MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

LECTURE 8 (OCTOBER 28, 2014) DAVID HANSEN

AN INTRODUCTION TO p-ADIC HODGE THEORY, I

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ABSTRACT. p-adic Hodge theory, broadly speaking, is the study of representations of the absolute Galois group of $\mathbb{Q}_p$ acting on $\mathbb{Q}_p$-vector spaces. Classical Hodge theory concerns the relationship between the singular and de Rham cohomologies of a compact Kähler manifold. p-adic Hodge theory began as the search for a similar theory relating the étale and de Rham cohomologies of varieties over p-adic fields. Over time, however, it's grown more broadly into a subject with its own rich inner life and with many applications in number theory.

During these two talks, I hope to explain some of the motivation and philosophy of p-adic Hodge theory, as well as some of its successes. Some applications which I hope to touch on include: properties of Galois representations associated with modular forms, good reduction of abelian varieties, modularity of elliptic curves, the Fontaine–Mazur conjecture, special values of L-functions, and some results on Hodge numbers of varieties over number fields.

Let us start with some notations which will be in effect the whole time. Let p be a prime, and let K be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}_p$ in K, with maximal ideal $\mathfrak{m}_K = (\varpi_K)$ and residue field $k = \mathcal{O}_K/\mathfrak{m}_K$. Let $\overline{K}$ be an algebraic closure of K. Our main object of study is $G_K = \mathrm{Gal}(\overline{K}/K)$.

Definition 1. A p-adic representation of G_K is a $\mathbb{Q}_p$-vector space V equipped with a (continuous) action of G_K.

There are cases where V is not finite-dimensional, and continuity requires a bit more care.

Let us recall the structure of G_K. There is an exact sequence

$$1 \to I_K \to G_K \twoheadrightarrow G_k \cong \hat{\mathbb{Z}} \to 1$$

where I_K is the inertia subgroup. It sits in another exact sequence

$$1 \to P_K \to I_K \overset{t_p}{\twoheadrightarrow} \prod_{\ell \neq p} \mathbb{Z}_\ell \to 1$$

where P_K is the "wild" inertia subgroup. The main thing for now is that P_K is a pro-p group. We can think of G_K as having a filtration with three pieces: $\hat{\mathbb{Z}}$, almost $\hat{\mathbb{Z}}$ with p-part stripped away, and P_K. P_K is what makes the representations of G_K subtle in p-adic situations.

As a consequence, any continuous $\rho : G_K \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ with $\ell \neq p$ satisfies that $\rho(P_K)$ is finite. p-adic representations are much wilder than ℓ-adic representations.

Let me give some examples.

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Example 2. Let $\chi : G_K \to L^\times$ be any continuous character, where $L/\mathbb{Q}_p$ is a finite extension. Take $V = L$ with action given by $g \cdot v = \chi(g)v$. We denote this as "$L(\chi)$".

Example 3. $\mathbb{Z}_p(1) := \{(1 = x_0, x_1, x_2, \cdots) \in K, x_{i+1}^p = x_i \text{ for all } i \geq 0\}$. This is non-canonically isomorphic to $\mathbb{Z}_p$, and has an action by G_K. There exists a unique $\chi_{\text{cyc}} : G_K \to \mathbb{Z}_p^\times$ such that

$$g \cdot x = (x_0^{\chi_{\text{cyc}}(g)}, x_1^{\chi_{\text{cyc}}(g)}, \cdots).$$

In other words, $g(\zeta) = \zeta^{\chi_{\text{cyc}}(g)}$ for any p^∞-th root of unity $\zeta \in \overline{K}$.

Define $\mathbb{Q}_p(1) = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $M(n) := M \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1)^{\otimes n}$ where M is any $\mathbb{Q}_p[G_K]$-module.

Note $\mathbb{Q}_p(1) \simeq \mathbb{Q}_p(\chi_{\text{cyc}})$, but non-canonically so.

Example 4. Let A be an abelian variety over K. The ℓ-adic Tate module is

$$T_\ell A := \varprojlim_n A(\overline{K})[\ell^n]$$

which is a finite free $\mathbb{Z}_\ell$-module of rank $2 \dim A$. The rational Tate module is

$$V_\ell A := T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

We will start posing some questions which we will eventually answer using p-adic Hodge theory. First we have a theorem of Serre and Tate, sometimes called the Néron–Ogg–Shafarevich criterion.

Theorem 5 (Serre–Tate)**.** *A has good reduction if and only if I_K acts trivially on $V_\ell A$ for some (any) $\ell \neq p$.*

A natural question is:

Question 1. Is there a similar good reduction criterion involving $V_p A$?

Example 6. Let $\Delta = q \prod_{n=1}^\infty (1 - q^n)^{24} = \sum_{n=1}^\infty \tau(n) q^n$.

Theorem 7 (Deligne)**.** *There exists a unique representation $\rho_{\Delta,p} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{Q}_p)$ such that*

$$\operatorname{tr} \rho_{\Delta,p}(\mathrm{Frob}_\ell) = \tau(\ell)$$

for any prime $\ell \neq p$.

Note $\mathrm{Frob}_\ell \in G_{\mathbb{Q}_\ell} \hookrightarrow G_\mathbb{Q}$.

Question 2. Can we recover $\tau(p)$ from $\rho_{\Delta,p}|_{G_{\mathbb{Q}_p}}$ somehow?

Let me give one more example which leads to more general things.

Example 8. Let X be any smooth proper variety over K. Then

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) := \left(\varprojlim_j H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^j\mathbb{Z}) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a p-adic representation.

If $X = A$ and $n = 1$, then $H^1_{\text{ét}}(A, \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathbb{Q}_p}(V_p A, \mathbb{Q}_p)$.

What sort of structure might we expect from such an object? Let us recall the complex analogue. Let X be a compact Riemannian manifold. Then the singular cohomology is related to differential objects via the de Rham isomorphism

$$H^n(X, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C} \cong H^n_{\text{dR}}(X, \mathbb{C}).$$

If X has more structure, say X is Kähler, then

$$H^n_{\mathrm{dR}}(X, \mathbb{C}) = \bigoplus_i H^{n-i}(X, \Omega^i_X)$$

where Ω^j_X is the sheaf of holomorphic j-forms.

Question 3. Is there some relationship between the étale cohomology groups $H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ and $H^*_{\mathrm{dR}}(X)$ or $H^*(X, \Omega^j_X)$?

The latter groups are very coherent and should really be thought of as the analogous objects of holomorphically defined cohomology.

Conjecture 9 (Tate (1967), proved by Faltings (1989))**.** *For any X as above, there is a canonical isomorphism*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i=0}^{n} H^{n-i}(X, \Omega^i_X) \otimes_K \mathbb{C}_p(-i)$$

where $\mathbb{C}_p$ is the p-adic completion of $\overline{K}$.

Tate proved this when $X = A$ is an abelian variety and $n = 1$.

There is a natural action of G_K on both sides: on the left it acts diagonally, and on the right it is trivial on the Hodge cohomology groups. We may ask if this isomorphism is G_K-equivariant. There is a remarkable corollary. Before stating it we need another theorem.

Theorem 10 (Tate (1967))**.**

$$H^0(G_K, \mathbb{C}_p(j)) = \begin{cases} K & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

Then the conjecture implies that

Corollary 11.

$$\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(j)\right)^{G_K} \cong \left(\bigoplus_{i=0}^{n} H^{n-i}(X, \Omega^i_X) \otimes_K \mathbb{C}_p(-i+j)\right)^{G_K} = H^{n-j}(X, \Omega^j_X).$$

Thus we have recovered the Hodge cohomology groups in some Galois-theoretic way.

The general philosophy and goals of p-adic Hodge theory are:

- Define and study interesting subcategories of the category of all p-adic representations of G_K.
- Relate them to representations occuring "in nature".
- Use these ideas to solve actual problems.

One overarching philosophy of how to do these is due to Fontaine: define "interesting" period rings B, which are topological $\mathbb{Q}_p$-algebras with a G_K-action and some extra structures. For any p-adic representation V, one can form

$$\mathbb{D}_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{G_K}.$$

This is a module over B^{G_K}, which tends to be a field. This module inherits whatever extra structure B has.

Let me try to recast the Hodge–Tate conjecture.

Example 12.

$$B_{\mathrm{HT}} := \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i).$$

By the theorem of Tate and the relation $\mathbb{C}_p(i) \otimes_{\mathbb{C}_p} \mathbb{C}_p(j) = \mathbb{C}_p(i+j)$,

$$B_{\mathrm{HT}}^{G_K} = \bigoplus_i \mathbb{C}_p(i)^{G_K} = K.$$

We can restate the conjecture as follows. If $V = H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, then

$$\mathbb{D}_{B_{\mathrm{HT}}}(V) \cong \bigoplus_j H^{n-j}(X, \Omega^j_X).$$

The interesting period rings are $B_\bullet$, where $\bullet \in \{\mathrm{HT}, \mathrm{dR}, \mathrm{st}, \mathrm{crys}\}$. These stand for "Hodge–Tate, de Rham, semistable and crystalline" respectively. For each of these adjectives, one defines

$$\mathbb{D}_\bullet(V) := \mathbb{D}_{B_\bullet}(V).$$

$\mathbb{D}$ is for Dieudonné.

For any one of these, one has a natural map

$$\alpha_\bullet : \mathbb{D}_\bullet(V) \otimes_{B_\bullet^{G_K}} B_\bullet \to V \otimes_{\mathbb{Q}_p} B_\bullet$$

which is always injective. The proof is this is not trivial and requires a careful analysis of each of these rings.

Definition 13. V is "blah" if α_{blah} is an isomorphism.

Let me spell out what it means to be a Hodge–Tate representation, which is the simplest category of all.

Example 14. V is Hodge–Tate if and only if

$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \bigoplus_i \mathbb{C}_p(i)^{\oplus n_i}$$

with $\sum_i n_i = \dim_{\mathbb{Q}_p} V$.

For any variety X, $H^n(X_{\overline{K}}, \mathbb{Q}_p)$ is Hodge–Tate. Indeed we have a more precise decomposition in terms of Hodge cohomology.

An answer to Question 1 is given by the

Theorem 15 (Coleman–Iovita)**.** *A has good reduction if and only if V_pA is crystalline.*