M, N$ .

*Proof.* Clearly **1** implies **2**.

**2** implies **3**: Consider

$$0 \rightarrow M \rightarrow U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_{n-1} \rightarrow C \rightarrow 0$$

such that  $U_0, \dots, U_{n-1}$  are injective. Now we have an exact sequence

$$0 \rightarrow M \rightarrow U_0 \rightarrow V_0 \rightarrow 0$$

where  $V_0$  is the kernel of  $U_1 \rightarrow U_2$ . For any ideal  $I$ , we know that  $\text{Ext}_A^{n+1}(A/I, M) = \text{Ext}_A^n(A/I, V_0)$  because  $U_0$  is injective. Now we obtain that  $\text{Ext}_A^n(A/I, V_0) = \text{Ext}_A^{n-1}(A/I, V_1)$  where  $V_1 = \ker(U_2 \rightarrow U_3)$ . This implies that

$$\text{Ext}_A^{n+1}(A/I, M) = \text{Ext}_A^1(A/I, C).$$

By **2**, we know that  $\text{Ext}_A^{n+1}(A/I, M) = 0$  and therefore  $C$  is injective. Thus there is an injective resolution of  $M$  of length  $n$ .

**3** implies **4**: This is trivial.

**4** implies **1**: The proof is similar to the proof that **2** implies **3**. □

**Corollary 4.3.4.** *The two numbers*

$$\sup_M(\text{proj. dim } M) = \sup_M(\text{inj. dim } M)$$

*are equal and are called the global homological dimension of  $A$ .*

**Lemma 4.3.5.** *Suppose  $A$  is Noetherian and  $M$  is an  $A$ -module of finite type. Then  $M$  is projective if and only if  $\text{Ext}_A^1(M, N) = 0$  for all  $N$  of finite type.*

*Proof.* Because  $M$  is finite type, we can write

$$0 \rightarrow K \xrightarrow{\psi} A^n \rightarrow M \rightarrow 0$$

where  $K$  is finite type. Then if we apply  $\text{Ext}_A(-, K)$ , we see that  $\text{Ext}_A^1(M, K) = 0$  and thus  $\text{Hom}(A^n, K) \rightarrow \text{Hom}(K, K)$  is surjective. Thus there exists  $\phi: A^n \rightarrow K$  such that  $\phi \circ \psi = \text{id}_K$  and thus the exact sequence is split. Thus  $M$  is a direct factor of a free module and is thus projective. □

**Lemma 4.3.6.** *Let  $A$  be a local Noetherian ring and  $M$  be a finite  $A$ -module. Then  $\text{proj. dim } M \leq n$  if and only if  $\text{Tor}_{n+1}^A(M, k) = 0$ .*

*Proof.* Let  $P_\bullet \rightarrow M \rightarrow 0$  be a projective resolution of length at most  $n$ . Then using this resolution, we compute  $\text{Tor}_{n+1}(M, k) = 0$ .

In the other direction, first assume that  $n = 0$ . If  $\text{Tor}_1^A(M, k) = 0$ , we want to show that  $M$  is projective. Because  $M$  is finite type, then there exists  $A^r \twoheadrightarrow M$  such that  $A^r \otimes k \rightarrow M \otimes k$  is an isomorphism. Applying the  $- \otimes k$  functor to

$$0 \rightarrow N \rightarrow A^r \rightarrow M \rightarrow 0,$$

we obtain the exact sequence

$$0 \rightarrow \text{Tor}_1^A(M, k) \rightarrow N \otimes k \rightarrow k^r \rightarrow M \otimes k \rightarrow 0$$

and this implies  $N \otimes k = 0$ . By Nakayama's lemma, we see that  $N = 0$  and thus  $M$  is free. The general case is similar. We construct

$$0 \rightarrow C \rightarrow F_n \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

where each  $F_i$  is projective, and then we see that  $\text{Tor}_0(C, k) = \cdots = \text{Tor}_{n+1}(M, k) = 0$ . This implies  $C \otimes k = 0$ , so  $C = 0$  and thus we have a projective resolution of length  $n$ .  $\square$

**Corollary 4.3.7.** *Let  $A$  be Noetherian and  $M$  be a finite  $A$ -module. Then*

1. *The projective dimension of  $M$  is the supremum of the projective dimensions of  $M_{\mathfrak{m}}$  for all maximal ideals  $\mathfrak{m}$  of  $A$ .*
2.  *$\text{proj. dim } M \leq n$  if and only if  $\text{Tor}_{n+1}(M, A/\mathfrak{m}) = 0$  for all maximal ideals  $\mathfrak{m}$  of  $A$ .*

*Proof.* 1. This follows from the fact that  $\text{Ext}_A^i(M, N)_{\mathfrak{m}} = \text{Ext}_{A_{\mathfrak{m}}}^i(M_{\mathfrak{m}}, N_{\mathfrak{m}})$  and the fact that for an  $A$ -module  $M$  we have  $M = 0$  if and only if  $M_{\mathfrak{m}} = 0$  for all  $\mathfrak{m}$  maximal.

2. This follows from **1**, the previous lemma, and the fact that  $\text{Tor}$  commutes with localization.  $\square$

**Lemma 4.3.8.** *Let  $A$  be noetherian. The following are equivalent:*

1.  *$\text{gl. dim } A \leq n$ .*
2.  *$\text{proj. dim } M \leq n$  for all finite  $A$ -modules  $M$ .*
3.  *$\text{inj. dim } M \leq n$  for all finite  $A$ -modules  $M$ .*
4.  *$\text{Ext}_A^{n+1}(M, N) = 0$  for all finite  $A$ -modules  $M, N$ .*
5.  *$\text{Tor}_{n+1}^A(M, N) = 0$  for all finite  $A$ -modules  $M, N$ .*

*Proof.* We know that **1** is equivalent to **2**, which implies **3**. We know that **3** implies **4** and **2** implies **5** are trivial. The previous corollary gives us **5** implies **2**, so we need to prove that **4** implies **2**. Let

$$0 \rightarrow C \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

such that  $F_i$  are free of finite type. Then we know that

$$\text{Ext}_A^1(C, N) = \cdots = \text{Ext}_A^{n+1}(M, N) = 0.$$

This implies that  $C$  is projective and thus  $\text{proj. dim } M \leq n$ .  $\square$

**Corollary 4.3.9.** *We have the local-global identity*

$$\text{gl. dim } A = \sup_{\mathfrak{m} \text{ maximal}} \text{gl. dim } A_{\mathfrak{m}}.$$

**Theorem 4.3.10.** *Let  $A$  be local Noetherian with residue field  $k$ . Then  $\text{gl. dim } A \leq n$  if and only if  $\text{Tor}_{n+1}^A(k, k) = 0$ . In particular,  $\text{gl. dim } A = \text{proj. dim}_A k$ .*

*Proof.* Suppose that  $\text{Tor}_{n+1}^A(k, k) = 0$ . Thus  $\text{proj. dim } k \leq n$ , and thus  $\text{Tor}_{n+1}(M, k) = 0$  for all  $M$  of finite type. This implies that  $\text{proj. dim } M \leq n$  for all finite  $A$ -modules  $M$ . Thus  $\text{gl. dim } A \leq n$ .  $\square$

**Lemma 4.3.11.** *Let  $(A, \mathfrak{m}, k)$  be local Noetherian and  $M$  a finite  $A$ -module. Let  $x \in A$  be an  $M$ -regular element. Then if  $\text{proj. dim } M \leq \infty$ , so is  $\text{proj. dim } M/xM$  and  $\text{proj. dim } M/xM = \text{proj. dim } M + 1$ .*

*Proof.* Let  $r = \text{proj. dim } M < \infty$ . Then we have the long exact sequence

$$0 \rightarrow M \rightarrow M \rightarrow M/xM \rightarrow 0,$$

so after applying the functor  $- \otimes k$ , we know that  $\text{Tor}_i(M, k) = 0$  if  $i > r$ . Thus if  $i > r + 1$ , then  $\text{Tor}_i(M/xM, k) = 0$ . For  $i = r + 1$ , we have an exact sequence

$$0 \rightarrow \text{Tor}_{r+1}(M/xM, k) \rightarrow \text{Tor}_r(M, k) \rightarrow \text{Tor}_r(M, k)$$

but  $x \in M$  annihilates  $k$  and thus  $\text{Tor}_{r+1}(M/xM, k) \simeq \text{Tor}_r(M, k) \neq 0$ , so  $\text{proj. dim}(M/xM) = r + 1$ .  $\square$

**Theorem 4.3.12.** *Let  $(A, \mathfrak{m}, k)$  be a regular local ring of dimension  $n$ . Then  $\text{gl. dim } A = n$ .*

*Proof.* Let  $x_1, \dots, x_n$  be a regular system of parameters. Then we know  $k = A/(x_1, \dots, x_n)$ . This implies that

$$\text{proj. dim } k = n + \text{proj. dim } A = n$$

and thus  $\text{gl. dim } A = n$ .  $\square$

**Corollary 4.3.13.** *Let  $k$  be a field. Then  $\text{gl. dim } k[x_1, \dots, x_n] = n$ .*

*Proof.* Let  $A = k[x_1, \dots, x_n]$ . Then  $A_{\mathfrak{m}} \cong k'[T_1, \dots, T_n]$  for some other field  $k'$  and is thus regular of dimension  $n$ . Thus  $\text{gl. dim } A_{\mathfrak{m}} = n$  for all maximal ideals  $\mathfrak{m}$ , and thus  $\text{gl. dim } A = n$ .  $\square$

## 4.4 Koszul Complex

Our goal is to prove the following result:

**Theorem 4.4.1** (Serre). *A local Noetherian ring is regular if and only if its global dimension is finite.*

To do this, we will introduce the *Koszul complex*. First, recall that if  $L_{\bullet}, M_{\bullet}$  are two complexes,  $(L_{\bullet} \otimes M_{\bullet})_n = \bigoplus_{i+j=n} L_i \otimes M_j$  and the differential satisfies the graded Leibniz rule

$$d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db.$$

For  $x \in A$ , define  $K(x) = \dots \rightarrow 0 \rightarrow A \xrightarrow{\times x} A \rightarrow 0 \rightarrow \dots$ . We can treat any  $A$ -module  $M$  as a complex by putting it in degree 0. Now for  $x_1, \dots, x_r \in A$  and  $M$  an  $A$ -module, define

$$K(x_1, \dots, x_r, M) = K(x_1) \otimes K(x_2) \otimes \dots \otimes K(x_r) \otimes M.$$

Then for an integer  $p$ , we have

$$K_p(x_1, \dots, x_r, M) = \bigoplus_{1 \leq i_1 \leq \dots \leq i_p \leq r} Me_{i_1, \dots, i_p} \cong M^{\binom{r}{p}}.$$

This satisfies

$$d(me_{i_1, \dots, i_p}) = \sum_{j=1}^p (-1)^{j-1} x_{i_j} me_{i_1, \dots, \hat{i}_j, \dots, i_p}.$$

Now consider the exact sequence

$$0 \rightarrow A \rightarrow K(x) \rightarrow A[-1] \rightarrow 0$$

Now if  $C$  is a complex, write  $C_\bullet(x) = C_\bullet \otimes K(x)$ , so we have a short exact sequence

$$0 \rightarrow C_\bullet \rightarrow C_\bullet(x) \rightarrow C_\bullet[-1] \rightarrow 0.$$

This gives us a long exact sequence of modules

$$\dots \rightarrow H_{p+1}(C) \rightarrow H_{p+1}(C(x)) \rightarrow H_p(C) \xrightarrow{\delta} H_p(C) \rightarrow \dots$$

By definition of  $K(x)$  it is easy to see that  $\delta_p = (-1)^p x$ . As a consequence, if  $H_p(C_\bullet)$  is trivial, then so is  $H_p(C_\bullet(x))$  for  $p > 1$ . In addition, we have

$$0 \rightarrow H_1(C(x)) \rightarrow H_0(C) \xrightarrow{x} H_0(C) \rightarrow H_0(C(x)) \rightarrow 0.$$

Thus if  $x$  is  $H_0(C)$ -regular, then  $H_1(C(x)) = 0$  and  $H_0(C(x)) = H_0(C)/xH_0(C)$ . If we iterate this, we have now proven the following result:

**Theorem 4.4.2.** *Let  $M$  be an  $A$ -module and  $x_1, \dots, x_r$  an  $M$ -regular sequence. Then*

$$H_p(\underline{x}, M) := H_p(K(x_1, \dots, x_r, M)) = 0$$

for all  $p > 0$  and  $H_0(\underline{x}, M) = M/(x_1, \dots, x_r)M$ .

**Corollary 4.4.3.** *If  $x_1, \dots, x_r$  is an  $A$ -regular sequence, then  $K(x_1, \dots, x_r, A)$  is a finite free resolution of  $A/(x_1, \dots, x_r)A$ .*

Now we will discuss minimal resolutions. Suppose  $(A, \mathfrak{m}, k)$  is a local ring. Then recall from the homework that  $u: L \rightarrow M$  is called *minimal* when  $u \otimes 1_k$  is an isomorphism. When  $M$  is finite, this is equivalent to  $u$  surjective and  $\ker(u) \subset \mathfrak{m}L$ . We now say that a **resolution**  $L_\bullet \rightarrow M$  is *minimal* if  $L_i \otimes k \simeq \ker(di) \otimes k$ , where  $L_i \xrightarrow{d_i} L_{i-1}$ . In particular, writing  $\bar{L}_i = L_i \otimes k$ , then the complex

$$\bar{L}_i \rightarrow \bar{L}_{i-1} \rightarrow \dots$$

satisfies  $d = 0$ , so  $\text{Tor}_i(M, k) = H_i(\bar{L}_\bullet) = \bar{L}_i$ . Therefore if  $L_\bullet$  is a minimal resolution of  $M$ , then  $\text{rk}(L_i) = \dim_k \text{Tor}_i(M, k)$ .

**Proposition 4.4.4.** *Now assume  $A$  is local Noetherian and  $M$  is a finite  $A$ -module. Then there exists a minimal free resolution of  $M$  and any two such resolutions are isomorphic.*

*Proof.* To prove existence, begin with  $\overline{M}$ . Choose a basis  $e_1, \dots, e_{n_0} \in M$  of  $M \otimes k = \overline{M}$ . This gives us a morphism  $0 \rightarrow K_0 \rightarrow A^{n_0} \rightarrow M \rightarrow 0$  that gives an isomorphism  $k^{n_0} \simeq \overline{M}$ . Now  $K_0$  is a finite  $A$ -module, so we can construct a minimal  $A^{n_1} \rightarrow K_0$  with kernel  $K_1$ . Iterating this procedure, we have a minimal resolution.

To prove uniqueness, if  $L_\bullet, L'_\bullet$  are minimal free resolutions, we will construct a map

$$\begin{array}{ccc} L_\bullet & \longrightarrow & M \\ \downarrow & & \downarrow \text{id} \\ L'_\bullet & \longrightarrow & M \end{array}$$

that makes this diagram commute. In degree zero, we have  $f_0: L_0 \rightarrow L'_0$  where  $\overline{f_0}$  is an isomorphism, so  $L_0 \cong L'_0 = A^{n_0}$ . Thus  $\det f_0 \in A^\times$ , so  $f_0$  is an isomorphism. The rest follows by iterating this procedure.  $\square$

**Lemma 4.4.5.** *Let  $F_\bullet \rightarrow M$  be a complex such that*

1.  $F_p$  is a finite free  $A$ -module for all  $p$ ;
2.  $\overline{F_0} \rightarrow \overline{M}$  is surjective.
3.  $d(F_i) \subset \mathfrak{m}F_{i-1}$  for all  $i > 0$  and induces a surjection  $\overline{F_i} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2 \otimes \overline{F_{i-1}} = \mathfrak{m} \otimes \overline{F_{i-1}}$ .

Then  $\text{rk}_A(F_i) \leq \dim_k \text{Tor}_i(M, k)$ .

*Proof.* Consider a minimal free resolution  $L_\bullet \rightarrow M$  of  $M$ . Because free implies projective, we can find a map

$$\begin{array}{ccc} F_\bullet & \longrightarrow & M \\ \downarrow & & \downarrow \text{id} \\ L_\bullet & \longrightarrow & M. \end{array}$$

Then the map  $\overline{F_0} \rightarrow \overline{L_0}$  must be injective because  $\overline{F_0} \twoheadrightarrow \overline{M}$  and  $\overline{L_0} \simeq \overline{M}$ . Therefore, we can split  $L_0 \cong F_0 \oplus F'_0$ . In particular, this means that  $\text{rk}_A(F_0) \leq \text{rk}_A(L_0)$ .

Now we have a diagram

$$\begin{array}{ccc} F_1 & \longrightarrow & \mathfrak{m}F_0 \\ \downarrow & & \downarrow \\ L_1 & \longrightarrow & \mathfrak{m}L_0 \end{array}$$

where we have  $\mathfrak{m}L_0 = \mathfrak{m}F_0 \oplus \mathfrak{m}F'_0$ , so downstairs we now have

$$\begin{array}{ccc} \overline{F_1} & \twoheadrightarrow & \overline{\mathfrak{m}} \otimes F_0 \\ \downarrow & & \downarrow \\ \overline{L_1} & \longrightarrow & \overline{\mathfrak{m}} \otimes L_0 \end{array}$$

and therefore  $\overline{F_1} \rightarrow \overline{L_1}$  is surjective. This gives us a splitting  $L_1 = F_1 \oplus F'_1$  and thus iterating this, we obtain the desired result.  $\square$

**Theorem 4.4.6.** *Let  $(A, \mathfrak{m}, k)$  be a local Noetherian ring and let  $s = \dim \mathfrak{m}/\mathfrak{m}^2$ . Then*

$$\mathrm{rk} \mathrm{Tor}_i(k, k) \geq \binom{s}{i}.$$

*In particular,  $\mathrm{Tor}_i(k, k) \neq 0$ .*

*Proof.* Write  $F_i := K_i(x_1, \dots, x_s; A)$  where  $x_1, \dots, x_s \in \mathfrak{m}$  is a basis for  $\mathfrak{m}/\mathfrak{m}^2$ . Thus we have  $F_0 = A \rightarrow H_0(\underline{x}, A) = A/(x_1, \dots, x_s) = k$ . Thus the  $F_\bullet$  satisfies the first two hypotheses of the previous lemma, so we check the third one. Here, we have

$$d\left(\sum \lambda_{i_1, \dots, i_p} e_{i_1, \dots, i_p}\right) = \sum_{j=1}^s (-1)^j x_j \left(\sum \lambda_{i_1, \dots, \hat{i}_j, \dots, i_p} e_{i_1, \dots, \hat{i}_j, \dots, i_p}\right).$$

Reducing modulo  $\mathfrak{m}$ , because  $\bar{x}_1, \dots, \bar{x}_s$  is a basis of  $\mathfrak{m} = \mathfrak{m}/\mathfrak{m}/\mathfrak{m}^2$ , we see that this differential is injective. Now we can apply the previous lemma to obtain the desired result.  $\square$

We are now ready to prove the theorem of Serre.

*Proof of Theorem 4.4.1.* We proved one direction last time. Now set  $s = \dim \mathfrak{m}/\mathfrak{m}^2$ . We know that  $\mathrm{Tor}_s(k, k) \neq 0$  and thus  $s \leq \mathrm{gl.dim} A = \mathrm{proj.dim}_A(k)$ . On the other hand, by the Auslander-Buchsbaum formula, we have

$$\mathrm{proj.dim}_A(k) + \mathrm{depth} k = \mathrm{depth} A$$

and because  $\mathrm{depth} k = 0$ , we see that  $\dim A \geq \mathrm{depth} A \geq s$ . However, we know that  $s \geq \dim A$ , and thus  $s = \dim A = \mathrm{depth} A$ , so  $A$  is regular.  $\square$

**Corollary 4.4.7.** *Let  $A$  be a regular local ring and choose  $\mathfrak{p} \in \mathrm{Spec} A$ . Then  $A_{\mathfrak{p}}$  is regular.*

*Proof.* This requires showing that  $A_{\mathfrak{p}}$  has finite global dimension. If  $M$  is an  $A_{\mathfrak{p}}$ -module, then  $M$  is an  $A$ -module. Then if  $F_\bullet \rightarrow M$  is a projective resolution of  $M$  of length at most  $\dim A$ , exactness of localization tells us that  $(F_\bullet)_{A_{\mathfrak{p}}} \rightarrow M$  is an  $A_{\mathfrak{p}}$ -projective resolution of  $M$ . Thus  $\mathrm{proj.dim}_{A_{\mathfrak{p}}}(M) < \dim A$ , so  $A_{\mathfrak{p}}$  has finite global dimension.  $\square$

## 4.5 Unique Factorization

Recall that a *unique factorization domain*, or a UFD, is a domain  $A$  such that every element has a unique factorization into irreducibles (up to multiplication of irreducibles by units). Here,  $\pi \neq 0$  is irreducible if  $\pi = ab$  implies  $a$  or  $b$  is a unit. For example, when  $A = \mathbb{Z}$ , the irreducibles are precisely the prime numbers.

**Exercise 4.5.1.** If  $A$  is Noetherian, then any element can be written as a product of irreducibles.

We say that two irreducibles  $\pi, \pi'$  are *equivalent* if  $(\pi) = (\pi')$ . Then for each equivalence class, we can choose one representative, and the unique factorization means that if

$$u' \pi_1^{n'_1} \cdots \pi_r^{n'_r} = u \pi_1^{n_1} \cdots \pi_r^{n_r},$$

then  $u = u'$  and  $n_i = n'_i$ .

**Lemma 4.5.2.** *A Noetherian domain is a UFD if and only if  $(\pi)$  is a prime ideal for all irreducibles  $\pi$ .*

*Proof.* Suppose  $ab \in (\pi)$ . Then if  $a = u \prod_i \pi_i^{n_i}, b = v \prod_i \pi_i^{m_i}$  where  $\pi = \pi_1$ , we know that  $n_1 + m_1 \geq 1$ , so at least one of them is at least 1, and thus either  $a \in (\pi)$  or  $b \in (\pi)$ .

In the other direction, we will prove that Gauss's lemma is satisfied. This says that if  $\pi \mid ab$  and  $\pi \nmid a$ , then  $\pi \mid b$ . This implies that our domain is a UFD, and is equivalent to  $(\pi)$  being a prime ideal for all units.  $\square$

**Lemma 4.5.3.** *Let  $A$  be a Noetherian domain. Then  $A$  is a UFD if and only if every prime ideal of height 1 is principal.*

*Proof.* Suppose  $A$  is a unique factorization domain. Then let  $\mathfrak{p}$  be a prime ideal of height 1. Because  $\mathfrak{p} \neq 0$ , there exists  $a = u \pi_1^{n_1} \cdots \pi_r^{n_r}$ . Thus there exists an irreducible  $\pi$  such that  $\pi \in \mathfrak{p}$ , and thus because  $0 \subsetneq (\pi) \subset \mathfrak{p}$  and  $\text{ht}(\mathfrak{p}) = 1$ , we see that  $(\pi) = \mathfrak{p}$ .

In the other direction, we will show that for any irreducible  $\pi \in A$ , the ideal  $(\pi)$  is prime. Let  $\mathfrak{p}$  be a minimal prime ideal containing  $(\pi)$ . We know that  $\text{ht}(\mathfrak{p}) \leq 1$ , and thus  $\text{ht}(\mathfrak{p}) = 1$ . But then we know that  $\mathfrak{p} = (\alpha)$ , so  $\pi = \alpha\beta$  for some  $\beta \in A$ . But because  $\pi$  is irreducible, this means that  $\alpha$  or  $\beta$  is a unit, and we know  $\alpha$  is not a unit, so  $\beta$  is a unit and thus  $(\pi) = (\alpha) = \mathfrak{p}$ .  $\square$

**Lemma 4.5.4.** *Let  $A$  be Noetherian and  $x \neq 0$  such that  $(x)$  is a prime ideal. Then  $A_x = A[x^{-1}]$  is a UFD if and only if  $A$  is.*

*Proof.* Suppose  $A$  is a UFD. Clearly  $A_x$  is a UFD. In the other direction, Then if

$$a = ux^n \pi_1^{n_1} \cdots \pi_r^{n_r} = u_x \pi_1^{n_1} \cdots \pi_r^{n_r} = u' x^{n'} \pi_1^{n'_1} \cdots \pi_r^{n'_r} = u'_x \pi_1^{n'_1} \cdots \pi_r^{n'_r},$$

we must have  $u_x = ux^n, u'_x = u'x^{n'}$ , then we obtain  $u = u', n = n'$ .  $\square$

**Theorem 4.5.5.** *Let  $A$  be a local Noetherian regular ring. Then  $A$  is a UFD.*

*Proof.* We induct on  $\dim A$ . If  $\dim A = 0$ , then  $A$  is a field. If  $\dim A = 1$ , then  $A$  is a DVR and the result is clear.

For the inductive step, choose  $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ . Then  $(x)$  is a prime ideal, and we want to show that  $A_x$  is a UFD. Choose a prime ideal  $\mathfrak{p}'$  of height 1 in  $A_x$ . We will show that  $\mathfrak{p}'$  is principal. Write  $\mathfrak{p} = \mathfrak{p}' \cap A$ , so  $\mathfrak{p}' = \mathfrak{p}A_x$ . Then because  $A$  is regular, it has finite global dimension, and thus  $\mathfrak{p}$  has a finite projective resolution

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow \mathfrak{p} \rightarrow 0,$$

where  $F_i = A^{m_i}$ . Now for any prime ideal  $\underline{\mathfrak{p}} \subset A_x$ , we have  $(A_x)_{\underline{\mathfrak{p}}} = A_{\underline{\mathfrak{p}} \cap A}$ . This implies that if  $x \notin \underline{\mathfrak{p}} \cap A$ , then  $\mathfrak{m} \neq \underline{\mathfrak{p}} \cap A$  and thus  $\text{ht}(\underline{\mathfrak{p}} \cap A) < \dim A$ . Then we know that  $A_{\underline{\mathfrak{p}} \cap A}$  is regular, so we can apply the inductive hypothesis and it is thus a UFD. Then we know that

$$\text{proj. dim}_{A_x}(\mathfrak{p}') = \sup_{\underline{\mathfrak{p}} \subset A_x} (\text{proj. dim}_{(A_x)_{\underline{\mathfrak{p}}}} \mathfrak{p}'_{\underline{\mathfrak{p}}})$$

and so because  $\mathfrak{p}'_{\underline{\mathfrak{p}}}$  is principal, it is free of rank 1 and thus has projective dimension 0. Thus  $\mathfrak{p}'$  is a projective  $A_x$ -module, and so we have an exact sequence

$$0 \rightarrow F'_n \rightarrow \cdots \rightarrow F'_1 \rightarrow F'_0 \rightarrow \mathfrak{p}' \rightarrow 0$$

where  $F'_i = A_x^{m_i}$ . Then we decompose this into short exact sequences

$$\begin{aligned} 0 \rightarrow K'_0 \rightarrow F'_0 \rightarrow \mathfrak{p}' \rightarrow 0 \\ 0 \rightarrow K'_1 \rightarrow F'_1 \rightarrow K'_0 \rightarrow 0 \\ \vdots \\ 0 \rightarrow F'_n \rightarrow F'_{n-1} \rightarrow K'_{n-2} \rightarrow 0. \end{aligned}$$

Because  $\mathfrak{p}'$  is projective, the exact sequences split and all  $K'_i$  are projective. Therefore

$$\bigoplus_{i \text{ even}} F'_i = \bigoplus_{i \text{ odd}} \oplus \mathfrak{p}'.$$

Writing the first direct sum as  $F$  and the second as  $G$ , we have  $F = G \oplus \mathfrak{p}'$ . Thus  $F, G$  are free  $A_x$ -modules, and  $\text{rank } F = \text{rank } G + 1$ . Write  $\text{rank } G = r$ . Now we have  $\bigwedge^{r+1} F = \bigwedge^r G \otimes \mathfrak{p}'$  because  $\bigwedge^i \mathfrak{p}' = 0$  if  $i > 1$  (after localization, we obtain a free module of rank 1). In particular, we have  $A_x \cong \bigwedge^{r+1} F = \mathfrak{p}'$  and thus  $\mathfrak{p}'$  is free of rank 1 as an  $A_x$ -module, so it is a principal ideal.  $\square$

**Definition 4.5.6.** Let  $A$  be a Noetherian ring. We say that  $A$  is *regular* if for all  $\mathfrak{p} \in \text{Spec } A$ ,  $A_{\mathfrak{p}}$  is a local Noetherian regular ring.

## 4.6 Criterion for Normality

Let  $A$  be a ring,  $k \geq 0$  an integer, and consider the conditions

$(R_k)$ : For all  $\mathfrak{p} \in \text{Spec } A$  such that  $\text{ht}(\mathfrak{p}) \leq k$ , then  $A_{\mathfrak{p}}$  is regular.

$(S_k)$ : For all  $\mathfrak{p} \in \text{Spec } A$ ,  $\text{depth}(A_{\mathfrak{p}}) \geq \inf(k, \text{ht}(\mathfrak{p}))$ .

**Theorem 4.6.1.** *Let  $A$  be a Noetherian ring. Then  $A$  is normal if and only if  $(S_2)$  and  $(R_1)$  hold.*

Before we prove this, we will prove several lemmas.

**Lemma 4.6.2.** *The ideal  $(0)$  is unmixed if and only if  $(S_1)$  holds. In other words, for all  $\mathfrak{p} \in \text{Spec } A$ , we have  $\text{depth}(A_{\mathfrak{p}}) \geq \inf(1, \text{ht}(\mathfrak{p}))$ .*

*Proof.* Assume that  $(0)$  is unmixed and let  $\mathfrak{p} \in \text{Spec } A$  with  $\text{ht}(\mathfrak{p}) \geq 1$ . Then  $(0)$  is also unmixed in  $A_{\mathfrak{p}}$ , so  $\text{Ass}(A_{\mathfrak{p}}) \cong \{\mathfrak{q} \subsetneq \mathfrak{p} \mid \mathfrak{q} \text{ minimal}\}$ . Therefore there exists  $x \in \mathfrak{p}A_{\mathfrak{p}} \setminus \bigcup_{\mathfrak{q}} \mathfrak{q}A_{\mathfrak{p}}$ . Thus  $x$  is not a zero divisor of  $A_{\mathfrak{p}}$ , so it must be  $A_{\mathfrak{p}}$ -regular, and thus  $\text{depth}(A_{\mathfrak{p}}) \geq 1$ .

Conversely, suppose  $(0)$  is not unmixed. Let  $\mathfrak{p} \in \text{Ass}(A)$  with  $\text{ht}(\mathfrak{p}) \geq 1$ . But then  $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}(A_{\mathfrak{p}})$ , and thus  $\mathfrak{p}A_{\mathfrak{p}}$  is contained in the set of zero-divisors. Thus  $\text{depth}(A_{\mathfrak{p}}) = 0$ .  $\square$

**Corollary 4.6.3.** *The property  $(S_k)$  holds if and only if for all  $i < k$ , any  $A$ -regular sequence  $(a_1, \dots, a_i)$  is unmixed. For  $i = 0$ , this is saying that  $(0)$  is unmixed.*

*Proof.* Use the previous lemma and an induction argument. Also use the fact that  $\text{depth}(A_{\mathfrak{p}}/aA_{\mathfrak{p}}) = \text{depth}(A_{\mathfrak{p}}) - 1$  and the same for dimension.  $\square$

Recall that a domain  $A$  is normal if it is integrally closed. Also,  $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ , so  $A$  is a normal domain if and only if all localizations at maximal ideals are normal domains.

**Definition 4.6.4.** A ring  $A$  is called *normal* if and only if for all  $\mathfrak{p} \in \text{Spec } A$ ,  $A_{\mathfrak{p}}$  is a normal domain.

It is not hard to see that it suffices to check this condition for maximal ideals.



**Corollary 4.6.5.** *Assume that  $A$  is reduced and that  $(S_2)$  holds. Then*

$$A = \bigcap_{\substack{\mathfrak{p} \in \text{Spec } A \\ \text{ht}(\mathfrak{p})=1}} A_{\mathfrak{p}}.$$

Here, the intersection is well-defined because the intersection of all minimal prime ideals is  $\{0\}$ .

*Proof.* Assume that  $(S_2)$  holds. Then consider  $\frac{\alpha}{\beta} \in \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$ . We will consider the ideal  $I = \{x \in A \mid x\frac{\alpha}{\beta} \in A\} = \text{Ann}(\bar{\alpha} \in A/\beta A)$ . Therefore we have an embedding  $A/I \hookrightarrow A/\beta A$  given by  $x \mapsto x\bar{\alpha}$ . Assume  $I \neq A$ , so  $A/I \neq 0$ . Then if  $Q \in \text{Ass}(A/I)$ ,  $Q$  must be of height 1. Also, because  $I \subset Q$ , then  $I_Q \subset QA_Q$ . However, we know that  $\frac{\alpha}{\beta} \in A_Q$ . But then this implies that  $I_Q = A_Q$ , which is a contradiction.  $\square$

**Lemma 4.6.6.** *Assumes that  $A$  satisfies  $(S_1)$  and  $(R_0)$ . Then  $A$  is reduced.*

*Proof.* Choose  $x \in A$  be nilpotent, so  $x^n = 0$  for some  $n \geq 1$ . We will show that  $x = 0$ . If  $\mathfrak{p} \in \text{Spec } A$  with  $\text{ht}(\mathfrak{p}) = 0$ , then by  $(R_0)$  we know that  $A_{\mathfrak{p}}$  is a field. This implies that  $x_{\mathfrak{p}} = 0$ , so there exists  $s \notin \mathfrak{p}$  such that  $sx = 0$ . This implies that  $\text{Ann}(x) \not\subseteq \mathfrak{p}$  for any minimal prime  $\mathfrak{p}$ .

Now assume that  $x \neq 0$  and let  $\mathfrak{q}$  be a maximal element of the set of ideals  $\{\text{Ann}(ax) \mid a \in A\}$ . Then we know that  $\mathfrak{q}$  is a non-minimal prime ideal, so by  $(S_1)$ , we have  $\text{depth}(A_{\mathfrak{q}}) \geq 1$ . Also,  $\mathfrak{q} = \text{Ann}(y)$  for some  $y = ax$ . Therefore  $A/\mathfrak{q} \hookrightarrow A$ . But this implies that  $\text{Ext}^{\bullet}(A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}, A_{\mathfrak{q}}) \neq 0$  and thus  $\text{depth}(A_{\mathfrak{q}}) = 0$ , which is a contradiction.  $\square$

*Proof of Theorem 4.6.1.* First assume that  $A$  is normal and choose  $\mathfrak{p} \in \text{Spec } A$ . Then if  $\text{ht}(\mathfrak{p}) = 0$ , we know  $A_{\mathfrak{p}}$  is a field. If  $\text{ht}(\mathfrak{p}) = 1$ , then  $A_{\mathfrak{p}}$  has dimension 1 and is integrally closed, so it must be a DVR and is thus regular. Thus  $(R_1)$  is satisfied. Now if  $\text{ht}(\mathfrak{p}) = 2$ , then  $A_{\mathfrak{p}}$  is normal of dimension 2 and is thus Cohen-Macaulay. Therefore,  $\text{depth}(A_{\mathfrak{p}}) = 2$ . If  $\text{ht}(\mathfrak{p}) > 2$ , we can choose  $\mathfrak{q} \subset \mathfrak{p}$  of height 2 and then  $\text{depth}(A_{\mathfrak{p}}) \geq \text{depth}(A_{\mathfrak{q}}) = 2$ , so  $(S_2)$  is satisfied.

Now assume that  $(R_1)$  and  $(S_2)$  hold. By the previous lemma, we know that  $A$  is reduced. This embeds  $A \hookrightarrow \text{Tot}(A) = \prod_i K_i$ , where  $K_i = \text{Frac}(A/\mathfrak{p}_i)$  with  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  the minimal primes of  $A$ . It suffices to prove that  $A$  is integrally closed in  $\text{Tot}(A)$ . This is because if  $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$ , then  $\varepsilon_i^2 = \varepsilon_i$  and thus  $\varepsilon_i = A$  and thus  $A = \prod_{i=1}^r A/\mathfrak{p}_i$  and then each  $A/\mathfrak{p}_i$  is a normal domain.

Let  $x \in \text{Tot}(A)$  be integral over  $A$ . Then  $x_{\mathfrak{p}} \in \text{Tot}(A)_{\mathfrak{p}} = \text{Tot}(A)$ . Then we know  $x_{\mathfrak{p}}$  is integral over  $A_{\mathfrak{p}}$ , so if  $\text{ht}(\mathfrak{p}) = 1$ , then  $A_{\mathfrak{p}}$  is regular, so it is a DVR and is thus integrally closed. Therefore  $x_{\mathfrak{p}} \in A_{\mathfrak{p}}$  for all  $\mathfrak{p}$  of height 1. This implies that

$$x \in \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}} = A. \quad \square$$