MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

LECTURE 9 (NOVEMBER 11, 2014)
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AN INTRODUCTION TO p-ADIC HODGE THEORY, II

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ABSTRACT. p-adic Hodge theory, broadly speaking, is the study of representations of the absolute Galois group of $\mathbb{Q}_p$ acting on $\mathbb{Q}_p$-vector spaces. Classical Hodge theory concerns the relationship between the singular and de Rham cohomologies of a compact Kähler manifold. p-adic Hodge theory began as the search for a similar theory relating the étale and de Rham cohomologies of varieties over p-adic fields. Over time, however, it's grown more broadly into a subject with its own rich inner life and with many applications in number theory.

During these two talks, I hope to explain some of the motivation and philosophy of p-adic Hodge theory, as well as some of its successes. Some applications which I hope to touch on include: properties of Galois representations associated with modular forms, good reduction of abelian varieties, modularity of elliptic curves, the Fontaine–Mazur conjecture, special values of L-functions, and some results on Hodge numbers of varieties over number fields.

Let me recall the setup from last time. Let $K/\mathbb{Q}_p$ be a finite extension, with ring of integers $\mathcal{O}_K$ and residue field $k = \mathcal{O}_K/\text{maximal ideal}$. $\overline{K}$ denotes an algebraic closure of K, and $\mathbb{C}_p = \hat{\overline{K}}$ is the p-adic completion of $\overline{K}$. It is a theorem that this is still algebraically closed. G_K denotes the Galois group $\text{Gal}(\overline{K}/K)$. The things we are primarily interested in are p-adic representations, i.e., a $\mathbb{Q}_p$-vector space V with a (continuous $\mathbb{Q}_p$-linear) action of G_K

To shed some light on these objects, Fontaine introduced the following formalism. He defined topological $\mathbb{Q}_p$-algebras $B_\bullet$ with actions of G_K which "classify" interesting p-adic representations. Given one of these rings, one can form

$$\mathbb{D}_\bullet(V) := (V \otimes_{\mathbb{Q}_p} B_\bullet)^{G_K}$$

which is a module over $B_\bullet^{G_K}$. We say V is "$\bullet$" if $\mathbb{D}_\bullet(V) \otimes_{B_\bullet^{G_K}} B_\bullet \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} B_\bullet$.

Since V and $B_\bullet$ are $\mathbb{Q}_p$-algebras, $\mathbb{Q}_p$ sits inside $V \otimes_{\mathbb{Q}_p} B_\bullet$. If we look at the preimage of V in $\mathbb{D}_\bullet(V) \otimes_{B_\bullet^{G_K}} B_\bullet$, since the Galois action is trivial on $\mathbb{D}_\bullet(V)$, we get that V embeds into a direct sum of finitely many $B_\bullet$'s.

The goals today are:

- B_{dR} and B_{crys},
- applications and current directions in p-adic Hodge theory.

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The Fontaine rings are hard to define and involve the Witt vectors, so I will focus on their properties. Let us begin l'anatomie des l'éléphants B_{dR} and B_{crys}.

B_{dR} is filtered by subspaces

$$\cdots \subset \mathrm{Fil}^{i+1} \subset \mathrm{Fil}^{i} \subset \cdots$$

such that $\mathrm{Fil}^i \cdot \mathrm{Fil}^j = \mathrm{Fil}^{i+j}$. We see that

$$B_{\mathrm{dR}}^+ := \mathrm{Fil}^0 B_{\mathrm{dR}}$$

is a subring of B_{dR}, and

$$B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[\xi^{-1}]$$

for any $\xi \in B_{\mathrm{dR}} \backslash B_{\mathrm{dR}}^+$. Why are we isolating this subring? B_{dR}^+ is a local ring with maximal ideal $\mathrm{Fil}^1 B_{\mathrm{dR}}$ and residue field $\mathbb{C}_p$. In particular, the power series ring $\mathbb{C}_p[[x]][x^{-1}]$ is a good toy model for B_{dR}. In fact this field is isomorphic to B_{dR}, but not canonically or continuously or G_K-equivariantly. Note that $\bigcup_i \mathrm{Fil}^i = B_{\mathrm{dR}}$, $\bigwedge_i \mathrm{Fil}^i = \{0\}$, and $B_{\mathrm{dR}}^{G_K} = K$.

$\mathbb{D}_{\mathrm{dR}}(V)$ is a K-vector space, filtered by K-vector subspaces

$$\mathrm{Fil}^i \mathbb{D}_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^i B_{\mathrm{dR}})^{G_K}.$$

The functor $\mathbb{D}_{\mathrm{dR}}$ takes p-adic Galois representaions to filtered K-vector spaces, which are much simpler. Inside B_{dR}, one has B_{crys}. If we think of B_{dR} as the Laurent series, then B_{crys} is roughly the space of meromorphic functions convergent inside some fixed disk. The ring B_{crys} has much more structure than B_{dR}. It is dense but not complete for the induced filtration. If we take Galois invariants, we don't usually get all of K but only

$$B_{\mathrm{crys}}^{G_K} = K_0,$$

the maximal subfield of K unramified over $\mathbb{Q}_p$. On K_0, one has the automorphism φ lifting $x \mapsto x^p$ on $\mathcal{O}_{K_0}/(p) = k$. There exists a canonical action of φ on B_{crys} compatible with φ on K_0.

Last time I was talking about various comparison theorems for different cohomologies. Let $X_{/K}$ be a smooth proper variety. We have:

- étale cohomology $H^*_{\text{ét}}(X_{/\overline{K}}, \mathbb{Q}_p)$. This is a p-adic representation of G_K. This should be thought of as the p-adic analogue of singular cohomology, and *a priori* doesn't see anything coherent on X.
- de Rham cohomology $H^*_{\mathrm{dR}}(X_{/K}) := \mathbb{H}^*(X_{/K}, \Omega^\bullet_X)$. This is the algebraic analogue of de Rham cohomology of a complex manifold. This is a filtered K-vector space, filtered by $\mathbb{H}^*(X_{/K}, \Omega^{\geq i}) \to \mathbb{H}^*(X_{/K}, \Omega^\bullet)$.

These two things live in very different universes.

Conjecture 1 (Fontaine (1980); proved by Faltings (1989–2002), Tsuji, Kisin, Scholze)**.** *There exists a canonical and functorial isomorphism of filtered K-vector spaces*

$$\mathbb{D}_{\mathrm{dR}}(H^n_{\text{ét}}(X_{/\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X_{/K}).$$

We can think of the ring B_{dR} as giving a bridge between the étale world and the coherent world.

If $n = 1$ and $X = E$ is an elliptic curve, then $H^1(X_{/\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathbb{Q}_p}(V_p E, \mathbb{Q}_p)$ is identified with the linear dual of the Tate module, and $H^1_{\mathrm{dR}}(E_{/K})$ sits in a sequence

$$0 \to H^0(E, \Omega_E) \to H^1_{\mathrm{dR}}(E_{/K}) \to H^1(E, \mathcal{O}_E) \to 0.$$

This case is a strong motivation for Fontaine.

There is a third cohomology, which I'm definitely not going to define:

- $H^*_{\mathrm{crys}}(\mathfrak{X}_k)$ if $X = \mathfrak{X} \times_{\mathcal{O}_K} K$ where $\mathfrak{X}_{/\mathcal{O}_K}$ is smooth and proper. This is a finite-dimensional K_0-vector space with an endomorphism φ.

Conjecture 2 (same persons as above)**.**

$$\mathbb{D}_{\mathrm{crys}}(H^n_{\text{ét}}(X_{/\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{crys}}(\mathfrak{X}_k).$$

Now we will apply this formalism to concrete examples. Recall a question from last time. Let

$$\Delta = q \prod_{n\geq 1}(1-q^n)^{24} =: \sum_{n\geq 1} \tau(n) q^n$$

be the Ramanujan delta function. By Deligne, for all p there exists a unique

$$\rho_{\Delta,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p)$$

such that $\operatorname{tr} \rho_{\Delta,p}(\mathrm{Frob}_\ell) = \tau(\ell)$ for any prime $\ell \neq p$. We may ask:

Question 1. Can one recover $\tau(p)$ from $\rho_{\Delta,p}|_{G_{\mathbb{Q}_p}}$?

The answer is yes:

$$\tau(p) = \text{trace of the action of } \varphi \text{ on } \mathbb{D}_{\mathrm{crys}}(\rho_{\Delta,p}|_{G_{\mathbb{Q}_p}})$$

due to Saito. This is secretly a comparison theorem for the crystalline cohomology.

Question 2. Suppose V is an irreducible representation of $G_{\mathbb{Q}}$ on some finite-dimensional L-vector space, where $L/\mathbb{Q}_p$ is a finite extension. When is V a subquotient of some $H^*_{\mathrm{dR}}(X_{/\overline{\mathbb{Q}}}, L)$?

This is the natural source of global Galois representations. There is a conjectural answer to this question.

Conjecture 3 (Fontaine–Mazur ($\sim$ 1993))**.** *V arises in this manner if and only if:*

- *$V|_{I_{\mathbb{Q}_\ell}}$ is trivial for all but finitely many ℓ (recall that $I_{\mathbb{Q}_\ell} \subset G_{\mathbb{Q}_\ell} \subset G_{\mathbb{Q}}$), and*
- *$V|_{G_{\mathbb{Q}_p}}$ is de Rham.*

One cannot even state this conjecture without defining de Rham Galois representations. Of course the name has nothing to do with de Rham cohomology directly.

In the 1-dimensional case, this theorem is pretty trivial and boils down to class field theory.

Theorem 4 (Wiles, Taylor–Wiles, Breuil, Kisin, Emerton, Khare–Wintenberger)**.** *This conjecture is true when $\dim_L V = 2$ and $\det(c_\infty : V \to V) = -1$, where $\langle c_\infty \rangle = G_{\mathbb{C}/\mathbb{R}} \hookrightarrow G_{\mathbb{Q}}$.*

The second condition is sometimes written as "V is odd".

The proof of this involves actually constructing X and associating certain modular forms (of which the modularity of elliptic curves is a propotypical example), and uses p-adic Hodge theory in crucial ways. Wiles introduced the idea of studying deformations of Galois representations. Already he needed the notion of flat deformation functors, but these are not enough in general. The contributions of Breuil and Kisin were to borrow techniques from integral p-adic Hodge theory.

Theorem 5 (Tsuji). *Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite type schemes over $\mathbb{Z}[\frac{1}{s}]$ whose generic fibers $X = \mathcal{X} \times_{\mathbb{Z}} \mathbb{C}$ and $Y = \mathcal{Y} \times_{\mathbb{Z}} \mathbb{C}$ are smooth and proper, and suppose that $|\mathcal{X}(\mathbb{F}_p)| = |\mathcal{Y}(\mathbb{F}_p)|$ for almost all p. Then*

$$\dim_{\mathbb{C}} H^i(X, \Omega_X^j) = \dim_{\mathbb{C}} H^i(Y, \Omega_Y^j)$$

for all i, j.

Idea of proof. Define

$$V := \bigoplus_{i \text{ even}} H^i_{\text{ét}}(\mathcal{X}_{/\overline{\mathbb{Q}}}, \mathbb{Q}_p) \oplus \bigoplus_{i \text{ odd}} H^i_{\text{ét}}(\mathcal{Y}_{/\overline{\mathbb{Q}}}, \mathbb{Q}_p)$$

and W the same with $\mathcal{X}$ and $\mathcal{Y}$ swapped. Then as a virtual representation,

$$V - W = \left(\sum (-1)^i H^i(\mathcal{X}_{/\mathbb{Q}})\right) - \left(\sum (-1)^i H^i(\mathcal{Y}_{/\mathbb{Q}})\right).$$

The trace of Frobenius at almost all p is the same on these two pieces. By Chebotarev density, we have $V^{\text{ss}} \simeq W^{\text{ss}}$. By purity, we get

$$H^i(\mathcal{Y}_{/\overline{\mathbb{Q}}})^{\text{ss}} \simeq H^i(\mathcal{X}_{/\overline{\mathbb{Q}}})^{\text{ss}}.$$

Now use de Rham comparison. □

Our last topic is the moduli of abelian varieties, which are things of current interest.

Let D be a fixed finite-dimensional K_0-vector space with a φ, and let $h : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function such that $\sum_{i \in \mathbb{Z}} h(i) = \dim_{K_0} D$. Let $\mathcal{M}_{D,h}$ be the functor which assigns to any finite L/K_0 the set of pairs (V, ι) where V is a crystalline representation of G_L with $\dim_L(\text{Fil}^i / \text{Fil}^{i+1})\mathbb{D}_{\text{dR}}(V) = h(i)$ and $\iota : \mathbb{D}_{\text{crys}}(V) \xrightarrow{\sim} D \otimes_{K_0} L_0$.

This is analogous to the fact that abelian varieties over $\mathbb{C}$ are determined by their singular cohomology and Hodge structure.

Theorem 6 (Rapoport–Zink). *If $h(i) \neq 0$ only for $i \in \{0, -1\}$, then $\mathcal{M}_{D,h}$ is an open subspace of a flag variety.*

What about general h? Scholze is currently teaching a course at Berkeley, and based on what he has done so far, my guess is that he has proved the following

Theorem 7 (Scholze). *$\mathcal{M}_{D,h}$ is representable by a "diamond".*

A diamond can come from a variety over $\mathbb{Q}_p$ as a topological space can come from a complex manifold.