p-adic modular forms

TCC (Spring 2021), Lecture 4

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Administrative issues

Slides:

- Lectures 1-3 available on webpage

Problem sheets:

- 3 sets for assessment
  1. 22nd February (Monday of Week 6): posted!
  2. 5th March (Friday of Week 7): tentative
  3. 19th March (Friday of Week 9): tentative
- available two weeks before deadlines

Plans

Today:

- Recap
- Properties of Serre's p-adic modular forms
- Application 1: p-adic zeta functions
- Hecke operators on p-adic modular forms

Next week:

- Hecke operators, continued
- Application 2: constant terms; congruences
- Geometric modular forms

Recap: p-adic modular forms a là Serre

- $f \in \mathbf{Q}_p[[q]]$ is a p-adic modular form if it is the limit of a sequence of classical modular forms $f_i \in M_{k_i,\mathbf{Q}}$.
- f has a well-defined notion of weight: k_i converges to $k \in \mathfrak{X} = \mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$ (group of characters of $\mathbf{Z}_p^\times$).

Example

- For any odd p, $E_{p-1}^{-1} \in M_{1-p}^\dagger$.
- Problem Sheet 1: For $p = 5$, $M_0^\dagger = \mathbf{Q}_5\langle j^{-1}\rangle$.
- Today: p-adic Eisenstein series

Properties of p-adic modular forms

Our previous results on congruences and weights carry over to p-adic modular forms, by a basic limiting argument:

Theorem (théorème 1', P.203)

Suppose $f \in M_k^\dagger$ and $f' \in M_{k'}^\dagger$ satisfy $f \neq 0$ and

$$v_p(f - f') \geq v_p(f) + m$$

for some $m \geq 1$. Then k and k' have the same image in $\mathfrak{X}_m$.

Properties of p-adic modular forms

Theorem (P.202)

Suppose $f \neq 0 \in \mathbf{Q}_p[[q]]$, and there is a sequence of p-adic modular forms $f_i \in M^{\dagger}_{k_i}$ with $f_i \to f$. Then:

- $k = \lim k_i \in \mathfrak{X}$ *exists;*
- f *is a p-adic modular form of weight* k.

Thus $M^{\dagger}$ is a p-adic Banach space (as a closed subspace of $\mathbf{Z}_p[[q]] \otimes \mathbf{Q}_p$), equipped with a continuous map $M^{\dagger} \to \mathfrak{X}$ (weight map).

Remark

This is useful for f which is more easily seen as a limit of p-adic (rather than classical) modular forms.

Properties of p-adic modular forms

Corollary (corollaire 1, P.203)

Let $f = \sum a_n q^n \in M_k^\dagger$ *with* $k \neq 0 \in \mathfrak{X}_{m+1}$ *for some* m *(i.e.* $p^m(p-1) \nmid k$*). Then*

$$v_p(a_0) + m \geq \inf_{n\geq 1} v_p(a_n).$$

Proof.

If $a_0 = 0$ there is nothing to prove. Otherwise, set $f' = a_0 \in M_0^\dagger$, so that

$$v_p(f - f') = \inf_{n\geq 1} v_p(a_n).$$

Since $k \neq 0 \in \mathfrak{X}_{m+1}$, the contrapositive of the theorem shows

$$v_p(f - f') < v_p(f') + (m+1) \leq v_p(a_0) + m + 1.$$

□

Properties of p-adic modular forms

As an immediate (but non-trivial?) consequence, this shows that non-zero constants cannot be p-adic modular form of weights $k \neq 0$.

Example

Suppose $f = a_0$ is a p-adic modular form of weight $k \neq 0$. Then $k \neq 0 \in \mathfrak{X}_{m+1}$ for sufficiently large m, so the corollary gives

$$v_p(a_0) + m \geq \inf_{n \geq 1} v_p(a_n) = \infty.$$

This forces $a_0 = 0$.

This will be used in the proof of the next theorem.

Properties of p-adic modular forms

The corollary gives a condition on the p-divisibility of a_0 in terms of a_n for $n \geq 1$. More concretely:

Example

In the setting of the corollary:

- If a_n are p-integral for all $n \geq 1$, then so is $p^m a_0$.
- When $(p-1) \nmid k$, m can be taken to be 0.

Even for classical modular forms, this is a new result!

Idea

Slogan: For p-adic modular forms, the non-constant Fourier coefficients govern the constant term.

Properties of p-adic modular forms

This is already remarkable, but the following is even more drastic:

Theorem (corollaire 2, P.204)

Let $f_i = \sum_{n=0}^{\infty} a_n^{(i)} q^n \in M_{k_i}^{\dagger}$ be a sequence of p-adic modular forms of weights k_i such that

- $\lim_{i \to \infty} a_n^{(i)} = a_n \in \mathbf{Q}_p$ *uniformly for all* $n \geq 1$;
- $\lim_{i \to \infty} k_i = k \neq 0 \in \mathfrak{X}$.

Then:

- $a_0 = \lim_{i \to \infty} a_0^{(i)} \in \mathbf{Q}_p$ *exists;*
- $f = \sum_{n=0}^{\infty} a_n q^n$ *is a p-adic modular form of weight* k.

Properties of p-adic modular forms

Remark

- Thus to prove that a sequence of p-adic modular forms converges, it is enough to check that all the non-constant terms and the weights converge.
- At first glance this might not seem very useful...
- We will apply this to the Eisenstein series, for which the conditions on $a_n^{(i)}$ and $k^{(i)}$ are easy to check – not so much for the constant terms (zeta values)!

Proof of theorem

Idea

How to find a_0?

- Use previous corollary to bound $\{a_0^{(i)}\} \subset \mathbf{Q}_p$.
- Use a compactness argument to find the limit.

Proof of theorem:

- Since $k \neq 0$, there exists $m \geq 1$ such that $k \neq 0 \in \mathfrak{X}_{m+1}$.
- By $\lim k_i = k$, the same holds for all $i \gg 0$:

$$k_i \neq 0 \in \mathfrak{X}_{m+1}.$$

- By uniform convergence, there exists $t \in \mathbf{Z}$ such that

$$v_p(a_n^{(i)}) \geq t$$

for all $n \geq 1$ and all $i \gg 0$.

Proof of theorem

- Applying the corollary to $f_i = \sum a_n^{(i)} q^n \in M_{k_i}^\dagger$ with $k_i \neq 0 \in \mathfrak{X}_{m+1}$, we get

  $$v_p(a_0^{(i)}) + m \geq \inf_{n \geq 1} v_p(a_n^{(i)})$$

  and hence

  $$v_p(a_0^{(i)}) \geq t - m$$

  for all $i \gg 0$.
- Thus $\{a_0^{(i)}\}_i$ lies in the compact subset $p^{t-m}\mathbf{Z}_p \subset \mathbf{Q}_p$.
- To prove $a_0^{(i)}$ converges, it suffices to show every convergent subsequence has the same limit.

Proof of theorem

- Suppose some subsequence $a_0^{(i_j)}$ converges to $a_0 \in \mathbf{Q}_p$. Then

$$f := \lim_{j\to\infty} f_{i_j} = a_0 + \sum_{n=1}^{\infty} a_n q^n$$

  is a p-adic modular form of weight k.
- If $a_0^{(i'_j)}$ is another subsequence convergent to $a'_0 \in \mathbf{Q}_p$, then

$$f' := \lim_{j\to\infty} f_{i'_j} = a'_0 + \sum_{n=1}^{\infty} a_n q^n$$

  is also a p-adic modular form of weight k.
- Now their difference

$$f - f' = a_0 - a'_0$$

  is a p-adic modular form of weight k and weight 0.
- Since $k \neq 0$, we conclude $a_0 - a'_0 = 0$ as desired.

Application: p-adic zeta function

For k even, recall the Eisenstein series

$$G_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$
$$= \frac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Idea

For suitable sequences $k_i \to k \in \mathfrak{X}$, the non-constant coefficients converge for all $n \geq 1 \implies$ the constant terms converge to the p-adic zeta function!

p-adic weights

Remark (p-adic weights)

- Recall that a p-adic weight $k \in \mathfrak{X}$ is a continuous character $k : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$.
- For $d \in \mathbf{Z}_p^\times$,
$$d^k := k(d)$$
makes sense; this notation extends d^k for integral weights $k \in \mathbf{Z}$.
- For each fixed $d \in \mathbf{Z}_p^\times$, this gives a *continuous* map
$$\begin{aligned} \mathfrak{X} &\to \mathbf{Q}_p^\times \\ k &\mapsto d^k \end{aligned}$$
with respect to the p-adic topology (check this!).

p-adic divisor sums

Definition

For $k \in \mathfrak{X}$ and $n \in \mathbf{Z}_{\geq 1}$, define

$$\sigma^*_{k-1}(n) := \sum_{\substack{d \mid n \\ (d,p)=1}} d^{k-1}.$$

Remark

The condition $(d, p) = 1$ ensures $d \in \mathbf{Z}_p^\times$, so that d^{k-1} is well-defined for general $k \in \mathfrak{X}$.

- Suppose $k_i \in \mathbf{Z}$ converges to $k \in \mathfrak{X}$.
- Replacing k_i by $k_i + p^{m_i}(p-1)$ (where $m_i \gg 0$), we may assume *additionally* that $k_i \to \infty$ in $\mathbf{R}$.

p-adic divisor sums

- For each fixed $n \in \mathbf{Z}_{\geq 1}$,

$$\sigma_{k_i-1}(n) = \sum_{d|n} d^{k_i-1} = \sum_{\substack{d|n \\ (d,p)=1}} d^{k_i-1} + \sum_{\substack{d|n \\ p|d}} d^{k_i-1}.$$

- As $i \to \infty$:
  - For d prime to p, $d^{k_i-1} \to d^{k-1}$ because $k_i \to k$ in $\mathfrak{X}$.
  - For d divisible by p, $d^{k_i-1} \to 0$ because $k_i \to \infty$ in $\mathbf{R}$.

  Thus

$$\sigma_{k_i-1}(n) \to \sum_{\substack{d|n \\ (d,p)=1}} d^{k-1} + 0 = \sigma^*_{k-1}(n).$$

- Moreover, this convergence is uniform in n (check this!).

p-adic Eisenstein series

- Suppose $k \neq 0 \in \mathfrak{X}$ is even (i.e. $k \in 2\mathfrak{X}$).
- Pick a sequence of even integers $k_i \geq 4$ as above.
- By the theorem, the sequence

$$G_{k_i} = \frac{1}{2}\zeta(1-k_i) + \sum_{n=1}^{\infty} \sigma_{k_i-1}(n)q^n$$

converges to a p-adic modular form of weight k.

Proposition (p-adic Eisenstein series)

For even $k \in \mathfrak{X} - \{0\}$, there is a p-adic modular form of weight k

$$G_k^* := a_0 + \sum_{n=1}^{\infty} \sigma_{k-1}^*(n)q^n$$

where $a_0 = \lim_{i\to\infty} \frac{1}{2}\zeta(1-k_i)$.

p-adic zeta function

Denote the constant term a_0 by

$$\frac{1}{2}\zeta^*(1-k) := \lim_{i\to\infty} \frac{1}{2}\zeta(1-k_i).$$

Definition (p-adic zeta function)

$\zeta^*(s)$ defines a function on the odd elements of $\mathfrak{X} - \{1\}$.

Continuity:

- Suppose $k_i \to k$, all of which are even elements of $\mathfrak{X} - \{0\}$.
- Then the non-constant coefficients of $G^*_{k_i}$ tend to those of G^*_k uniformly.
- By the theorem, $\frac{1}{2}\zeta^*(1-k_i) \to \frac{1}{2}\zeta^*(1-k)$.

p-adic zeta function: special values

Question

What are the special values $\zeta^*(1-k)$ for $k \in \mathbf{Z}_{\geq 2}$ even?

Warning:

- $\zeta^*(1-k)$ is constructed as the limit of $\zeta(1-k_i)$ for a sequence $k_i \to k$ (in $\mathfrak{X}$) satisfying $k_i \to \infty$ (in $\mathbf{R}$), so *a priori*

$$\zeta^*(1-k) \neq \zeta(1-k)$$

  even when $k \in \mathbf{Z}_{\geq 2}$!
- Here is a formal (but bogus!) calculation:

$$\zeta^*(1-k) = \lim_{i\to\infty} \zeta(1-k_i) \overset{?!}{=} \lim_{i\to\infty} \prod_{\ell} \frac{1}{1-\ell^{k_i-1}}$$

$$= \prod_{\ell \neq p} \frac{1}{1-\ell^{k-1}} \overset{?!}{=} (1-p^{k-1})\zeta(1-k).$$

p-adic zeta function: special values

Proposition

For $k \in \mathbf{Z}_{\geq 2}$ even, $\zeta^*(1-k) = (1-p^{k-1})\zeta(1-k)$.

- Serre deduces this from théorème 3 on P.206-207, by identifying ζ^* as the Kubota–Leopoldt p-adic zeta function.
- However, it seems more natural to turn the development around: an alternative calculation of $\zeta^*(1-k)$ would **imply** théorème 3.
- It is possible to do this using the Hecke operator V_p, which acts on $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbf{Q}_p[[q]]$ via

$$f|V_p := \sum_{n=0}^{\infty} a_n q^{np}.$$

We shall see later that V_p preserves the space of p-adic modular forms $M_k^\dagger$.

p-adic zeta function: special values

Taking the operator V_p for granted, we can relate

$$G_k^* = \frac{1}{2}\zeta^*(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}^*(n)q^n$$

and

$$G_k = \frac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n.$$

Lemma

If $k \in \mathbf{Z}_{\geq 2}$ is even, then

$$G_k^* = G_k | (1 - p^{k-1}V_p).$$

See Problem Sheet 2.

p-adic zeta function

- Comparing the constant terms yields

$$\zeta^*(1-k) = (1-p^{k-1})\zeta(1-k).$$

- **Summary:** ζ^* is a **continuous** function on the odd elements of $\mathfrak{X} - \{1\}$ and interpolates the values $(1-p^{k-1})\zeta(1-k)$ at the negative odd integers (a **dense** subset of $\mathfrak{X}$!).
- Therefore, ζ^* must coincide with the **Kubota–Leopoldt p-adic zeta function** L_p – see théorème 3 on P.206 for a precise statement.

p-adic zeta function

Remark

- This gives a clean construction of the p-adic zeta function.
- But this doesn't come for free!
- In the development of mod p and p-adic modular forms:
  1. Clausen–von Staudt theorem ("elementary"): used extensively for E_{p-1} and $E_{p^m(p-1)}$
  2. Kummer congruence ("deep"): used to show $E_{p+1} \equiv E_2 \pmod{p}$
- In particular, it seems impossible to completely avoid the use of Kummer congruence; the whole theory relies upon the single instance of $E_{p+1} \equiv E_2 \pmod{p}$.

p-adic zeta function for totally real fields

Let K be a totally real number field.

- Dedekind zeta function of K:

$$\zeta_K(s) := \sum_{\substack{0 \neq \mathfrak{a} \subset \mathcal{O}_K \\ \text{ideal}}} N\mathfrak{a}^{-s} = \prod_{\substack{0 \neq \mathfrak{p} \subset \mathcal{O}_K \\ \text{prime}}} (1 - N\mathfrak{p}^{-s})^{-1}$$

- Special values:
  - $\zeta_K(1-n) = 0$ for all odd $n \in \mathbf{Z}_{\geq 1}$ (unless $K = \mathbf{Q}$ and $n = 1$)
  - $\zeta_K(1-n) \in \mathbf{Q} - \{0\}$ for all even $n \in \mathbf{Z}_{\geq 1}$, with generalized Clausen–von Staudt and Kummer
- Eisenstein series with constant terms $\sim \zeta_K(1-n)$
- Construction of p-adic zeta function $\zeta_K^*(s)$

See §5 for a detailed treatment.

Motivation: Congruences for j

- Consider the j-invariant

$$j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n) q^n, \quad c(n) \in \mathbf{Z}.$$

- For $p \in \{2, 3, 5, 7, 11\}$, we saw congruences (Lehner, 1949; Atkin, 1966) which imply

$$c(p^m n) \to 0 \text{ in } \mathbf{Q}_p \text{ as } m \to \infty.$$

- Conceptually $c(p^m n)$ is the n-th Fourier coefficient of $j | U_p^m$, but there are technical issues:
  1. j is a meromorphic modular function on $\mathrm{SL}_2(\mathbf{Z})$, with a pole at ∞.
  2. $j | U_p$ is a meromorphic modular function on $\Gamma_0(p)$, with a pole at 0.

Hecke operators on classical modular forms

Let $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ be a (classical) modular form of weight k and level 1.
For ℓ prime, recall the Hecke operator is given by

$$f|_k T_\ell = \sum_{n=0}^{\infty} a_{n\ell} q^n + \ell^{k-1} \sum_{n=0}^{\infty} a_n q^{n\ell} \in M_k.$$

Question

To what extent does this formula define an operator on p-adic modular forms?

Hecke operators on p-adic modular forms

Let $f = \sum a_n q^n \in M_k^\dagger$ be a p-adic modular form, and $f_i = \sum a_n^{(i)} q^n \in M_{k_i}$ be a sequence of (classical) modular forms with $f_i \to f$.

For any prime ℓ,

$$f_i|_{k_i} T_\ell = \sum_{n=0}^{\infty} a_{n\ell}^{(i)} q^n + \ell^{k_i - 1} \sum_{n=0}^{\infty} a_n^{(i)} q^{n\ell}$$

is a modular form of weight k_i.

The assumption $f_i \to f$ implies:

- $a_n^{(i)} \to a_n$ for all n.
- $k_i \to k$.

Question

When can we say "$f_i|_{k_i} T_\ell \to f|_k T_\ell$"?

Hecke operators T_ℓ for $\ell \neq p$

Remark (p-adic weights)

Recall that for each fixed $d \in \mathbf{Z}_p^\times$, the map $\mathfrak{X} \to \mathbf{Q}_p^\times$ given by $k \mapsto d^k$ is continuous.

If $\ell \neq p$, then $\ell^{k_i - 1} \to \ell^{k-1}$, so

$$f_i|_{k_i} T_\ell = \sum_{n=0}^{\infty} a_{n\ell}^{(i)} q^n + \ell^{k_i - 1} \sum_{n=0}^{\infty} a_n^{(i)} q^{n\ell} \in M_{k_i}$$

tends to

$$f|_k T_\ell := \sum_{n=0}^{\infty} a_{n\ell} q^n + \ell^{k-1} \sum_{n=0}^{\infty} a_n q^{n\ell}.$$

Hence this defines a p-adic modular form of weight k.