KATO'S EULER SYSTEMS

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ABSTRACT. These are notes from the (ongoing) Student Number Theory Seminar on Kato's Euler systems at Columbia University in Fall 2016, which is organized by David Hansen and myself.

Contents

Last updated: September 16, 2016. Please send corrections and comments to phlee@math.columbia.edu.

1. Lecture 1 (September 15, 2016): David Hansen

1.1. **Introduction.** Let us recall what Euler systems are and what they are good for.

Let E be an elliptic curve over $\mathbf{Q}$. There is no question that $E(\mathbf{Q})$ is a very interesting group. How do we study it? Of course we have the BSD conjecture, but it is very hard to prove in any generality. The difficulty is captured in the following diagram (borrowed from Kato):

$$\begin{array}{c} \text{groups of cycles} \\ \text{(arithmetic)} \end{array} \xleftrightarrow[\ ?\ ]{\text{too far}} \begin{array}{c} L\text{-functions} \\ \text{(analytic)} \end{array}$$

Kato's idea is that Euler systems are supposed to be "arithmetic shapes" of L-functions.

$$\begin{array}{ccc} & \text{Euler systems} & \\ \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow \\ \text{groups of cycles} & & L\text{-functions} \end{array}$$

The left arrow is a general machine due to Kolyvagin, Perrin-Riou, Kato, Rubin, etc. The right arrow requires p-adic Hodge theory and algebraic geometry to understand and culminates in the "explicit reciprocity laws". Euler systems are very hard to construct, and once they are constructed they remain hard to understand.

Fix T a continuous geometric representation of $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on a finite free $\mathbf{Z}_p$-module (or $\mathcal{O}_L$-module for a finite extension $L/\mathbf{Q}_p$). An *Euler system* for T is a collection of cohomology classes $c_m \in H^1(\mathbf{Q}(\zeta_m), T)$ which are related under corestriction: for any $m \geq 1$ and prime ℓ,

$$\mathrm{cores}_{\mathbf{Q}(\zeta_{m\ell})/\mathbf{Q}(\zeta_m)}(c_{m\ell}) = \begin{cases} c_m & \text{if } \ell \mid m \text{ (or maybe if } T \text{ is ramified at } \ell), \\ P_\ell(\sigma_\ell^{-1}) c_m & \text{if } \ell \nmid m \text{ and } T \text{ is unramified at } \ell; \end{cases}$$

here $P_\ell(X) = \det(1 - \sigma_\ell^{-1} X | T^*(1))$ and σ_ℓ is the arithmetic Frobenius.

Rough idea: An Euler system for T controls the *Selmer group* associated with $T^*(1)$, which is a subgroup of $H^1(\mathbf{Q}, T^*(1) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p)$.

Idealized very rough statement: The index of c_1 in $H^1_{\{p\}}(\mathbf{Q}, T)$ is a bound for the exponent of $\mathrm{Sel}(T^*(1))$. (In practice this only works if $\dim(T^{\sigma_\infty = -1}) = 1$.)

There are two basic cases:

- $T = \mathbf{Z}_p(1)$;
- $T = T_f \subset V_f$ the p-adic Galois representation associated to a holomorphic newform $f \in S_k(\Gamma_1(N))$, $k \geq 2$.

This seminar is concerned with the latter.

L-functions have nothing to do with the idealized statement above, and only come into the picture when we take into consideration *all* the Euler systems.

"The construction of Euler systems is a totally artisinal activity." — Colmez

1.2. **A basic example.** Take $T = \mathbf{Z}_p(1)$. There is a Kummer map

$$\mathbf{Q}(\zeta_m)^\times \to H^1(\mathbf{Q}(\zeta_m), \mathbf{Z}_p(1)).$$

For $p \mid m$, define c_m as the image of $(1 - \zeta_m)(1 - \zeta_m^{-1})$, where $\zeta_m \in \overline{\mathbf{Q}}^\times$ are consistent m-th roots of unity.

This factors through the étale cohomology of an affine scheme

$$H^1_{\text{ét}}\left(\operatorname{Spec}\mathbf{Z}\left[\zeta_m, \frac{1}{p}\right], \mathbf{Z}_p(1)\right) \to H^1(\mathbf{Q}(\zeta_m), \mathbf{Z}_p(1)).$$

In this case

$$\operatorname{Sel}(\mathbf{Q}(\zeta_m), T^*(1)) \simeq \operatorname{Cl}(\mathbf{Q}(\zeta_m)) \otimes \mathbf{Q}_p/\mathbf{Z}_p.$$

By the class number formula, class numbers are given by the p-adic valuation of some L-function, hence the index of c_1 is controlled by L-functions. How do we get L-values out of $(c_m)_m$?

Here is the idea. For $p \nmid m$, the system $c_{mp^\infty} = (c_{mp^n})_{n\geq 1}$ defines an element of

$$\varprojlim_n H^1(\mathbf{Q}(\zeta_{mp^n}), \mathbf{Z}_p(1)) \cong H^1(\mathbf{Q}(\zeta_m), \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1))$$

where $\Lambda := \mathbf{Z}_p[[\operatorname{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})]]$. There is a specialization map along $\Lambda \twoheadrightarrow \mathbf{Z}_p(k-1)$ which gives

$$\begin{aligned} H^1(\mathbf{Q}(\zeta_m), \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)) &\to H^1(\mathbf{Q}(\zeta_m), \mathbf{Z}_p(k)) \\ \gamma &\mapsto \chi_{\text{cyc}}^{k-1}(\gamma). \end{aligned}$$

We have a restriction map $H^1(\mathbf{Q}(\zeta_m), \mathbf{Z}_p(k)) \otimes \mathbf{Q}_p \to H^1(\mathbf{Q}_p(\zeta_m), \mathbf{Q}_p(k))$.

Finally, if $k < 0$, the Bloch–Kato exponential gives an isomorphism

$$\mathbb{D}_{\text{dR}}(\mathbf{Q}_p(k)) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\zeta_m) \xrightarrow{\sim} H^1(\mathbf{Q}_p(\zeta_m), \mathbf{Q}_p(k))$$

and the left-hand side is canonically $\mathbf{Q}_p(\zeta_m)$.

Composing all these maps, we obtain

$$\varprojlim_n H^1(\mathbf{Q}(\zeta_{mp^n}), \mathbf{Z}_p(1)) \to \mathbf{Q}_p(\zeta_m).$$

Miracle (Coates–Wiles, Coleman): The image in $\mathbf{Q}_p(\zeta_m)$ of c_{mp^∞} under this map is a linear combination of numbers $L(k,\eta) \in \mathbf{Q}$, where η's are Dirichlet characters of conductor dividing m.

The magic here is the "Bloch–Kato exponential"

$$\exp_V : \mathbb{D}_{\text{dR}}(V) \to H^1(\mathbf{Q}_p, V)$$

which factors through $\mathbb{D}_{\text{dR}}/\mathbb{D}^+_{\text{dR}}$; here V is any p-adic representation of $G_{\mathbf{Q}_p}$, and the map is the connecting map obtained by applying the functor $(-)^{G_{\mathbf{Q}_p}}$ to the "fundamental exact sequence of p-adic Hodge theory"

$$\left(0 \to \mathbf{Q}_p \to (B_{\text{crys}})^{\varphi=1} \to B_{\text{dR}}/B^+_{\text{dR}} \to 0\right) \otimes_{\mathbf{Q}_p} V.$$

1.3. **Kato's Euler system(s).** First, Kato constructs (following Beilinson) some interesting units

$${}_cg_{1/M}, {}_dg_{1/N} \in \mathcal{O}^\times(Y(M,N)),$$

where $Y(M,N)$ is the affine modular curve over $\mathbf{Q}$ parametrizing triples (E, P_1, P_2) where P_1 and P_2 are points of exact order M and N respectively such that $\mathbf{Z}/M \times \mathbf{Z}/N \to E$ sending $(a,b) \mapsto aP_1 + bP_2$ is injective. Here c and d are arbitrary parameters chosen to avoid some finite list of conditions. Thus we get

$${}_{c,d}z_{M,N} = \{{}_cg_{1/M}, {}_dg_{1/N}\} \in K_2(Y(M,N)).$$

These already satisfy various norm compatibilities as M, N vary.

There is a natural map $\mathcal{O}^\times(Y) \to H^1_{\text{ét}}(Y, \mathbf{Z}_p(1))$, and a compatible map $K_2(Y) \to H^2(Y, \mathbf{Z}_p(2))$. In particular, we can apply this for $Y = Y(M, N)$ and feed in ${}_{c,d}z_{M,N}$ to obtain

$$\left({}_{c,d}z^{(p)}_{Mp^n, Np^n}\right) \in \varprojlim_n H^2_{\text{ét}}(Y(Mp^n, Np^n), \mathbf{Z}_p(2)).$$

By some magic mystery manipulations, this cohomology group maps to

$$\cdots \to H^2(Y(M,N), \operatorname{sym}^{k-2}(\mathcal{H}_p)(k-r)) \overset{\text{edge}}{\to} H^1(\mathbf{Q}, V_{k,\mathbf{Z}_p}(M,N)(k-r)),$$

where $\mathcal{H}_p = T_pE(-1)$ and $V_{k,\mathbf{Z}_p}(M,N)(k-r) = H^1_{\text{ét}}(Y(M,N)_{\overline{\mathbf{Q}}}, \operatorname{sym}^{k-2}(\mathcal{H}_p)) \otimes \mathbf{Z}_p(k-r)$. This class is labeled ${}_{c,d}z^{(p)}_{M,N}(k, r, r')$, where $k \geq 2$, $1 \leq r' \leq k-1$, and r is arbitrary.

Note that the space $V_{k,\mathbf{Z}_p}(M,N)$ is basically a direct sum of Galois representations of weight k modular forms, so $\mathbb{D}_{\text{dR}}(-)$ should be a coherent cohomology space of weight k modular forms $\approx M_k(M,N)$.

The rough statement of Kato's explicit reciprocity laws: The image of ${}_{c,d}z^{(p)}_{M,N}(k, r, r')$ in this space (after applying a Bloch–Kato dual exponential) is a product of two Eisenstein series.

By Rankin–Selberg, the projection of this to any f-eigenspace gives L-values.