TCC (SPRING 2021): p-ADIC MODULAR FORMS

PROBLEM SHEET 3

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This problem sheet is due at 11:59 PM on **Friday 9th April 2021**. Please submit your work as a single PDF file (either typeset in LaTeX or a scan of legible handwriting) by email.

Problem 0. (**NOT FOR SUBMISSION**) Suppose R is a ring containing $\frac{1}{6}$. Recall that any pair $(E/R, \omega)$ can be written in terms of the Weierstrass equation:

$$E : y^2 = x^3 + a_4 x + a_6, \quad \omega = \frac{dx}{y}.$$

(a) Prove that the rules

$$E_4(E/R, \omega) := -12 a_4,$$
$$E_6(E/R, \omega) := 216 a_6$$

define modular forms of weights 4 and 6 respectively.

(b) Show that E_4 and E_6 have q-expansions

$$E_4(\mathrm{Tate}(q), \omega_{\mathrm{can}}) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$
$$E_6(\mathrm{Tate}(q), \omega_{\mathrm{can}}) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

In particular, they define holomorphic modular forms over $\mathbf{Z}$ and agree with the classical definitions of Eisenstein series.

Problem 1. Let R be an $\mathbf{F}_p$-algebra. Consider an elliptic curve E/R given by the Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

with a differential form

$$\omega = \frac{dx}{2y + a_1 x + a_3} \in H^0(E/R, \Omega^1).$$

(a) Suppose $p = 2$. Show that a_1 is the Hasse invariant of (E, ω).

(b) Suppose $p = 3$. Show that[1] $a_1^2 + a_2$ is the Hasse invariant of (E, ω).

Problem 2. Recall that U_p and V_p act on the space M of *convergent* p-adic modular forms as follows: if $f = \sum a_n q^n \in M$, then

$$U_p f = \sum a_{np} q^n,$$
$$V_p f = \sum a_n q^{np}.$$

Last updated: March 29, 2021. Please send questions and comments to Pak-Hin.Lee@warwick.ac.uk.

[1] This corrects a typo in Exercise 3.0.6 of Calegari's notes.

(For concreteness, you may think of M with growth condition $r = 1$, level $N = 1$, weight $k \in \mathbf{Z}$ and coefficients $\mathbf{C}_p$ – although the precise choices don't matter for this problem – and assume that M is a p-adic Banach space.)

Let $f \in M$ and set $g = (1 - V_p U_p) f$.

(a) Show that $U_p g = 0$.

(b) For any $\lambda \in \mathbf{C}_p$ with $|\lambda| < 1$, prove that

$$f_\lambda := \sum_{i=0}^{\infty} (\lambda V_p)^i g \in M$$

satisfies $U_p f_\lambda = \lambda f_\lambda$.

In particular, this shows that U_p has a *continuous* spectrum on M.