FARGUES–FONTAINE CURVE

NOTES TAKEN BY PAK-HIN LEE

Abstract. These are notes from the ongoing learning seminar on the Fargues–Fontaine curve at Columbia University during Winter Break 2015, which is organized by Wei Zhang.

The seminar closely follows Sean Howe's notes for Laurent Fargues' lectures at the University of Chicago. These notes merely serve to reflect the topics covered at the learning seminar at Columbia and are not carefully edited; the reader who wishes to learn the materials should instead consult the original papers by Fargues and Fontaine and the excellent notes by Howe.

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Last updated: December 19, 2015. Please send corrections and comments to phlee@math.columbia.edu.

1. Lecture 1 (December 15, 2015): Daniel Gulotta

1.1. **Introduction.** The Fargues–Fontaine curve is a geometric object that can be used to study local Langlands. Let E and K be local fields (i.e., finite extensions of $\mathbb{Q}_p$, or $\mathbb{F}_p((t))$) with the same residue characteristic. Local Langlands studies representations[1]

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell}).$$

We can replace GL_n by general reductive groups but we will stick to this simpler case. The Fargues–Fontaine curve X will be a "curve" over E, with a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-action. We will study vector bundles on X.

Remark. X is not of finite type over E, but it behaves like a curve in many ways: regular, noetherian, dimension 1, "complete" in the sense that there is a degree function on divisors so that nonzero effective divisors have degree > 0 and principal divisors have degree 0.

Examples of objects that have a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-action are: K^{sep}, $\mathcal{O}_{K^{\mathrm{sep}}}$, $\widehat{\overline{K}}$, $\widehat{\mathcal{O}_{\overline{K}}}$. What we want is roughly

$$\text{“}\operatorname{Spa}(E, \mathcal{O}_E) \times_{\overline{\mathbb{F}_p}} \operatorname{Spa}(\widehat{\overline{K}}, \mathcal{O}_{\widehat{\overline{K}}})\text{”}.$$

This doesn't literally make sense: E and K could have characteristic 0, so E never contains $\overline{\mathbb{F}_p}$. The fundamental group of this object should be the Galois group of E.

1.2. **Fundamental groups.** For nice connected topological spaces X and Y,

$$\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y).$$

This is also true for X, Y varieties over algebraically closed fields of characteristic 0, or in characteristic p if X or Y is proper. But if $X = Y = \operatorname{Spec} \mathbb{F}_p$, then $X \times_{\mathbb{F}_p} X \cong X$, so

$$\pi_1(X \times_{\mathbb{F}_p} Y) \cong \widehat{\mathbb{Z}} \text{ but } \pi_1(X) \times \pi_1(Y) \cong \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}.$$

The issue is that there is only one Frobenius acting on $X \times Y$.

Theorem 1.1 (Drinfeld). *Let X, Y be connected schemes of finite type over $\mathbb{F}_q$. Let $(X \times Y)/\mathrm{p.Fr.}$ be the category of finite étale maps $S \to X \times Y$ along with an isomorphism $S \xrightarrow{\sim} F_X^* S$, where F_X is the Frobenius on X. Then*

$$\pi_1(X \times Y/\mathrm{p.Fr.}) \cong \pi_1(X) \times \pi_1(Y).$$

This suggests that we should take the product of the spaces $\operatorname{Spa}(E, \mathcal{O}_E)$ and $\operatorname{Spa}(\widehat{\overline{K}}, \mathcal{O}_{\widehat{\overline{K}}})$ and quotient out by the Frobenius. More precisely,

- If E and K have characteristic p, then we will take

$$\operatorname{Spa}(E, \mathcal{O}_E) \times_{\mathbb{F}_q} \operatorname{Spa}(\widehat{\overline{K}}, \mathcal{O}_{\widehat{\overline{K}}})/\phi^{\mathbb{Z}}$$

  where ϕ is the Frobenius on K.
- If E has characteristic 0, then we take

$$\mathcal{O}_E \widehat{\otimes}_{\mathbb{Z}_p} W(\mathcal{O}_{\widehat{\overline{K}}})/\phi^{\mathbb{Z}}.$$

[1]PH: During the lecture there was some confusion as to what the coefficient field should be. After discussion with Rahul, we believe the correct treatment is to take $\overline{\mathbb{Q}_\ell}$-coefficients (where $\ell \neq p$), i.e., we are studying *classical* local Langlands, not p-adic local Langlands.

- If K has characteristic 0, we replace $\widehat{\overline{K}}$ with $\widehat{\overline{K}}^\flat$.

The Fargues–Fontaine curve satisfies

$$X^\diamond = \mathrm{Spd}\, E \times \mathrm{Spd}\, \widehat{\overline{K}}^\flat / \text{p.Fr.}$$

but we won't be talking about diamonds.

1.3. **The adic Fargues–Fontaine curve.** Let E be a local field with residue field $\mathbb{F}_q$ and uniformizer π. Let F be a characteristic p perfectoid field over $\mathbb{F}_q$ (i.e., perfect, complete with respect to a nontrivial valuation $F \to \mathbb{R}_{\geq 0}$; for example, we can take $\widehat{\overline{K}}^\flat$), and ϖ_F be a nonzero nonunit element of $\mathcal{O}_F$.

Let $\mathbb{A}$ be the unique $\mathcal{O}_E$-algebra satisfying:

1. $\mathbb{A}$ is π-adically complete,
2. $\mathbb{A}$ is flat over $\mathcal{O}_E$,
3. $\mathbb{A}/\pi\mathbb{A} \cong \mathcal{O}_F$.

More explicitly,

- If E has characteristic p, this is $\mathcal{O}_E \widehat{\otimes}_{\mathbb{F}_p} \mathcal{O}_F$ $(= \mathcal{O}_F[[\pi]]$ since $\mathcal{O}_E \cong \mathbb{F}_q[[\pi]])$.
- If E has characteristic 0, this is $W(\mathcal{O}_F) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$. (In classical p-adic Hodge theory, this is called A_{inf}.)

Consider the adic space $\mathrm{Spa}(\mathbb{A}, \mathbb{A})$. Points are equivalence classes of continuous valuations $|\cdot| : \mathbb{A} \to \Gamma \cup \{0\}$ where Γ is a totally ordered abelian group (which we will write multiplicatively). Furthermore, we require that $|a| \leq 1$ for all $a \in \mathbb{A}$.

A rational subset of $\mathrm{Spa}(\mathbb{A}, \mathbb{A})$ is a subset defined by inequalities

$$|f_1| \leq g, |f_2| \leq g, \cdots, |f_n| \leq g, |g| \neq 0$$

for $f_1, \cdots, f_n, g \in \mathbb{A}$ such that $(f_1, \cdots, f_n, g)$ is open.

There is a "Teichmüller lift" $[\cdot] : \mathcal{O}_F \to \mathbb{A}$, which is multiplicative. The maximal ideal of $\mathbb{A}$ is $(\pi, [\varpi_F]^{1/p^\infty})$.

$\mathrm{Spa}(\mathbb{A}, \mathbb{A})$ has one closed point, corresponding to the maximal ideal $\mathfrak{m} = (\pi, [\varpi_F^{1/p^\infty}])$ (the valuation sends $\mathfrak{m} \mapsto 0$, and everything else to 1). All remaining points are analytic (kernel of valuation not open).

Let

$$\mathcal{Y} = \mathrm{Spa}(\mathbb{A}, \mathbb{A})_a = \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \backslash \{\text{closed point}\}.$$

For all points in $\mathcal{Y}$, either $|\pi| \neq 0$ or $|[\pi_F]| \neq 0$. We can define a continuous map $\mathcal{Y} \to [0, 1]$ such that for any rank 1 valuation $|\cdot| : \mathbb{A} \to \mathbb{R}$,

$$|\cdot| \mapsto q^{-\frac{\log |\pi|}{\log |[\varpi_F]|}}$$

In equal characteristic, this actually is a disc since $\mathbb{A} \cong \mathcal{O}_F[[\pi]]$.

Let $I \subset [0, 1]$ be an interval with endpoints in $q^{\mathbb{Q}} \cup \{0\}$, not equal to $\emptyset$, $\{0\}$, $\{1\}$ or $[0, 1]$. Define Y_I to be the inverse image of I. If $I = [q^r, q^s]$ is closed, this is a rational subset defined by the inequalities

$$|[\varpi_F]|^r \leq \pi \leq |[\varpi_F]|^s.$$

In general, Y_I is an increasing union of rational subsets.

For any $\rho \in [0,1]$, define the "Gauss norm" $|\cdot|_\rho$ as follows. Let $f = \sum_n [x_n]\pi^n \in \mathbb{A}$. (Each element of $\mathbb{A}$ can be represented uniquely in this way.) Define

$$|f|_p = \begin{cases} \sup_n |x_n|\rho^n & \text{if } \rho > 0, \\ \sup_{n \mid |x_n|=1} q^{-n} & \text{if } \rho = 0. \end{cases}$$

This norm extends:

- to $\mathbb{A}[\pi^{-1}, [\varpi_F]^{-1}]$ if $\rho \notin \{0,1\}$;
- to $\mathbb{A}[\pi^{-1}]$ if $\rho = 1$;
- to $\mathbb{A}[[\varpi_F]^{-1}]$ if $\rho = 0$.

Then $\mathcal{O}(Y_I)$ is the completion of $\mathbb{A}[\pi^{-1}, [\varpi_F]^{-1}]$ with respect to $|\cdot|_\rho$ for $\rho \in I$ when $0, 1 \notin I$. If $0 \in I$, we take $\mathbb{A}[[\varpi_F]^{-1}]$ instead. If $1 \in I$, we take $\mathbb{A}[\pi^{-1}]$ instead. If I is closed, we only need to take the endpoint norms.

Now we are ready to define the adic Fargues–Fontaine curve. The Frobenius action on $\mathbb{A}$

$$\sum_n [x_n]\pi^n \mapsto \sum_n [x_n^q]\pi^n$$

defines a map $\phi : \mathcal{Y} \to \mathcal{Y}$. This map sends the radius $\rho \mapsto \rho^q$ and fits in the commutative diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "\phi"'] & I \arrow[d, "\rho \mapsto \rho^q"] \\ \mathcal{Y} \arrow[r] & I \end{tikzcd}$$

The fixed points of I are 0 and 1. $\mathcal{Y}$ has a point above 0 with $|\pi| = 0$, and a point above 1 with $|[\varpi_F]| = 0$. ϕ does not fix any other points of $\mathcal{Y}$.

Let $Y = \mathcal{Y}_{(0,1)}$ and $X = Y/\phi$. This is an adic space over E.

Definition 1.2. X is the Fargues–Fontaine curve.

Eventually we will study vector bundles on X. It will turn out that the only semistable vector bundle of slope 0 is the trivial bundle. For example, if we want to study $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-representations, we can look at the Galois action on the Fargues–Fontaine curve.

1.4. **Connections to p-adic Hodge theory.** Let $E = \mathbb{Q}_p$, $K/\mathbb{Q}_p$ finite, $F = \mathbb{C}_p^\flat$. $\mathbb{A}$ is the ring denoted A_{inf} in p-adic Hodge theory. There is a map $\mathbb{A} \to \mathbb{C}_p$

$$\sum_n [x_n]p^n \mapsto \sum_n x_n^{(0)} p^n,$$

which defines a point $x_C \in X$ by pulling back from $\mathbb{C}_p$. x_C is the only $\mathrm{Gal}(\overline{K}/K)$-fixed point of X. The completed local ring at x_C is B_{dR}^+. The global sections of $Y = \mathcal{Y}_{(0,1)}$ are $\mathcal{O}_Y = B_{\mathrm{rig}}^+$.

There are also connections with (ϕ, Γ)-modules, by considering $F = (\mathbb{Q}_p^{\mathrm{cyc}})^\flat$. In this case $\mathcal{O}_Y = \mathcal{R}$.

2. Lecture 2 (December 18, 2015): Qirui Li

2.1. **Perfectoid fields.** We want to relate the characteristic p local field $\mathbb{F}_q((\pi))$ and the mixed characteristic local field $\mathbb{Q}_p$. The idea is that if we keep adjoining p-power roots of the uniformizer to $\mathbb{Q}_p$

$$\mathbb{Q}_p \subseteq \mathbb{Q}_p(p^{1/p}) \subseteq \mathbb{Q}_p(p^{1/p^2}) \subseteq \cdots$$

these roots "converge" to π and the field $\mathbb{Q}_p(p^{1/p^\infty})$ "behaves like" $\mathbb{F}_p((\pi^{1/p^\infty}))$. Roughly speaking, the process of passing from $\mathbb{Q}_p(p^{1/p^\infty})$ to $\mathbb{F}_q((\pi^{1/p^\infty}))$ will be known as "tilting", and the reverse will be "untilting".

Let K be a complete non-discrete valuation field, with ring of integers $\mathcal{O}_K$ and uniformizer ϖ satisfying $0 < |\varpi| < |p|$.

Definition 2.1. K is called a perfectoid field if $\mathrm{Frob} : \mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi$ given by $x \mapsto x^p$ is surjective.

Example 2.2. $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is a perfectoid field.

Example 2.3. If p is odd, then $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ is a perfectoid field since the map

$$\begin{aligned} \widehat{\mathbb{Z}_p(\zeta_{p^\infty})}/p &\to \widehat{\mathbb{Z}_p(\zeta_{p^\infty})}/p \\ \overline{\zeta_{p^{n+1}} - 1} &\mapsto \overline{\zeta_{p^n} - 1} \end{aligned}$$

is an isomorphism.

Example 2.4. $\mathbb{F}_q\widehat{((\pi^{1/p^\infty}))}$ is a perfectoid field.

2.2. **Tilting.** Suppose K is a perfectoid field. Define

$$K^\flat = \lim_{x \mapsto x^p} K = \{(x^{(0)}, x^{(1)}, \cdots) \mid x^{(n)} = (x^{(n+1)})^p\},$$

with multiplication

$$(x^{(0)}, x^{(1)}, \cdots) \times (y^{(0)}, y^{(1)}, \cdots) = (x^{(0)}y^{(0)}, x^{(1)}y^{(1)}, \cdots)$$

and addition

$$(x^{(0)}, x^{(1)}, \cdots) + (y^{(0)}, y^{(1)}, \cdots) = (c^{(0)}, c^{(1)}, \cdots)$$

where

$$c^{(n)} = \lim_{N \to \infty} (x^{(n+N)} + y^{(n+N)})^{p^N}.$$

These make $K^\flat$ into a ring. We define a valuation on $K^\flat$ as follows: if $x = (x^{(0)}, x^{(1)}, \cdots) \in K^\flat$, then

$$|x| = |x^{(0)}|.$$

$K^\flat$ is a complete valuation field, called the tilting of K.

Proposition 2.5. *There exists an isomorphism*

$$\theta : \mathcal{O}_{K^\flat} \xrightarrow{\cong} (\mathcal{O}_K/\varpi)^\flat.$$

Proof. The map is given by
$$(a^{(0)}, a^{(1)}, \cdots) \mapsto (\overline{a^{(0)}}, \overline{a^{(1)}}, \cdots).$$
For the inverse map, given $(\overline{a^{(0)}}, \overline{a^{(1)}}, \cdots) \in (\mathcal{O}_K/\varpi)^\flat$, we let $(\widetilde{a}^{(0)}, \widetilde{a}^{(1)}, \cdots)$ be an arbitrary lifting, and
$$a^{(n)} = \lim_{N\to\infty} (\widetilde{a}^{(n+N)})^{p^N}.$$
Then the sequence $(a^{(0)}, a^{(1)}, \cdots)$ lies in $\mathcal{O}_{K^\flat}$. $\square$

Example 2.6. The tilting of $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is $\widehat{\mathbb{F}_p((\pi^{1/p^\infty}))}$, by tilting the isomorphism
$$\mathbb{Z}_p[p^{1/p^\infty}]/p \xrightarrow{\sim} \mathbb{F}_p[\pi^{1/p^\infty}]/\pi$$
and using Proposition 2.5.

Example 2.7. The tilting of $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ is $\widehat{\mathbb{F}_p((\pi_\epsilon^{1/p^\infty}))}$, where π_ϵ reduces to $\overline{\pi_\epsilon} = (\overline{1-\zeta_p}, \overline{1-\zeta_{p^2}}, \cdots)$.

Theorem 2.8. *If K is a perfectoid field, and L/K is a field extension of degree n, then*

(1) *L is a perfectoid field.*
(2) *$L^\flat$ is a field extension of degree n of $K^\flat$.*
(3) *$\mathcal{O}_L$ is* almost étale *over $\mathcal{O}_K$, i.e., for any $\epsilon > 0$, there exists $x_i \in \mathcal{O}_L$ for all $1 \le i \le n$ such that*
$$1 - \epsilon < |\det(\mathrm{Tr}(x_i x_j))| < 1$$
(equivalently, $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is killed by arbitrarily small powers of p, or by the maximal ideal $\mathfrak{m}$).
(4) *There is an equivalence of categories*
$$\{\text{finite étale extensions of } K\} \leftrightarrow \{\text{finite étale extensions of } K^\flat\}.$$

Corollary 2.9.

(1) *A perfectoid field K is algebraically closed if and only if $K^\flat$ is algebraically closed.*
(2) $\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K^\flat}/K^\flat)$.

Example 2.10. The absolute Galois groups of $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{F}_p((\pi^{1/p^\infty}))}$ are isomorphic.

2.3. **Geometric picture.** Recall the definition from last time. Let E be a discrete valuation field with ring of integers $\mathcal{O}_E$ and residue field $\mathbb{F}_q$, and F be a perfectoid field over $\mathbb{F}_q$ with pseudo-uniformizer ϖ of $\mathcal{O}_E$. We want to study perfectoid fields over E that tilt to F.

Define
$$\mathbb{A} = W_{\mathcal{O}_E}(\mathcal{O}_F) = W(\mathcal{O}_F) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$$
and
$$\mathcal{Y} = \mathrm{Spa}(\mathbb{A}) \backslash V(\pi, \varpi).$$
There is a Teichmüller lifting $[\cdot] : \mathcal{O}_F \to \mathbb{A}$. Define
$$Y = \mathrm{Spa}(\mathbb{A}) \backslash V(\pi[\varpi]) = \mathcal{Y}_{(0,1)}.$$

Next we will define the classical points $|Y|^{\mathrm{cl}}$. An $f = \sum_{i=0}^\infty [x_i]\pi^i \in \mathbb{A}$ is called primitive if $x_0 \neq 0$ and there exists $d \ge 0$ such that $x_d \in \mathcal{O}_F^\times$. The minimum such d is called the degree of f.

It is clear that $\deg(fg) = \deg(f) + \deg(g)$.

Example 2.11. {Primitive degree 0 elements} $\simeq \mathbb{A}^\times$.

Example 2.12. If $a \notin \mathcal{O}_F^\times$, then $\deg(\pi - a) = 1$.

Definition 2.13.

$$|Y|^{\mathrm{cl}} = \{\text{irreducible primitive elements}\}/\mathbb{A}^\times = \mathrm{Irr}/\sim.$$

Recall that $B = \mathcal{O}(Y) = \mathcal{O}(\mathcal{Y}_{(0,1)})$. If $E = \mathbb{F}_q((\pi))$, then

$$B = \left\{ \sum_{i\in\mathbb{Z}} [a_i]\pi^i \mid \lim_{x\to\infty} |a_i|\rho^i = 0 \text{ for any } \rho \in (0,1) \right\}.$$

Theorem 2.14. *Let $y = (f) \in |Y|^{\mathrm{cl}}$ where f is primitive of degree d. Let $k(y) = B/(f)$ and $\theta : B \twoheadrightarrow k(y)$. Then*

(1) *$k(y)$ is a perfectoid field over E.*
(2) *There is a map $F \to k(y)^\flat$ given by $a \mapsto \theta([a^{1/p^N}])$, and $k(y)^\flat$ is a degree d extension of F.*
(3) *There is a 1-1 correspondence*

$$\mathrm{Irr}^{\deg=1}/\sim \leftrightarrow \{(K,i) \mid K \textit{ is perfectoid over } E, i : F \xrightarrow{\cong} K^\flat\}/\sim$$

(4) *If F is algebraically closed, then every primitive degree 1 element f is $(f) = (\pi - [a])$ for some $0 < |a| < 1$.*

Warning: The choice of a is not unique, i.e., it might be the case that

$$(\pi - [a_1]) = (\pi - [a_2]).$$

By considering the diagram

$$\begin{tikzcd}
B \arrow[r] & B/(\pi - [a_1]) \\
{[F]} \arrow[u, hook, "{[a]\mapsto[a]}"] \arrow[r, "{[a_1]\mapsto\pi}"] & k(y) \arrow[u] \\
F \arrow[u, "{a\mapsto[a]}"] \arrow[r, "\cong"] & k(y)^\flat \arrow[u, "{(x^{(0)},x^{(1)},\cdots)\mapsto x^{(0)}}"']
\end{tikzcd}$$

we see that the ambiguity is the $\mathbb{Z}_p(1)$-torsor

$$\{(x^{(0)}, x^{(1)}, \cdots) \in k(y)^\flat \mid x^{(0)} = \pi\}.$$

For any $y = (f) \in |Y|^{\mathrm{cl}}$, $k(y) = B/(f)$, we have the map $\theta : \mathbb{A}[\frac{1}{\pi}] \to k(y)$ with $\ker\theta = f$. The completion of the local ring at y is

$$\widehat{\mathcal{O}_{Y,y}} = \lim_{n\to\infty} \mathbb{A}\left[\frac{1}{\pi}\right]/(\ker\theta)^n.$$

We define

$$B_{\mathrm{dR}}^+(k(y)) = \lim_{n\to\infty} \mathbb{A}\left[\frac{1}{\pi}\right]/(f)^n.$$

For $y = \alpha$, the local ring is

$$\lim_{\rho \to 0} \mathcal{O}_{\mathcal{Y}_{[0,\rho]}} = \left\{ \sum_{i=0}^{\infty} [x_i]\pi^i \mid \text{there exists } \epsilon > 0 \text{ such that } \lim |x_i|\epsilon^i = 0 \right\}$$

and the completion is $\widehat{\mathcal{O}_{\mathcal{Y},\alpha}} = W_{\mathcal{O}_E}(F)$. These are also denoted as $\mathcal{O}_{\mathcal{E}^\dagger}$ and $\mathcal{O}_{\mathcal{E}}$ respectively, where $\dagger$ means "overconvergent".